Nonlinear waves in ice sheets

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Ice sheets

“Mathematics of Sea Ice Phenomena” (Isaac Newton Institute 2017)

Outline

- Motivation
- Mathematical formulation
- Hamiltonian equations
- Direct numerical simulation (solitary waves)
  - Steady solutions in a reference moving frame
  - Time-evolving solutions (stability?)
- Weakly nonlinear modeling
  - Modulational regime (near $k = k_0 > 0$)
  - Long-wave/shallow-water regime (near $k = 0$)

Climate change

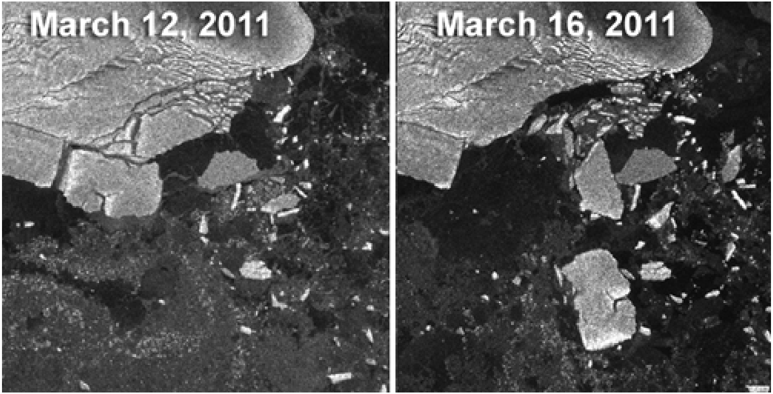
It is now recognized that ocean waves may have a significant impact on sea ice in polar regions

Expect to see ...

- weakened and more compliant sea ice because of warmer temperatures
- less compact sea ice
- larger ocean waves generated by more powerful storms
  - will penetrate further into the sea ice
  - have greater destructive payload to fracture the ice canopy
  - promote further melting by
    - breaking up the sea ice into smaller chunks
    - wave-induced melting

(e.g. Squire 2011; Thomson & Rogers 2014; Kohout et al. 2014)

Sea ice and tsunamis



ESA/Envisat

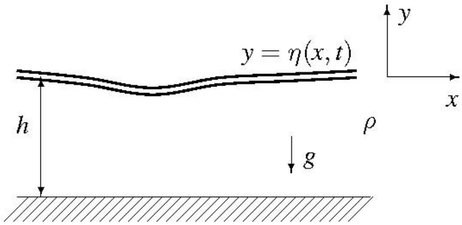
March 2011 Tohoku tsunami caused large Manhattan-size icebergs to break off the Sulzberger Ice Shelf in Antarctica

Other applications: VLFS

Megafloat - Floating airport - Tokyo Bay

- Sea ice floes/sheets and compliant pontoon type VLFS are very similar in geometry and properties
- Same mathematical problems to be solved
- Climate change induced wave intensification important for both

(2D) Governing equations



Incompressible, inviscid fluid and irrotational flow

$$\begin{aligned}
\Delta\varphi &= 0\,, \quad \text{for} \quad -h < y < \eta \\
\partial_y\varphi &= 0\,, \quad \text{on} \quad y = -h \\
\partial_t\eta + \partial_x\eta\,\partial_x\varphi - \partial_y\varphi &= 0\,, \quad \text{on} \quad y = \eta \quad \text{(kinematic)} \\
\partial_t\varphi + \tfrac{1}{2}|\nabla\varphi|^2 + g\eta + \tfrac{1}{\rho}P &= 0\,, \quad \text{on} \quad y = \eta \quad \text{(Bernoulli)}
\end{aligned}$$

where P is the pressure exerted by the ice sheet on the fluid surface

Model for the ice sheet

- Assumptions:
  - Thin elastic plate (ice thickness $\ll$ wavelength)
  - Ice sheet bends in unison with ocean waves
  - No inertia, no stretching, only bending
  - Continuous ice sheet (no ice floes ... yet)
- The **linear** Euler–Bernoulli model

$$P = \mathcal{D}\,\partial_x^4 \eta\ , \qquad \mathcal{D} = \text{coefficient of ice rigidity}$$

has been extensively used (Squire et al. 1996, etc.)

