Counting unstable eigenvalues in Hamiltonian spectral problems via commuting operators

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Hamiltonian spectral problems

- **Hamiltonian PDEs:** *linear operators of the form* $\boxed{\mathcal{J}\mathcal{L}}$
  - $\mathcal{J}$ skew-adjoint operator
  - $\mathcal{L}$ self-adjoint operator

- **Question:** *find the unstable spectrum of* $\boxed{\mathcal{J}\mathcal{L}}$

Count unstable eigenvalues

- **Hamiltonian structure:** *linear operator of the form* $\boxed{\mathcal{J}\mathcal{L}}$
  - $\mathcal{J}$ skew-adjoint operator
  - $\mathcal{L}$ self-adjoint operator

- **Under suitable conditions:**

$$\boxed{\mathbf{n_u}(\mathcal{J}\mathcal{L}) \le \mathbf{n_s}(\mathcal{L})}$$

  - $\mathbf{n_u}(\mathcal{J}\mathcal{L})$ = number of unstable eigenvalues of $\mathcal{J}\mathcal{L}$
  - $\mathbf{n_s}(\mathcal{L})$ = number of negative eigenvalues of $\mathcal{L}$

[well-known result, extensively used in stability problems . . .]

[does not work very well for periodic waves . . .]

An extended eigenvalue count

- **Hamiltonian structure:** *linear operator of the form* $\boxed{\mathcal{J}\mathcal{L}}$
  - $\mathcal{J}$ skew-adjoint operator
  - $\mathcal{L}$ self-adjoint operator
- **There exists a self-adjoint operator** $\mathcal{K}$ *such that*

$$\boxed{(\mathcal{J}\mathcal{L})(\mathcal{J}\mathcal{K}) = (\mathcal{J}\mathcal{K})(\mathcal{J}\mathcal{L})}$$

- **Under suitable conditions:**

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Stability of periodic waves

- **classical result:** *allows to show (orbital) stability of periodic waves with respect to co-periodic perturbations*
- **particular case $n_s(\mathcal{K}) = 0$:** *used to show nonlinear (orbital) stability of periodic waves with respect to subharmonic perturbations (for the KdV and NLS equations)*

[Deconinck, Kapitula, 2010; Gallay, Pelinovsky, 2015]

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  [Deconinck, Kapitula, 2010; Gallay, Pelinovsky, 2015]

- **key step: construction of a nonnegative operator $\mathcal{K}$**
  - relies upon the existence of a higher order conserved functional (due to integrability)

General result

Hypotheses

- $\mathcal{J}$, $\mathcal{L}$, $\mathcal{K}$ closed linear operators acting in a Hilbert space **H**
  - **$\mathcal{J}$ skew-adjoint operator with bounded inverse**
  - **$\mathcal{L}$, $\mathcal{K}$ self-adjoint operators**

$$\boxed{(\mathcal{J}\mathcal{L})(\mathcal{J}\mathcal{K})\mathbf{u} = (\mathcal{J}\mathcal{K})(\mathcal{J}\mathcal{L})\mathbf{u}, \quad \forall\, \mathbf{u} \in \mathcal{D}}$$

  - the nonpositive spectrum $\boldsymbol{\sigma}_{\mathbf{s}}(\mathcal{K}) \cup \boldsymbol{\sigma}_{\mathbf{c}}(\mathcal{K})$ consists of a finite number of isolated eigenvalues with finite multiplicities
  - the unstable spectrum $\boldsymbol{\sigma}_{\mathbf{u}}(\mathcal{J}\mathcal{L})$ consists of isolated eigenvalues with finite algebraic multiplicities

Main result

- *The number* $\mathbf{n_u}(\mathcal{J}\mathcal{L})$ *of unstable eigenvalues of the operator* $\mathcal{J}\mathcal{L}$ *and the number* $\mathbf{n_{sc}}(\mathcal{K})$ *of nonpositive eigenvalues of the self-adjoint operator* $\mathcal{K}$ *satisfy*

$$\boxed{\mathbf{n_u}(\mathcal{J}\mathcal{L}) \leq \mathbf{n_{sc}}(\mathcal{K}).}$$

