Hamiltonians and normal forms for water waves

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of Nonlinear Surface Waves
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Outline

Free surface water waves

Birkhoff normal forms

Mapping properties of normal forms transformations

Almost global existence

A Hamiltonian for overturning waves

Critique

Joint work with Catherine Sulem (University of Toronto)

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Contrast two ODEs

- Quadratic case

$$\dot{z} = z^2 \ , \qquad z(0) = \varepsilon$$
$$z(t) = \frac{\varepsilon}{1 - \varepsilon t} \ , \qquad T = \frac{1}{\varepsilon}$$

where $T = T(\varepsilon)$ is the time of existence of a nonsingular solution

- Cubic case

$$\dot{w} = w^3 \ , \qquad w(0) = \varepsilon$$
$$w(t) = \sqrt{\frac{\varepsilon^2}{1 - 2\varepsilon^2 t}} \ , \qquad T = \frac{1}{2\varepsilon^2}$$

- The general time of existence does not change when these ODE are replaced by

$$\dot{z} = i\omega z + z^2 + h^{(3)}(z) \ , \qquad \dot{w} = i\omega w + w^3 + k^{(4)}(w)$$

Free surface water waves

- Incompressible and irrotational flow

$$\nabla \cdot u = 0 \;, \qquad \nabla \wedge u = 0$$

which is a potential flow in the fluid domain $S(\eta)$

$$u = \nabla \varphi \;, \quad \Delta \varphi = 0$$

Fluid domain $S(\eta)$: $-h < y < \eta(x,t)$, $\quad x \in \mathbb{R}^{d-1}$
Bottom boundary conditions $\quad \partial_N \varphi = 0$

- Free surface conditions on $y = \eta(x,t)$

$$\begin{aligned}
\partial_t \eta &= \partial_y \varphi - \partial_x \eta \cdot \partial_x \varphi \quad \text{kinetic BC} \\
\partial_t \varphi &= -g\eta - \tfrac{1}{2}|\nabla \varphi|^2 \quad \text{Bernoulli condition}
\end{aligned}$$

Free surface water waves with surface tension

- Incompressible and irrotational flow

$$\nabla \cdot u = 0 \, , \qquad \nabla \wedge u = 0$$

which is a potential flow in the fluid domain $S(\eta)$

$$u = \nabla \varphi \, , \quad \Delta \varphi = 0$$

Fluid domain $S(\eta)$: $-h < y < \eta(x,t), \quad x \in \mathbb{R}^{d-1}$
Bottom boundary conditions $\partial_N \varphi = 0$

- Free surface conditions on $y = \eta(x,t)$ with surface tension

$$\begin{aligned} \partial_t \eta &= \partial_y \varphi - \partial_x \eta \cdot \partial_x \varphi \quad \text{kinetic BC} \\ \partial_t \varphi &= -g\eta - \tfrac{1}{2}|\nabla \varphi|^2 + \sigma \kappa(\eta) \quad \text{Bernoulli condition} \end{aligned}$$

where $\kappa(\eta)$ is the mean curvature of the free surface

Figure : Great waves off the Oregon coast

Zakharov's Hamiltonian

- The energy functional

$$
\begin{aligned}
H &= K + P \\
&= \int_x \int_{y=-h}^{\eta(x)} \tfrac{1}{2} |\nabla\varphi|^2 \, dy dx + \int_x \tfrac{g}{2} \eta^2 \, dx
\end{aligned}
$$

- Zakharov's choice of variables

$$
z := (\eta(x), \xi(x)) \, , \qquad \text{where} \quad \xi(x) := \varphi(x, \eta(x))
$$

That is $\varphi = \varphi[\eta, \xi](x, y)$

- Expressed in terms of the Dirichlet – Neumann operator $G(\eta)$

$$
H(\eta, \xi) = \int \tfrac{1}{2} \xi G(\eta) \xi + \tfrac{g}{2} \eta^2 \, dx
$$