$$\mathcal{D} = \frac{\mathcal{E}\theta^3}{12(1-\nu^2)}$$

where θ is ice thickness, $\mathcal{E} \approx 6$ GPa is Young's modulus and $\nu \approx 0.3$ is Poisson's ratio

Model for the ice sheet

- The (nonlinear) Kirchhoff–Love model

$$P = \mathcal{D}\,\partial_x^2 \left(\frac{\eta_{xx}}{(1+\eta_x^2)^{3/2}} \right)$$

has also been used in a number of studies

- A more recent **nonlinear** model is given by the special Cosserat theory of hyperelastic shells (Plotnikov & Toland 2011)

$$P = \frac{\mathcal{D}}{\sqrt{1+\eta_x^2}} \partial_x \left[\frac{1}{\sqrt{1+\eta_x^2}} \partial_x \left(\frac{\eta_{xx}}{(1+\eta_x^2)^{3/2}} \right) \right] + \frac{\mathcal{D}}{2} \left(\frac{\eta_{xx}}{(1+\eta_x^2)^{3/2}} \right)^3$$

where the term in red is the mean curvature at any point on the ice sheet.

Model for the ice sheet

The Cosserat model has a number of interesting features:

- is highly nonlinear (suitable for large-amplitude ice deflections)
- satisfies Kirchhoff's hypothesis for elastic sheets
- conserves **elastic energy** (unlike Kirchhoff–Love)

$$E = \frac{1}{2}\int_{-\infty}^{\infty}\int_{-h}^{\eta} |\nabla\varphi|^2 dydx + \frac{1}{2}\int_{-\infty}^{\infty}\left[g\eta^2 + \frac{\mathcal{D}}{\rho}\frac{\eta_{xx}^2}{(1+\eta_x^2)^{5/2}}\right]dx$$

- suggests a Hamiltonian formulation for nonlinear **ice-covered ocean waves** *à la Zakharov*
- has implications for asymptotics, numerics, etc.
- ... check out Walter and Onno's talks this week

Recent literature

For example ...

- Haragus-Courcelle & Ilichev (1998): Euler–Bernoulli, asymptotics
- Părău & Dias (2002): Love–Kirchhoff, asymptotics & numerics
- Hegarty & Squire (2008): Love–Kirchhoff, asymptotics & numerics
- Bonnefoy, Meylan & Ferrant (2009): Love–Kirchhoff, numerics
- Milewski, Vanden-Broeck & Wang (2011, 2013): Cosserat, asymptotics & numerics
- Guyenne & Părău (2012, 2014): Cosserat, asymptotics & numerics
- Ambrose & Siegel (2015?): Cosserat, analysis
- Groves, Hewer & Wahlén (2016): Cosserat, analysis
- Dan Ratliff (soon?), Olga Trichtchenko (soon?), ...

Hamiltonian formulation

Following Zakharov (1968) and Craig & Sulem (1993), define

$$\xi(x,t) = \varphi(x, \eta(x,t), t)$$

then the Hamiltonian equations are given by

$$\partial_t \begin{pmatrix} \eta \\ \xi \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \delta H/\delta\eta \\ \delta H/\delta\xi \end{pmatrix}$$

with Hamiltonian (i.e. energy)

$$H = \frac{1}{2}\int_{-\infty}^{\infty} \left[\xi G(\eta)\xi + g\eta^2 + \frac{\mathcal{D}}{\rho}\frac{\eta_{xx}^2}{(1+\eta_x^2)^{5/2}} \right] dx$$

where

$$G(\eta)\xi = (-\eta_x, 1)^\top \cdot \nabla\varphi\Big|_{y=\eta}$$

denotes the **Dirichlet–Neumann operator** (DNO) that returns the normal fluid velocity at the interface, given η and ξ.