Main result

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- *If, in addition,* $\mathbf{ker}(\mathcal{K}) \subset \mathbf{ker}(\mathcal{J}\mathcal{L})$, *then*

$$\boxed{\mathbf{n_u}(\mathcal{J}\mathcal{L}) \leq \mathbf{n_s}(\mathcal{K}).}$$

Corollary

Assume that $\mathcal{K}$ **is a nonnegative operator.**

- $\boxed{\mathbf{n_u}(\mathcal{J}\mathcal{L}) \leq \mathbf{n_c}(\mathcal{K})}$

- *If, in addition,* $\mathbf{ker}(\mathcal{K}) \subset \mathbf{ker}(\mathcal{J}\mathcal{L})$, *then*

$$\boxed{\mathbf{n_u}(\mathcal{J}\mathcal{L}) = \mathbf{0}}$$

  i.e., the spectrum of $\mathcal{J}\mathcal{L}$ *is purely imaginary.*

Proof

- *$\boldsymbol{\lambda}$ and $\boldsymbol{\sigma}$ isolated eigenvalues of $\mathcal{J}\mathcal{L}$; spectral subspaces $\mathbf{E}_{\boldsymbol{\lambda}}$ and $\mathbf{E}_{\boldsymbol{\sigma}}$*

$$\boxed{(\lambda+\overline{\sigma})\langle\mathcal{K}\mathbf{u},\mathbf{v}\rangle = 0, \quad \forall\ \mathbf{u}\in\mathbf{E}_{\boldsymbol{\lambda}},\ \mathbf{v}\in\mathbf{E}_{\boldsymbol{\sigma}}}$$

  - $\mathbf{E}_{\mathbf{u}}$ unstable spectral subspace of $\mathcal{J}\mathcal{L}$

$$\boxed{\langle\mathcal{K}\mathbf{u},\mathbf{u}\rangle = 0, \quad \forall\ \mathbf{u}\in\mathbf{E}_{\mathbf{u}}}$$

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- spectral decomposition of the Hilbert space $\mathbf{H}$

$$\boxed{\mathbf{H} = \mathbf{F}_{\mathrm{sc}}\oplus\mathbf{F}_\mathrm{u}, \quad \sigma(\mathcal{K}\big|_{\mathbf{F}_{\mathrm{sc}}}) = \sigma_{\mathrm{sc}}(\mathcal{K}), \quad \sigma(\mathcal{K}\big|_{\mathbf{F}_\mathrm{u}}) = \sigma_\mathrm{u}(\mathcal{K})}$$

  - spectral projector $\boxed{\mathbf{P}_{\mathrm{sc}}\big|_{\mathbf{E}_\mathrm{u}} : \mathbf{E}_\mathrm{u}\to\mathbf{F}_{\mathrm{sc}}}$ **is injective**

$$\boxed{\dim(\mathbf{E}_\mathrm{u}) = \mathbf{n}_\mathrm{u}(\mathcal{J}\mathcal{L}) \le \dim(\mathbf{F}_{\mathrm{sc}}) = \mathbf{n}_{\mathrm{sc}}(\mathcal{K})}$$

Proof

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  - $\mathbf{E}_\mathbf{u}$ unstable spectral subspace of $\mathcal{J}\mathcal{L}$

$$\boxed{\langle\mathcal{K}\mathbf{u},\mathbf{u}\rangle = 0, \quad \forall\ \mathbf{u}\in\mathbf{E}_\mathbf{u}}$$

- If $\mathbf{ker}(\mathcal{K}) \subset \mathbf{ker}(\mathcal{J}\mathcal{L})$: spectral decomposition

$$\boxed{\mathbf{H} = \mathbf{F}_\mathbf{s}\oplus\mathbf{F}_\mathbf{cu}, \quad \sigma(\mathcal{K}\big|_{\mathbf{F}_\mathbf{s}}) = \sigma_\mathbf{s}(\mathcal{K}), \quad \sigma(\mathcal{K}\big|_{\mathbf{F}_\mathbf{cu}}) = \sigma_\mathbf{cu}(\mathcal{K})}$$

  - spectral projector $\boxed{\mathbf{P}_\mathbf{s}\big|_{\mathbf{E}_\mathbf{u}} : \mathbf{E}_\mathbf{u}\to\mathbf{F}_\mathbf{s}}$ **is injective**

$$\boxed{\dim(\mathbf{E}_\mathbf{u}) = \mathbf{n}_\mathbf{u}(\mathcal{J}\mathcal{L}) \le \dim(\mathbf{F}_\mathbf{s}) = \mathbf{n}_\mathbf{s}(\mathcal{K})}$$