Dirichlet – Neumann operator

- Laplace's equation on the fluid domain $S(\eta)$

$$\xi(x) \mapsto \varphi(x,y) \mapsto N \cdot \nabla\varphi\,(1 + |\nabla_x\eta|^2)^{1/2} := G(\eta)\xi(x)$$

- Equations take the form of a Hamiltonian PDE, in Darboux coordinates

$$\begin{aligned} \partial_t\eta &= \mathrm{grad}_\xi H = G(\eta)\xi \\ \partial_t\xi &= -\mathrm{grad}_\eta H = -g\eta - \mathrm{grad}_\eta K \end{aligned}$$

That is

$$\partial_t z = J\mathrm{grad}_z H\ , \qquad J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

Expression for $\mathrm{grad}_\eta K$ related to the Hadamard variational formula [lectures on Green's functions, Collège de France (1911)(1916)]

Dirichlet – Neumann operator

Lemma 1 (properties of the Dirichlet - Neumann operator)

1. $G(\eta) \geq 0$ *and* $G(\eta)1 = 0$
2. $G(\eta)^* = G(\eta)$ *Hermitian symmetry*
3. $G(\eta) : H^1_\xi \to L^2_\xi$ *is analytic in* η *for* $\eta \in C^1$

$$G(\eta)\xi = G^{(0)}\xi + G^{(1)}(\eta)\xi + G^{(2)}(\eta)\xi + \dots$$

[Employs a theorem of Christ & Journé (1987)]

4. *Setting* $D_x := -i\partial_x$ *several terms of the Taylor series are:*

$$\begin{aligned} G^{(0)}\xi &= |D_x| \tanh(h|D_x|)\xi \\ G^{(1)}\xi &= (D_x \cdot \eta D_x - G^{(0)}\eta G^{(0)})\xi \end{aligned}$$

- Simulations using the Dirichlet – Neumann formulation

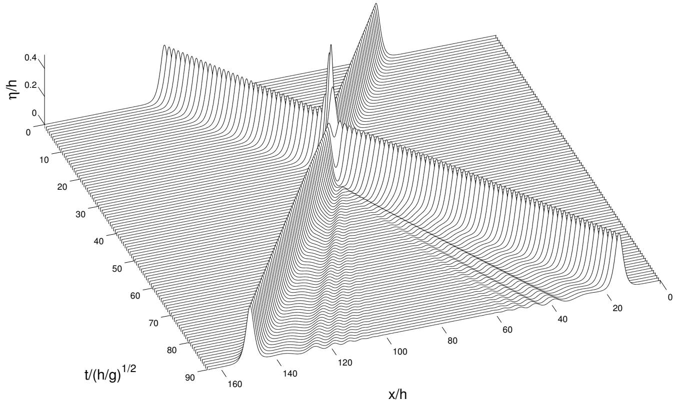


Figure : Head-on collision of two solitary waves, case $S/h = 0.4$
W. Craig, J. Hammack, D. Henderson, P. Guyenne & C. Sulem, Phys. Fluids 18, (2006)

Poisson brackets and conservation laws

One way to express conservation uses the Poisson bracket

$$\partial_t K(\eta(t,\cdot),\xi(t,\cdot)) = \{H,K\} := \int \mathrm{grad}_\eta K \,\mathrm{grad}_\xi H - \mathrm{grad}_\xi K \,\mathrm{grad}_\eta H \, dx$$

- Mass $\quad M(\eta) = \int \eta \, dx$

$$\begin{aligned} \{H,M\} &= \int \mathrm{grad}_\eta M \,\mathrm{grad}_\xi H - \mathrm{grad}_\xi M \,\mathrm{grad}_\eta H \, dx \\ &= \int 1 \, G(\eta)\xi \, dx \\ &= \int G(\eta) 1 \, \xi \, dx = 0 \end{aligned}$$

- Momentum $\quad I(\eta,\xi) = \int \eta \partial_x \xi \, dx \,, \qquad \partial_t I = \{H,I\} = 0$
- Energy $\quad H(\eta,\xi) \,, \qquad \partial_t H = \{H,H\} = 0$

