The EGH conjecture and the Sperner property of Complete intersections

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There is a strong parallelism between the theory of posets and the theory of Artinian rings.

Suppose that $P = \bigsqcup_{i=0}^{c} P_i$ is a finite graded poset.

If you do not know what this means, please consider the Boolean lattice:

$$P = 2^{\{1,2,\cdots,n\}}.$$

P is a graded poset with the graded piece

$$P_i = \{M \in P | \# M = i\}$$

at level i and it gives us the decomposition:

$$P = \bigsqcup_{i=0}^{n} P_i.$$

Note $2^{\{1,2,\cdots,n\}}$ can be thought of as the set of square free monomials. So the lattice

$$P = 2^{\{x_1,x_2,\cdots,x_n\}}$$

is a graded basis of the Artinian algebra

$$A = \mathbb{Q}[x_1, x_2, \cdots, x_n]/(x_1^2, \cdots, x_n^2),$$

$$A = \sum_{i=0}^{n} A_i.$$

The number of elements $|P_i|$ in a graded component of P is called a rank number. I would like to call the sequence

$$(|P_0|, |P_1|, \cdots, |P_{c-1}|, |P_c|)$$

$c = n$ in this case.

the rank vector of P.

$$\boxed{\begin{array}{c}\textbf{Definition}\\ \text{Let } P = \bigsqcup_{i=0}^{c} P_i \text{ be a finite graded poset. The rank numbers are the entries of the vector:}\\ (|P_0|, |P_1|, \cdots, |P_{c-1}|, |P_c|)\end{array}}$$

A rank vector is unimodal if $\exists j$

$$(|P_0| \leq |P_1| \leq \cdots \leq |P_j| \geq \cdots \geq |P_{c-1}| \geq |P_c|)$$

The $\boxed{\text{Sperner number}}$ of P is the most number of $|P_i|$.
The $\boxed{\text{Dilworth number}}$, denoted by $d(P)$, is

$$d(P) = \mathrm{Max}\{|I|; \text{antichain } I \subset P\}$$

Recall that P has the $\boxed{\text{Sperner property}}$ if the Dilworth number $d(P)$ is equal to the Sperner number.
We will state it as a definition:

Definition : Sperner property (for posets)

Let $P = \bigsqcup_{i=0}^{c} P_i$ be a graded poset. P has the Sperner property if

$$d(P) = \mathrm{Sperner}(P)$$

i.e., $\max_i |P_i| \geq |I|$ for any antichain I in P.

If we translate this definition into the language of commutative rings, then the definition is

Definition : Sperner property (for artinian algebras)

Let $A = \bigoplus_{i=0}^{c} A_i$ be a graded artinian algebra. A has the Sperner property if

$$d(A) = \mathrm{Sperner}(A)$$

i.e., $\max_i(\dim_K A_i) \geq \mu(\mathfrak{a})$ for any ideal $\mathfrak{a}$ in A.

The Dilworth number $d(A)$ is defined by

$$d(A) := \max\{\mu(I) | I \text{ ideal of } A\}.$$

One of the motivations for the definition of the weak Lefschetz property was the implication

$$\boxed{\boxed{\text{Weak Lefschetz property}} \Rightarrow \boxed{\text{Sperner Property}}}$$

The proof is easy. In some sense "Sperner property" is the goal, "weak Lefschetz property" is a method to prove it.

Although "the weak Lefschetz property" of Artinian rings is a very good definition, it has one weakness:

In many cases the assumption

"characteristic zero on the ground field"

is almost inevitable.