Waves induced by a moving load on sea ice

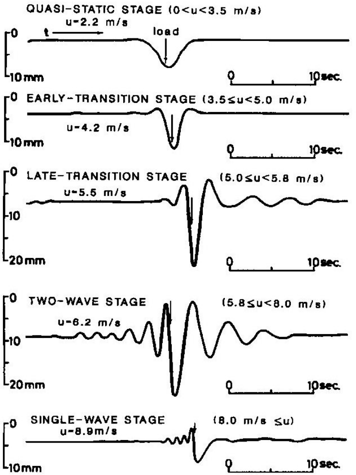


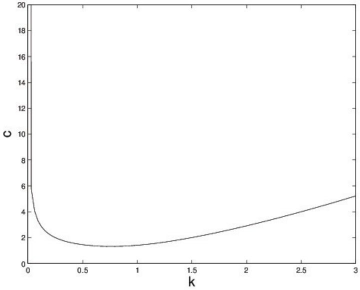
Fig. 1. Typical ice deflection records at various load speeds, February 5, 1981. (Reproduced by permission from *Takizawa* [1985].)

Experiments of Takizawa (1985) - Lake Saroma (Japan)

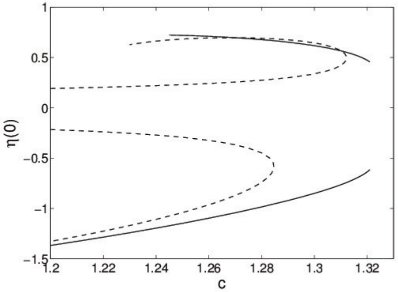
Numerical results: branches of solutions

Linear dispersion relation: wave speed c vs. wavenumber k

$$c^2 = \frac{g}{k} + \frac{\mathcal{D}k^3}{\rho} \quad \text{minimum at} \quad k_{\min} \approx 0.75\,, \quad c_{\min} \approx 1.32 \quad (h = \infty)$$



Linear dispersion relation

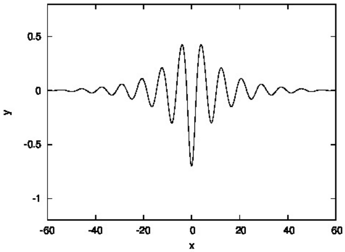


Nonlinear results

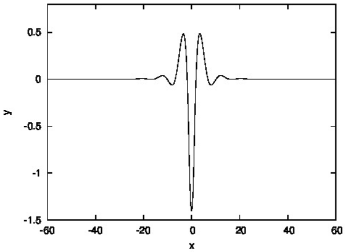
Free (solid line) and forced (dashed line) waves

Asymptotic regimes of interest: near $k = 0$ and near $k = k_{\min}$

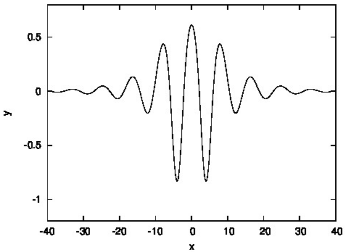
Numerical results: wave profiles (BIE, $h = \infty$)



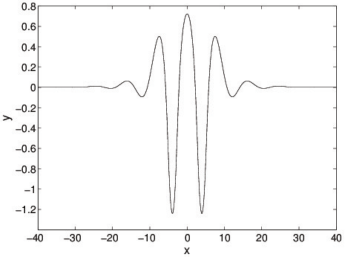
Solitary waves of depression $c = 1.316$



$c = 1.192$



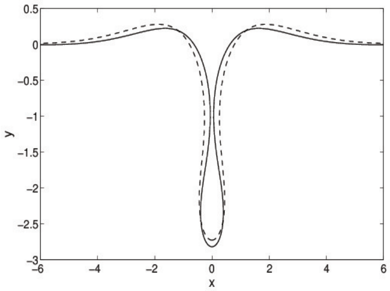
Solitary waves of elevation $c = 1.301$



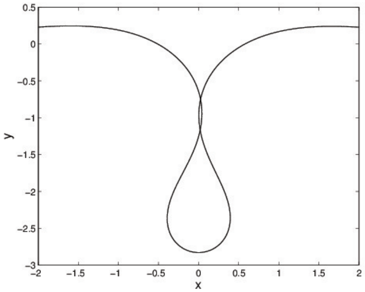
$c = 1.245$

Numerical results: limiting profiles (BIE, $h = \infty$)

Highly nonlinear solutions (as $c \to 0$) ...