Taylor expansion of the Hamiltonian

- From the analyticity of $G(\eta)$ (*e.g.* the case $\sigma = 0$)

$$
\begin{aligned}
H &= H^{(2)} + H^{(3)} + H^{(4)} + \dots \\
&= \frac{1}{2} \int \xi G^{(0)} \xi + g\eta^2 \, dx \; + \sum_{m \geq 3} \frac{1}{2} \int \xi G^{(m-2)}(\eta) \xi \, dx
\end{aligned}
$$

- The water wave equations linearized about $(\eta, \xi) = 0$ are

$$
\partial_t \begin{pmatrix} \eta \\ \xi \end{pmatrix} = J \, \mathrm{grad}_{\eta,\xi} H^{(2)}(\eta, \xi)
$$

namely

$$
\begin{aligned}
\partial_t \eta &= |D_x| \tanh(h|D_x|) \xi \\
\partial_t \xi &= -g\eta
\end{aligned}
$$

A harmonic oscillator with frequencies $\omega(k) = \sqrt{g|k| \tanh(h|k|)}$

Birkhoff normal form

- Normal form - transform the equations to retain only essential nonlinearities

$$\tau : z = \begin{pmatrix} \eta \\ \xi \end{pmatrix} \mapsto w$$

in a neighborhood $B_R(0) \subseteq H^r$

- Conditions:
  1. The transformation τ is canonical, so the new equations are

$$\partial_t w = J \,\mathrm{grad} \overline{H}(w) \;, \qquad \overline{H}(w) = H(\tau^{-1}(w))$$

  2. The new Hamiltonian is

$$\overline{H}(w) = H^{(2)}(w) + \left(Z^{(3)} + \cdots + Z^{(M)} \right) + \overline{TR}^{(M+1)}$$

     where each $Z^{(m)}$ retains only resonant terms $\{H^{(2)}, Z^{(m)}\} = 0$

- The transformed Hamiltonian $\overline{H}(w)$ is conserved by the flow

Significance of the normal form

- This transformation procedure and reduction to Birkhoff normal form is part of the theory of averaging for dynamical systems
- Fourier transform variables and complex symplectic coordinates

$$(\eta_k, \xi_k) := \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} e^{-ikx} (\eta(x), \xi(x))\, dx$$

$$z_k := \frac{1}{\sqrt{2}} \left(Q_k \eta_k + i Q_k^{-1} \xi_k \right) , \qquad Q_k = \sqrt[4]{\frac{g}{|k|}}$$

Action angle variables $z_k = \sqrt{I_k} e^{i\Theta_k} \quad I_k = |z_k|^2$

- A Hamiltonian $h(I)$ in action variables alone is integrable

$$\partial_t \Theta = \partial_I h(I) , \qquad \Theta(t) = \Theta(0) + t \partial_I h(I)$$
$$\partial_t I = -\partial_\Theta h(I) = 0 , \qquad I(t) = I(0)$$

Such flows conserve each I_k, hence every Sobolev norm

$$\|z(t)\|_r^2 = \sum_k \langle k \rangle^{2r} |z_k(t)|^2 = \|z(0)\|_r^2$$

Further reasons to study the normal form

- Zakharov's theory of wave turbulence. The reduction of a Hamiltonian PDE to its resonant manifold is a normal forms transformation. In Zakharov's notation, $Q(D_x) := \sqrt[4]{g/\omega(D_x)}$

$$a(x) := \frac{1}{\sqrt{2}}\Big(Q(D_x)\eta(x) + iQ^{-1}(D_x)\xi(x)\Big) \mapsto b(x)$$

- In KAM theory the Arnold condition depends upon the normal form

- special solutions. Resonant terms $Z^{(3)} + \ldots Z^{(M)}$ describe an averaged system, which often has particular solutions of interest. Example: Wilton ripples and three wave resonances