For example

$$\text{Let } A = K[x_1, x_2, \cdots, x_n]/(x_1^2, x_2^2, \cdots, x_n^2).$$

$$A \begin{cases} \text{has the WLP if char } K = 0, \\ \text{does not have the WLP if char } K > 0, \end{cases}$$

$$\text{if } p < n.$$

On the other hand,

$$A \text{ has the Sprener property for any } K.$$

So I would like to introduce a new definition:

> **Definition**
>
> **Let $A = \bigoplus_{i=0}^{c} A_i$ be a graded algebra over $K = A_0$ a field. A has the matching property if**
>
> $$\dim_K V \leq \dim_K(A_1 \cdot V) \text{ if } \dim_K A_i \leq \dim_K A_{i+1},$$
>
> **for any vector subspace $V \subset A_i$, where $\boxed{A_1 \cdot V}$ denotes the subspace of A_{i+1} spanned by**
>
> $$\{xy | x \in A_1, y \in V\}$$

If we borrow a term from Hall's Marriage Theorem, $A_1 \cdot V$ can be called the neighbor of V. If $\mathfrak{a}$ is an ideal generated by elements of the same degree, then

$$\dim_K V \leq \dim_K(A_1 \cdot V) \Leftrightarrow \mu(\mathfrak{a}) \leq \mu(\mathfrak{m}\mathfrak{a}).$$

With this definition we can prove three interesting theorems as follows:

Theorem 1

Suppose that $A = \bigoplus_{i=0}^{c} A_i$ is a graded Gorenstein algebra. If $A = \bigoplus_{i=0}^{c} A_i$ has the matching property and if A has a unimodal Hilbert function, then A has the Sperner property.

Theorem 2

Assume that the EGH conjecture is true. Then every complete intersection has the Sperner property. (I will explain the EGH conjecture shortly.)

Theorem 3

Suppose that $A = K[x_1, x_2, \cdots, x_n]/(F_1, F_2, \cdots, F_n)$ is a complete intersection, where F_i is a product of linear forms. Then A has the Sperner property.

(Abed Abedelfatah proved that the EGH conjecture is true for such A.)

Sketch of proof

There are implications:

$$\textbf{Theorem 1} \Rightarrow \textbf{Theorem 2} \Rightarrow \textbf{Theorem 3.}$$

Let me outline the proof for Theorem 1.
We use the following notation.

Notation

Suppose $(A, \mathfrak{m})$ is a local ring.

$$\mu(\mathfrak{a}) = \dim \mathfrak{a}/\mathfrak{m}\mathfrak{a}$$

$$\tau(\mathfrak{a}) = \dim(\mathfrak{a} : \mathfrak{m})/\mathfrak{a}$$

$$d(A) = \mathrm{Max}\{\mu(\mathfrak{a}) | \mathfrak{a} \subset A.\}$$

So $\mu(\mathfrak{a})$ is the minimal number of generators and $\tau(\mathfrak{a})$ is the type of $\mathfrak{a}$. ($d(A)$ is the Dilworth number of A.)

$\tau(\mathfrak{a})$ is the index of reduciblity.

For an Artinian (Gorenstein) local ring $(A, \mathfrak{m})$ we introduce two families of ideals:

$$\mathcal{F}(A) := \{\mathfrak{a} \mid \mu(\mathfrak{a}) = d(A)\}, \quad \mathcal{G}(A) := \{\mathfrak{a} \mid \tau(\mathfrak{a}) = d(A)\}.$$

If A is graded, we assume that $\mathfrak{a}$ runs over all graded ideals of A (although it does not make much difference). The following result was proved in Ikeda-Watanabe:

"Dilworth lattice of Artin rings" Journal of commut. Algebra vol. 1, no.2 (2009), 315–326.

Proposition

$\mathcal{F}(A)$ and $\mathcal{G}(A)$ are posets with the inclusion as an order, and moreover these are lattices with respect to $+$ and $\cap$ as join and meet, and they are isomorphic as lattices via the correspondence:

$$\mathcal{F}(A) \leftrightarrow \mathcal{G}(A)$$

$$\mathcal{F}(A) \ni \mathfrak{a} \mapsto \mathfrak{m}\mathfrak{a} \in \mathcal{G}(A), \text{ and}$$
$$\mathcal{G}(A) \ni \mathfrak{a} \mapsto (\mathfrak{a} : \mathfrak{m}) \in \mathcal{F}(A).$$

Assume that A is Gorenstein. Then the correspondence

$$\mathcal{F}(A) \to \mathcal{G}(A)$$

defined by $\mathfrak{a} \mapsto (0 : \mathfrak{a})$ gives an order reversing isomorphism of lattices. Since A is Gorenstein, we have $0 : (0 : \mathfrak{a}) = \mathfrak{a}$, which implies that the same correspondence $\mathfrak{a} \mapsto 0 : \mathfrak{a}$ gives us the isomorphism in the opposite direction also.