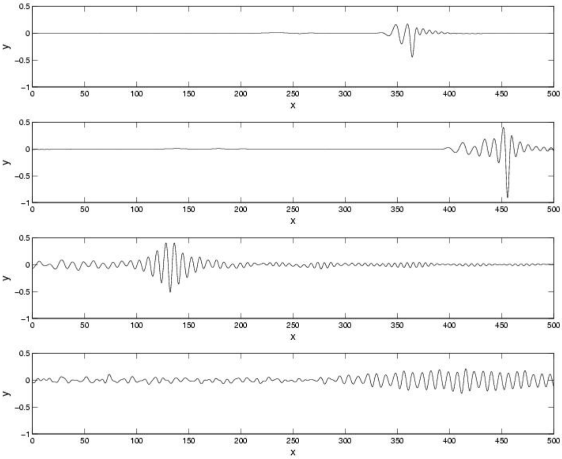
$c = 0.187$ (solid line) and $c = 0.475$ (dashed line)



$c = 0$

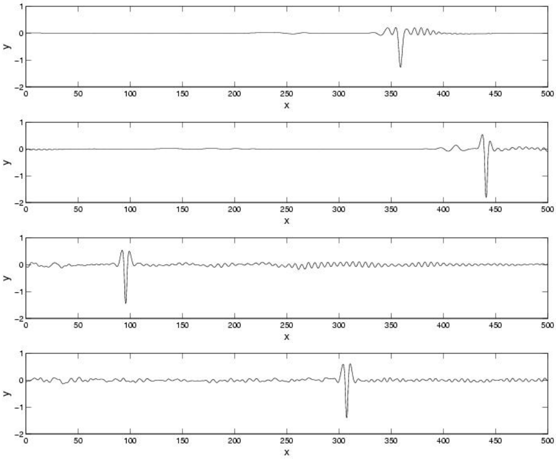
Looks like Crapper's (bubble-shaped) solutions for capillary waves

Numerical results: stability (HOS, $h = \infty$)



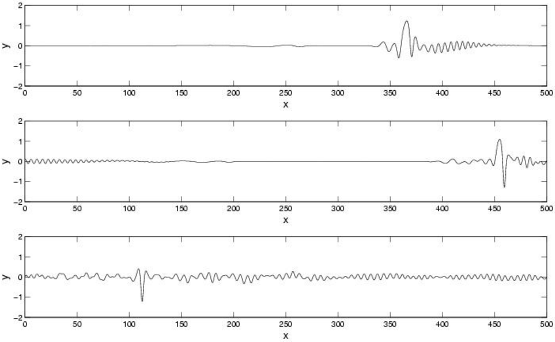
Snapshots of the ice-sheet deflection $\eta(x, t)$ at $t = 50, 120, 330, 1000$ (from top to bottom) for $c = 1.32$

Numerical results: stability (HOS, $h = \infty$)



Snapshots of the ice-sheet deflection $\eta(x, t)$ at $t = 50, 120, 330, 1000$ (from top to bottom) for $c = 1.21$

Numerical results: stability (HOS, $h = \infty$)



Snapshots of the ice-sheet deflection $\eta(x, t)$ at $t = 52, 123.5, 325$ (from top to bottom) for $c = 1.26$

Energetic consideration (BIE, $h = \infty$)

These observations on change of stability are consistent with

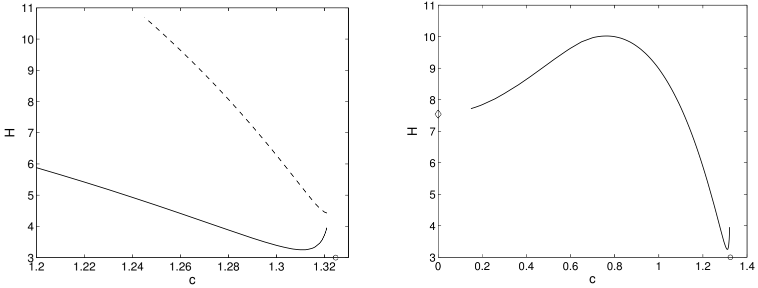


Figure: **Left:** energy for the depression solitary wave branch (solid line) and for the elevation branch (dashed line) for $1.2 < c < 1.321$. **Right:** energy of the entire depression solitary wave branch computed up to the limiting case for $0.15 < c < 1.321$. The value $c_{\min}$ is marked by a circle and the energy of the singular solution for $c = 0$ with self-intersecting profile is marked by ($\diamond$).