Resonant terms

- Cubic resonances: It is well known in the folklore of fluid dynamics that with surface tension $\sigma = 0$ there are no *three wave interactions*. Namely

$$\omega_{k_1} \pm \omega_{k_2} \pm \omega_{k_3} = 0 \,,$$
$$k_1 + k_2 + k_3 = 0$$

  implies that at least one of $k_1, k_2, k_3 = 0$

  In particular this means that $Z^{(3)} = 0$, and the new equations have no quadratic nonlinear terms; the lowest order nonlinear terms will be cubic.
- The question in PDEs: *mapping properties* of $\tau := \tau^{(3)}$: is the transformation well defined, and on which Banach spaces

Third order Birkhoff normal form

- Theorem 2 (C. Sulem & WC (2016))

  Let $d = 2$ *and* $h = +\infty$ *and fix* $r > 3/2$. *There exists* $R_0 > 0$ *such that for any* $R < R_0$, *on every neighborhood* $B_R(0) \subseteq H^r_\eta \oplus H^r_\xi$ *the canonical Birkhoff normal forms transformation* $\tau^{(3)}$ *is defined.*

  $$\tau^{(3)} \; : \; B_R(0) \to B_{2R}(0) \qquad (\tau^{(3)})^{-1} \; : \; B_{R/2}(0) \to B_R(0)$$

  The result is that $w = \tau^{(3)}(z)$ *transforms* $H(z)$ *to normal form*

  $$\overline{H}(w) = H^{(2)}(w) + 0 + \overline{TR}^{(4)}(w)$$

  - Clearly $\overline{H}(w) = H^{(2)}(w) + 0 + \overline{TR}^{(4)}(w)$ takes the form

    $$\overline{H} = H^{(2)}(I) + \mathcal{O}(\|w\|^4)$$

Fourth order Birkhoff normal form - formal calculation

Theorem 3 (Dyachenko, Lvov & Zakharov (1994), WC & Worfolk (1995))

In the case $d = 2$ *and* $h = +\infty$, *set* $I_1(k) = \frac{1}{2}(z_k \bar{z}_k + z_{-k}\bar{z}_{-k})$ *and* $I_2(k) = \frac{1}{2}(z_k \bar{z}_k - z_{-k}\bar{z}_{-k})$. *The formal second Birkhoff normal form is*

$$
\begin{aligned}
\overline{H}^{+} = & \sum_k \omega_k I_1(k) - \frac{1}{2\pi} \sum_k |k|^3 \big(I_1(k)^2 - 3I_2(k)^2\big) \\
& + \frac{4}{\pi} \sum_{|k_4|<|k_1|} I_2(k_1) I_2(k_4) + \overline{TR}^{(5)} \\
= & H^{(2)}(I) + \overline{H}^{(4)}(I) + \overline{TR}^{(5)}
\end{aligned}
$$

In particular there are no nonzero Benjamin - Feir resonant interactions. Specifically $c_{k_1k_2k_3k_4} = 0$ when

$$k_1 : k_2 : k_3 : k_4 = n^2 : (n+1)^2 : n^2(n+1)^2 : -(n^2+n+1)^2)$$

$$\omega_1 : \omega_2 : \omega_3 : \omega_4 = n : -(n+1) : -n(n+1) : (n^2+n+1)$$

Fourth order Birkhoff normal form

Define the energy space $E^r := H^r_\eta \oplus H^{r+1/2}_\xi$
Quartet interactions are indexed by

$$\{(k_1, k_2, k_3, k_4) \in \mathbb{Z}^4 \ : \Sigma_{j=1}^4 k_j = 0\}$$

The resonant set is

$$R = \{k_1k_4, k_2k_3 > 0 : k_1 + k_2 = 0 = k_3 + k_4 \text{ or } k_1 + k_3 = 0 = k_2 + k_4\}$$

A quasihomogeneous neighborhood of R is a set of near-resonant modes

$$C_R^+ := \{(k_1, k_2, k_3, k_4) \in \mathbb{Z}^4 : \Sigma_{j=1}^4 k_j = 0 \quad \text{satisfying} \\ |k_1 + k_2| < (|k_1| + |k_2|)^{1/4} \text{ and } |k_3 + k_4| < (|k_3| + |k_4|)^{1/4}\}$$