$\boxed{\text{Continue: Outline of proof of Theorem 1}}$

Proof of Theorem 1.

Suppose that $\mathfrak{a} \subset A$ is an ideal with $\mu(\mathfrak{a}) = d(A)$. We have to show that $\dim_K A_i = \mu(\mathfrak{a})\ \exists i$.

Compare two ideals $\mathfrak{a}$ and $\mathfrak{a}'$ defined as follows:

$$\mathfrak{a} = \sum_{j \geq \alpha} \mathfrak{a}_j \supset \mathfrak{a}' := \sum_{j > \alpha} \mathfrak{a}_j.$$

(α is the initial degree of $\mathfrak{a}$.)

Basis elements for $\mathfrak{a}_\alpha$, say,

$$f_1, f_2, \cdots, f_l$$

do not exist in $\mathfrak{a}'$ but basis elements chosen among

$$A_1 \cdot f_1, A_1 \cdot f_2, \cdots, A_1 \cdot f_l$$

are a part of a minimal generating set for $\mathfrak{a}'$.

If $\dim_K \mathfrak{a}_\alpha \leq \dim_K(A_1 \cdot \mathfrak{a}_\alpha)$, this implies that

$$\mu(\mathfrak{a}) \leq \mu(\mathfrak{a}'). \tag{1}$$

Let j_0 be the smallest integer such that

$$\dim_K A_{j_0} > \dim_K A_{j_0+1},$$

and (let α be the initial degree of $\mathfrak{a}$ and) assume for the moment

$$\alpha < j_0.$$

Then the matching property implies the inequality (1).

Since $\mu(\mathfrak{a}) < \mu(\mathfrak{a}')$ is not the case, this implies that

$$\mu(\mathfrak{a}) = \mu(\mathfrak{a}') = d(A).$$

This argument can be used repeatedly so we may assume that there exists an ideal $\mathfrak{a}$ such that

$$\mu(\mathfrak{a}) = d(A), \text{ and } \mathfrak{a} \subset \sum_{i \geq j_0} A_i = \mathfrak{m}^{j_0}.$$

By Ikeda's isomorphisms

$$\mathcal{F} \to \mathcal{G} \to \mathcal{F}$$

$$\mathfrak{a} \mapsto \mathfrak{m}\mathfrak{a} \mapsto 0 : \mathfrak{m}\mathfrak{a},$$

The ideal $0 : \mathfrak{m}\mathfrak{a}$ has the same number of generators as $\mathfrak{a}$. But

$$0 : \mathfrak{m}\mathfrak{a} \supset 0 : \mathfrak{m}^{j_0+1} \supset \mathfrak{m}^{c-j_0},$$

and since A has the unimodal Hilbert vector $c - j_0 \leq j_0$. So

$$0 : \mathfrak{m}\mathfrak{a} \supset 0 : \mathfrak{m}^{j_0+1} \supset \mathfrak{m}^{c-j_0} \supset \mathfrak{m}^{j_0}.$$

By the same argument as before, we may construct an ideal $\mathfrak{a}''$ containing $\mathfrak{m}^{j_0}$ with initial degree at least j_0, with $\mu(\mathfrak{a}'') = d(A)$. So proof is complete.

Proof for Theorem 2

The EGH conjecture is this:

A is a complete intersection (artinian) and B is a monomial complete intersection with the the same set of generator degrees as A. If $I \subset A$ is an ideal, then there should be an ideal $J \subset B$ such that A/I and B/J have the same Hilbert function.

In view of Theorem 1, it suffices to show that A has the matching property:

$$\dim_K V \leq \dim_K (A_1 \cdot V)$$

for any $V \subset A_j$ as long as $\dim_K A_j \leq \dim_K A_{j+1}$.

It is easy to see that the monomial complete intersection has the matching property. (For proof use the matching property of finite chain product in combinatorics. Or also the WLP of the monomial complete intersection over $\mathbb{Q}$ can be used.)