Application

KP-II equation

Kadomtsev-Petviashivili equation

$$\boxed{(u_t + 6uu_x + u_{xxx})_x + u_{yy} = 0}$$

- model equation for water waves (small surface tension)
- two-dimensional extension of the KdV equation

$$\boxed{u_t + 6uu_x + u_{xxx} = 0}$$

Question: *transverse stability of one-dimensional periodic traveling waves (spectral, linear, nonlinear)*

- the classical counting criterion does not allow to fully understand transverse stability for periodic waves

1D PERIODIC TRAVELING WAVES

- **one-parameter family of one-dimensional periodic traveling waves** (up to symmetries)

$$\boxed{u(x,t) = \phi_c(x + ct)}$$

  - **speed** $c > 1$
  - 2π**-periodic, even profile** ϕ_c satisfying the KdV equation

$$\boxed{v''(x) + cv(x) + 3v^2(x) = 0}$$

  - **known explicitly!**

Linearized equation

linearized KP-II equation

$$\boxed{(w_t + w_{xxx} + cw_x + 6(\phi_c(x)w)_x)_x + w_{yy} = 0}$$

- 2π-periodic coefficients in x
- Ansatz

$$\boxed{w(x,y,t) = e^{\lambda t + ipy} W(x), \quad \lambda \in \mathbb{C},\ p \in \mathbb{R}}$$

linearized equation for W(x)

$$\boxed{\lambda W_x + W_{xxxx} + cW_{xx} + 6(\phi_c(x)W)_{xx} - p^2 W = 0}$$

Spectral stability problem

- **linearized equation for W(x)**

$$\boxed{\lambda W_x + W_{xxxx} + cW_{xx} + 6(\phi_c(x)W)_{xx} - p^2 W = 0}$$

- *the periodic wave* ϕ_c *is* **spectrally stable iff** *the linear operator*

$$\boxed{\mathcal{A}_{c,p}(\lambda) = \lambda\partial_x + \partial_x^4 + c\partial_x^2 + 6\partial_x^2(\phi_c(x)\,\cdot) - p^2}$$

is invertible for $\mathbf{Re}\,\boldsymbol{\lambda} > \mathbf{0}$.

- **2D bounded perturbations: space** $C_b(\mathbb{R})$ and $\mathbf{p} \in \mathbb{R}$.
- continuous spectrum . . .

Floquet/Bloch decomposition

- $\mathcal{A}_{c,p}(\lambda)$ *is invertible in* $C_b(\mathbb{R})$ *iff the operators*

$$\boxed{\mathcal{A}_{c,p}(\lambda,\gamma) = \lambda(\partial_x + i\gamma) + (\partial_x + i\gamma)^4 + c(\partial_x + i\gamma)^2 + 6(\partial_x + i\gamma)^2(\phi_c(x)\,\cdot) - p^2}$$

are invertible in $L^2_{per}(0, 2\pi)$, *for any* $\gamma \in [0, 1)$.

- $\boxed{\gamma \in (0, 1)}$: **study the spectrum of the operator**

$$\boxed{\mathcal{B}_{c,p}(\gamma) = -(\partial_x + i\gamma)^3 - c(\partial_x + i\gamma) - 6(\partial_x + i\gamma)(\phi_c(x)\,\cdot) + p^2(\partial_x + i\gamma)^{-1}}$$

- $\boxed{\gamma = 0}$: **restrict to functions with zero mean**

Counting criterion

- apply the counting criterion to

$$\boxed{\mathcal{B}_{c,p}(\gamma) = \mathcal{J}(\gamma)\mathcal{L}_{c,p}(\gamma)}$$

  - **skew-adjoint operator** $\boxed{\mathcal{J}(\gamma) = (\partial_x + i\gamma)}$
  - **self-adjoint operator**

$$\boxed{\mathcal{L}_{c,p}(\gamma) = -(\partial_x + i\gamma)^2 - c - 6\phi_c(x) + p^2(\partial_x + i\gamma)^{-2}}$$

- construct **positive commuting operators** $\boxed{\mathcal{K}_{c,p}(\gamma)}$
  - find **commuting operators** $\mathcal{M}_{c,p}(\gamma)$
  - show that suitable linear combination of $\mathcal{M}_{c,p}(\gamma)$ and $\mathcal{L}_{c,p}(\gamma)$ **is a positive operator**

Commuting operators

- **natural candidate:** use a higher-order conserved functional
  - resulting operator satisfies the commutativity relation
  - cannot obtain positive operators ...