The neighborhood $C_R^- \subseteq \mathbb{Z}^4$ is similar with $k_2 \leftrightarrow k_3$

Fourth order Birkhoff normal form

Theorem 4 (WC & Sulem (2016))

Let $Q \subseteq \mathbb{Z}^4$ *be a set of quartet interactions, such that*

$$Q \backslash B_\rho(0) \cap C_R^{\pm} = \emptyset \qquad \rho < +\infty$$

is symmetric under $(k \leftrightarrow -k)$, $(k_2 \leftrightarrow k_3)$ *and* $(k_1 \leftrightarrow k_4)$.
Then for $r > 3/2$ *there exists a canonical transformation* $\tau_Q^{(4)}$ *on* $B_R(0) \subseteq E^r$ *such that*

$$\tau_Q^{(4)} : H^{(2)} + \overline{H}^{(4)} + R^{(5)} \to \widetilde{H} = H^{(2)} + \widetilde{Z}^{(4)} + \widetilde{R}^{(5)} \; .$$

such that $supp\, \tilde{Z}^{(4)} \subseteq C_R^{\pm}$. *For* $(k_1, k_2, k_3, k_4) \in R$ *then*

$$\widetilde{Z}^{(4)}_{k_1,k_2,k_3,k_4} = Z^{(4)}_{k_1,k_2,k_3,k_4}(I)$$

Surface tension $\sigma > 0$

Theorem 5 (C. Sulem & WC (2015))

In the case of positive surface tension, with $0 < h \leq +\infty$*, a similar statement holds for* $r > 1$*, namely* $\partial_z \tau^{(3)} \ : B_R(0) \subseteq H_\eta^{r+1} \oplus H_\xi^{r+1/2} \to H_\eta^{r+1} \oplus H_\xi^{r+1/2}$*. However it is possible that* $Z^{(3)}$ *is nonzero (Wilton ripples).*

NB Furthermore $\tau^{(3)}$ is smooth on a scale of Hilbert spaces. That is, in the case with surface tension the Jacobian maps energy spaces

$$\partial_z \tau^{(3)} \ : H_\eta^{r+1/2} \oplus H_\xi^{r} \to H_\eta^{r+1/2} \oplus H_\xi^{r}$$

In the case w/o surface tension the Jacobian maps

$$\partial_z \tau^{(3)} \ : H_\eta^{r-1} \oplus H_\xi^{r-1} \to H_\eta^{r-1} \oplus H_\xi^{r-1}$$

NB: The transformation mixes the domain η and the potential ξ.

Almost global existence theory

Consider $d = 2$, $\sigma > 0$, finite or infinite depth h, and $x \in [0, 2\pi]$ with periodic boundary conditions.

Theorem 6 (M. Berti & J.-M. Delort (2016))

There is a set of parameters $M_0 \subseteq M$ *of full measure, such that for* $(g, h, \sigma) \in M_0$ *and for any* $m \geq 3$ *there are* $s_0 > 0$ *and* $\varepsilon_0 > 0$ *such that for* $0 < \varepsilon < \varepsilon_0$ *and* $s \geq s_0$ *there is a transformation*

$$\tau = \tau^{(m)} \ : \ z(t,x) \in B_\varepsilon(0) \subseteq H^s \mapsto w(t,x) \in B_{2\varepsilon}(0)$$

(1) such that $z(t,x)$ *satisfies the water waves equations and*
(2) the energy estimate in the w *coordinates is satisfied*

$$\partial_t \|w(t)\|_s \leq C_s \varepsilon^{m-2} \|w(t)\|_s$$

This recent result is a time $T_\varepsilon > c\varepsilon^{-(m-2)}$ existence theorem for water waves in the spatially periodic case, for nonresonant parameter values. It depends upon nonresonance conditions but is not a normal form