Before I proceed, I would like to make a remark.

Remark

The Hilbert function of an ideal $I \subset A$ usually means the Hilbert function of A/I. So it is the vector:

$$(|(A/I)_0|, |(A/I)_1|, |(A/I)_2|, \cdots ,)$$

(I used the notation $|V| = \dim_K V$.)
I want to say the Hilbert funtion of I is the vector:

$$(0, \cdots , 0, |I_\alpha|, |I_{\alpha+1}|, |I_{\alpha+2}|, \cdots ,)$$

Of course this is determined by the Hilbert functions of A and A/I.

Given $V \subset A_j$, let I be the ideal generated by V. Then the first non-zero part of the Hilbert function for I is

$$(|V|, |A_1 \cdot V|, \cdots,)$$

where $|V| := \dim_K V$, $|A_1 \cdot V| := \dim_K(A_1 \cdot V)$.

The EGH Conjecture says that there is an ideal J in B (the monomial complete intersection) such that J and I have the same Hilbert function. So the proof is complete.

$\boxed{\text{Proof of Theorem 3}}$

A recent result of Abed Abedelfatah

"On the Eisenbud-Green-Harris Conjecture"

says that the EGH Conjecture is ture for a complete intersection whose defining ideal is generated by products of linear forms. Therefore we can apply Theorem 2.

$\boxed{\text{Outline of proof of Abedelfatah's theorem}}$

How can it be proved that the EGH Conjecture is true for a complete intersection if the defining ideal is generated by products of linear forms?

$$\boxed{\boxed{\textbf{This is how.}}}$$

If the defining ideal of A is generated by products of linear forms, then there exists a linear element $l \in A$ such that both A/lA and $A/(0:l)$ are complete intersections whose defining ideals are products of linear forms. Furthermore there is a short exact sequence:

$$0 \to A/(0:l) \to A \to A/(l) \to 0.$$

So we can use the induction to prove that the EGH conjecture is true for A.

Problem 1

Is the matching property inherited by a subring (with the same socle)? It is true that many complete intersections are subrings of quadratic complete intersections. If the answer to this problem is yes, then it encourages us to assume that generators are degree two to start with.

Problem 2

Is the EGH conjecture inherited by a subring (with the same socle)? Original EGH conjecture is about quadratic complete intersections. This fact and Chris McDaniel's result in mind, I want to emphasize this problem, because if the answer is yes, it might reduce the EGH conjecture to the original case.

This is the end of my talk. Thank you very much.

Some remarks on EGH Conjecture

The original EGH conjecture is the case all degrees of $F_1, \cdots, F_n$ are two so each generator. In this sense the complete intersection:

$$K[x_1, x_2, \cdots, x_n]/(f_1, f_2, \cdots, f_n)$$

where f_i is a product of two linear forms is interesting.

Assume that the Young subgroup

$$G := S_{n_1} \times S_{n_2} \times \cdots S_{n_r} \subset S_n$$

acts on $K[x_1, x_2, \cdots, x_n]$ block-wise (so S_{n_1} permutes the first n_1 variables, and S_{n_2} permutes the second n_2 variables, and so on...) and G permutes the set

$$f_1, f_2, \cdots, f_n$$

as it does the variables. Define a ring homomorphism

$$\phi : K[y_1, y_2, \cdots, y_r] \to K[x_1, \cdots, x_n]/(f_1, f_2, \cdots, f_n)$$

by

$$
\begin{gathered}
y_1 \mapsto x_1 + x_2 + \cdots + x_{n_1} \\
y_2 \mapsto x_{n_1+1} + x_{n_1+2} + \cdots + x_{n_1+n_2} \\
\vdots \\
y_r \mapsto x_{n_1+\cdots+n_{r-1}+1} + \cdots + \cdots + x_n
\end{gathered}
$$

By Goto's Theorem the kernel of ϕ is a complete intersection.

If we assume that, for every i, $f_i = l_i l'_i$ is a product of linear forms, then the kernel of ϕ can be generated by a product of linear forms.