Weakly nonlinear modeling (modulational regime)

We look for solutions in the form of quasi-monochromatic waves with carrier wavenumber $k_0 = k_{\min} > 0$ and with slowly varying amplitude depending on $X = \varepsilon x$ where $\varepsilon \ll 1$

- Normal mode decomposition

$$\eta \simeq \frac{1}{\sqrt{2}} a^{-1}(D)(z + \bar{z}) \,, \qquad \xi \simeq \frac{1}{\sqrt{2}\mathrm{i}} a(D)(z - \bar{z})$$

where

$$a(D) = \sqrt[4]{\frac{g + \mathcal{D}D^4/\rho}{G_0}} \,,$$

- Modulational Ansatz

$$z = \varepsilon\, u(X,t)\, e^{\mathrm{i}k_0 x} \,, \qquad \bar{z} = \varepsilon\, \bar{u}(X,t)\, e^{-\mathrm{i}k_0 x}$$

Weakly nonlinear modeling (modulational regime)

- Expansion in ε of the Hamiltonian

$$H = \varepsilon \int_{-\infty}^{\infty} \Big[\frac{\overline{u}}{2} \Big(\omega(k_0) + \varepsilon \partial_k \omega(k_0) D_X + \frac{\varepsilon^2}{2} \partial_k^2 \omega(k_0) D_X^2 \Big) u + \text{c.c.} + \frac{\varepsilon^2}{2} \Big(\frac{k_0^3}{2} - \frac{5\mathcal{D}k_0^7}{4\rho(g + \mathcal{D}k_0^4/\rho)} \Big) |u|^4 \Big] dX + O(\varepsilon^4)$$

where

$$\omega(k) = \sqrt{G_0(g + \mathcal{D}k^4/\rho)},$$

denotes the linear dispersion relation in terms of the angular frequency.

- The equations of motion become

$$\partial_t \begin{pmatrix} u \\ \overline{u} \end{pmatrix} = \begin{pmatrix} 0 & -i\varepsilon^{-1} \\ i\varepsilon^{-1} & 0 \end{pmatrix} \begin{pmatrix} \delta H/\delta u \\ \delta H/\delta \overline{u} \end{pmatrix}$$

Weakly nonlinear modeling ($k \simeq k_{\min}$, $h = \infty$)

Small-amplitude wavepackets satisfy the NLS equation

$$\mathrm{i}\partial_\tau u + \lambda \partial_X^2 u + \mu |u|^2 u = 0$$

where $\tau = \varepsilon^2 t$ and

$$\begin{aligned}
\lambda &= \frac{15(\mathcal{D}/\rho)^2}{8(gk_0 + \mathcal{D}k_0^5/\rho)^{3/2}} \left[k_0^4 + \left(1 + \frac{4}{\sqrt{15}}\right)\frac{g\rho}{\mathcal{D}}\right]\left[k_0^4 + \left(1 - \frac{4}{\sqrt{15}}\right)\frac{g\rho}{\mathcal{D}}\right] \\
&= \frac{15(\mathcal{D}/\rho)^2}{8(gk_0 + \mathcal{D}k_0^5/\rho)^{3/2}} \left[k_0^4 + \left(1 + \frac{4}{\sqrt{15}}\right)3k_{\min}^4\right]\left[k_0^4 + \left(1 - \frac{4}{\sqrt{15}}\right)3k_{\min}^4\right]
\end{aligned}$$

$$\mu = \frac{3\mathcal{D}k_0^3/\rho}{4\left(g + \mathcal{D}k_0^4/\rho\right)}\left(k_0^4 - \frac{2g\rho}{3\mathcal{D}}\right) = \frac{3\mathcal{D}k_0^3/\rho}{4\left(g + \mathcal{D}k_0^4/\rho\right)}\left(k_0^4 - 2k_{\min}^4\right)$$

Here $\lambda\mu < 0$ for $k_0 = k_{\min}$ (defocusing), which implies that localized NLS solitons do not exist in this case.