Hamiltonian in general coordinates

- Fluid domain $\Omega(t) \subseteq \mathbb{R}^2$ with free surface $\gamma(t,s)$.
  Evolution determined by free surface conditions

$$\mathbf{T}_{txy} \cdot \mathbf{N}_{txy} = 0 \,, \qquad p(t, \gamma(t,s)) = 0$$

- Energy = kinetic + potential

$$H = K + P \,, \qquad K = \tfrac{1}{2} \iint_\Omega |\nabla \varphi(x,y)|^2 \, dydx$$

  Potential energy

$$P = \iint_\Omega \nabla \cdot V \, dydx \,, \qquad V = (0, \frac{gy^2}{2})$$

- Dirichlet – Neumann operator, with $\varphi(\gamma(s)) = \xi(s)$

$$G(\gamma)\xi(s) := N \cdot \nabla \varphi(\gamma(s)) \,, \qquad K = \tfrac{1}{2} \int_\gamma \xi(s) G(\gamma) \xi(s) \, ds$$

Legendre transform

- Lagrangian

$$L = K - P$$

The kinematic boundary condition states that

$$N \cdot \dot{\gamma} = N \cdot \nabla \varphi(\gamma) = G(\gamma)\xi$$

Decmpose the vector field $\dot{\gamma}(s)$ along the curve $\gamma = (\gamma_1(s), \gamma_2(s))$ in terms of its Frenet frame $(T(s), N(s))$

$$n(t,s) = N \cdot \dot{\gamma}(t,s) \ , \quad \tau(t,s) = T \cdot \dot{\gamma}(t,s)$$

then the Lagrangian is

$$L = \tfrac{1}{2} \int_\gamma n(t,s) G^{-1}(\gamma) n(t,s) \, dS_\gamma - \int_\gamma V \cdot N \, dS_\gamma$$

- The Legendre transform

$$\delta_{\dot{\gamma}} L = G^{-1}(\gamma) n(t,s) = \xi(s) \ ,$$

$$H = \tfrac{1}{2} \int_\gamma \xi(s) G(\gamma) \xi(s) \, dS_\gamma + \int_\gamma \frac{g}{2} \gamma_2^2(s) \partial_s \gamma_1(s) \, ds$$

Hamilton's canonical equations

- variations $\delta\gamma$ and $\delta\xi$ of the Hamiltonian

$$N \cdot \dot{\gamma} = n = \delta_\xi K = G(\gamma)\xi$$

  the kinematic boundary conditions
- Decomposing boundary variations $\delta\gamma = N \cdot \delta\gamma + T \cdot \delta\gamma$ variations of the potential energy

$$\delta_\gamma P \cdot \delta\gamma = \int_\gamma g\gamma_2(s) N \cdot \delta\gamma \, dS_\gamma$$

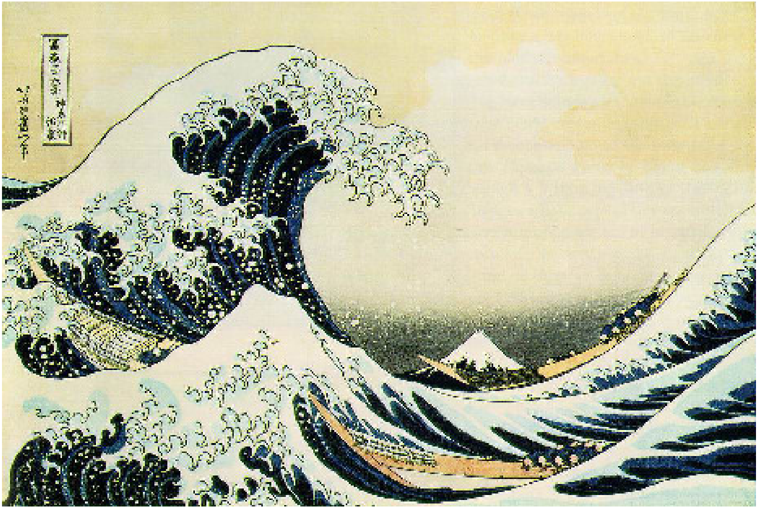
- Finally $\delta_\gamma K$ has normal and tangential components. The normal component gives the result that

$$\partial_t \xi = -g\gamma_2 - \tfrac{1}{2}\Big(\frac{1}{|\partial_s \gamma|^2} (\partial_s \xi)^2 - (G(\gamma)\xi)^2 - \frac{1}{|\partial_s \gamma|} \partial_s \xi \tau \Big)$$