Commuting operators

- **natural candidate:** use a higher-order conserved functional
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- **second option:** use the operators from the KdV equation
  - $\mathbf{p = 0}$ corresponds to the KdV equation
  - decompose:

$$\boxed{\mathcal{L}_{c,p} = \mathcal{L}_{\mathrm{KdV}} + p^2 \mathcal{L}_{\mathrm{KP}}, \; \mathcal{M}_{c,p} = \mathcal{M}_{\mathrm{KdV}} + p^2 \mathcal{M}_{\mathrm{KP}}}$$

  - $\mathcal{M}_{\mathrm{KdV}}$ is obtained from a higher order conserved functional:

$$\boxed{\mathcal{M}_{\mathrm{KdV}} = \partial_x^4 + 10\partial_x \phi_c(x) \partial_x - 10c\phi_c(x) - c^2}$$

  - compute $\mathcal{M}_{\mathrm{KP}}$ directly from the commutativity relation:

$$\boxed{\mathcal{M}_{\mathrm{KP}} = \frac{5}{3}\left(1 + c\partial_x^{-2}\right)}$$

Main result

Transverse spectral stability of periodic waves (with respect to bounded perturbations):

- *there exist constants* **b** *such that the operators* $\boxed{\mathcal{K}_{\mathbf{c},\mathbf{p},\mathbf{b}}(\gamma) = \mathcal{M}_{\mathbf{c},\mathbf{p}}(\gamma) - \mathbf{b}\mathcal{L}_{\mathbf{c},\mathbf{p}}(\gamma)}$ **are positive**[1]
- *the commutativity relation holds*
- *the general counting criterion implies that the spectra of* $\boxed{\mathcal{B}_{\mathbf{c},\mathbf{p}}(\gamma) = \mathcal{J}(\gamma)\mathcal{L}_{\mathbf{c},\mathbf{p}}(\gamma)}$ **are purely imaginary**

[1] except for $p = \gamma = 0$ when the operator is nonnegative with 1D kernel (spanned by the derivative of the wave)

Transverse linear stability

linearized problem

$$\boxed{w_t = \mathcal{B}_c w}$$

- $\boxed{\mathcal{B}_c = \mathcal{J}\mathcal{L}_c}$, $\boxed{\mathcal{J} = \partial_x,\ \mathcal{L}_c = -\partial_x^2 - c - 6\phi_c(x) - \partial_x^{-2}\partial_y^2}$
- $\mathcal{B}_c$ **acts in** $\dot{L}^2(N, p)$ (the space of locally square-integrable functions on $\mathbb{R}^2$ which are $2\pi N$-periodic and have zero mean in x, and are $2\pi/p$-periodic in y)

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linear operator

$$\boxed{\mathcal{K}_c = \mathcal{M}_c - b\mathcal{L}_c}$$

- **nonnegative operator with point spectrum and 1D kernel** (spanned by the derivative of the periodic wave)
- **the operators $\mathcal{J}\mathcal{K}_c$ and $\mathcal{J}\mathcal{L}_c$ commute**

Transverse linear stability

- **Lyapunov functional** $\boxed{\mathbf{w \mapsto \langle K_c w, w\rangle}}$
  - $\frac{\mathbf{d}}{\mathbf{dt}}\langle \mathbf{K_c w(t), w(t)}\rangle = \mathbf{0}$ (from commutativity)
  - $\langle \mathbf{K_c w, w}\rangle \geq \mathbf{c}\|\mathbf{w}\|^2$**, when** $\langle \mathbf{w}, \partial_x \phi_c\rangle = \mathbf{0}$ (from positivity)

- *implies* **transverse linear stability of the periodic waves**
  (with respect to doubly periodic perturbations)

Transverse nonlinear stability

Open problem ...

- *the nonnegative linear operator $\mathcal{K}_{\mathbf{c}}$ is not the Hessian operator of some conserved higher-order energy functional of the KP-II equation ...*
- *find a conserved higher-order energy functional of the KP-II equation for which the Hessian operator (at the periodic wave) is nonnegative?*