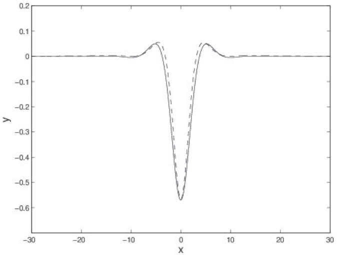
Weakly nonlinear modeling ($k \simeq k_{\min}$, finite depth)

Small-amplitude waves of carrier wavenumber k_0 satisfy
a defocusing/focusing NLS equation depending on water depth

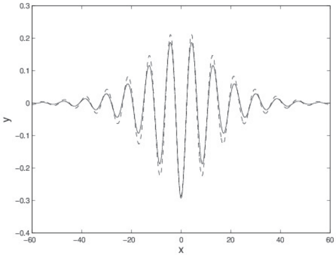
$$\mathrm{i}\partial_\tau u + \lambda \partial_X^2 u + \mu |u|^2 u = 0$$

where $\lambda = \frac{1}{2}\partial_k^2 \omega > 0$ and

$$\mu \gtrless 0 \quad \text{(focusing/defocusing) for} \quad h \lessgtr h_c$$



$c = 0.9, h = 1.5$

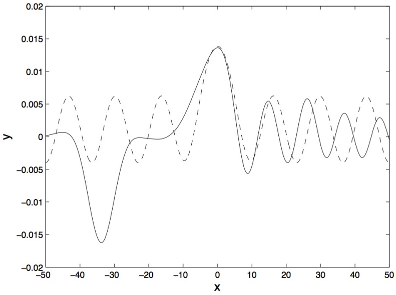


$c = 1.3, h = 3.095$

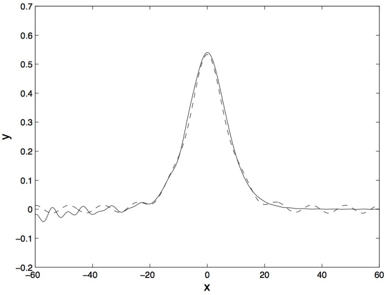
Weakly nonlinear modeling ($k \simeq 0$, shallow water)

Small-amplitude long waves (on shallow water) satisfy a 5th-order KdV equation

$$\partial_\tau u + 3c_2 u \partial_X u + c_3 \partial_X^3 u + 2c_4 \partial_X u \partial_X^2 u + c_4 u \partial_X^3 u + c_5 \partial_X^5 u = 0$$



$c = 0.722$, $h = 0.5$



$c = 1.905$, $h = 3.095$

Weakly nonlinear modeling ($k \simeq 0$, shallow water)

- Wavelength $\ell_d = 2\pi/k_d$ of these dispersive tails is determined by the resonance condition

$$c_d(k_d) = \sqrt{h}\left[1 - \frac{1}{6}h^2k_d^2 + \left(\frac{19}{360}h^4 + \frac{1}{2}\right)k_d^4\right] = c$$

  where c_d is the 5th-order KdV approximation of the linear dispersion relation.
- On which side the dispersive tail appears is determined by the value of its group velocity c_g relative to that of its phase velocity c_p. If $c_g < c_p$, then the ripples appear behind the solitary pulse. Otherwise, they appear ahead of it.
- Here $c_g < c_p$ if $k_d < k_{\min}$ and larger otherwise.
- For $(c, h) = (0.722, 0.5)$, we have $k_d = 0.501 > k_{\min} = 0.204$, so ripples are ahead (right side). For $(c, h) = (1.905, 3.095)$, we have $k_d = 0.586 < k_{\min} = 0.735$, so ripples are behind (left side).

Conclusions

- Hamiltonian formulation for fully nonlinear ice-covered ocean waves
- Direct numerical simulations
- Weakly nonlinear models in the modulational and long-wave limits
- Small-amplitude solitary waves of depression are unstable
- Large-amplitude solitary waves of depression are stable
- Solitary waves of elevation are unstable