  The tangential component $T \cdot \dot{\gamma}$ depends upon the manner in which the surface is parametrized

Critique

- This talk in on work in progress for several reasons
  Existence theorems depend upon solving the initial value problem in special variables.
  Nalimov (1971) shows that this is possible in Lagrangian coordinates
  In Eulerian coordinates such proofs depend upon Alinhac's 'good variables'.
  Question: is there a systematic symmetrization for the water wave equations, that is independent of coordinates
- Normal forms in the case $\sigma = 0$ and $0 < h < +\infty$ are not included
- In the case of a variable bathymetry $h(x)$, periodic for example, the dispersion relation is replaced by the Bragg frequencies $\omega_h(k)$
- Properties of the normal forms transformations $\tau^{(M)}$ on energy spaces $H^s_\eta \oplus H^{s+1/2}_\xi$



Thank you

proof of Theorem 2

Proposition 7

- *One can choose initial data* $\eta_0(x) = \eta(x,0)$ *such that* $M = 2\pi\hat{\eta}(0) = 0$
- *Unless* $\langle k, p-q\rangle = 0$ *the coefficients satisfy*

$$c(p,q) = 0$$

  (conservation of momentum)
- *There are no nonzero* $m = 3$ *resonances. Indeed*

$$\omega(k_1) \pm \omega(k_2) \pm \omega(k_3) = 0 \qquad \text{and} \qquad k_1 \pm k_2 \pm k_3 = 0$$

  implies $k_\ell = 0$ *for some* $\ell = 1,2,3$

The auxiliary Hamiltonian is determined by a cohomological equation

$$\{K^{(3)}, H^{(2)}\} + H^{(3)} = 0$$

to be solved for $K^{(3)}$. This is a linear equation

- The transformation $\tau^{(3)}$ is constructed as the time $s = 1$ flow of the Hamiltonian vector field of $K^{(3)}$

$$\frac{d}{ds} z = J \mathrm{grad}_z K^{(3)} := X^{K^{(3)}}(z)$$

- In case $h = +\infty$ the auxiliary Hamiltonian $K^{(3)}$ is remarkably simple

$$K^{(3)}(\eta, \xi) = \frac{1}{2} \int \big(i\mathrm{sgn}(D)\eta \big)^2 |D| \xi \, dx = \frac{1}{2} \int \tilde{\eta}^2 \partial_x \tilde{\xi} \, dx \qquad (1)$$

where $(\tilde{\eta}, \tilde{\xi}) := -i\mathrm{sgn}(D)(\eta, \xi)$ the Hilbert transform

- The auxiliary flow giving $\tau^{(3)}$ is the solution map of

$$\begin{aligned} \partial_s \tilde{\eta} = -\tilde{\eta} \partial_x \tilde{\eta} &\;=\; \mathrm{grad}_{\tilde{\xi}} K^{(3)} \\ \partial_s \tilde{\xi} = -\tilde{\eta} \partial_x \tilde{\xi} &\;=\; -\mathrm{grad}_{\tilde{\eta}} K^{(3)} \end{aligned}$$

NB This is Burgers flow for $\tilde{\eta}$, and its linearization for $\tilde{\xi}$ (WC & C. Sulem (2012), also Hunter & Ifrim (2012))

end of proof

- Check the expression (1)

$$\begin{aligned}
&\{H^{(2)}, K^{(3)}\} \\
&= \tfrac{1}{2} \int \eta |D| \big(i\mathrm{sgn}(D)\eta \big)^2 + |D|\xi \big(i\mathrm{sgn}(D)(i\mathrm{sgn}(D)\eta |D|\xi) \big) \, dx \\
&\quad = \int \frac{1}{6} \partial_x \big(i\mathrm{sgn}(D)\eta \big)^3 \, dx - \int (i\mathrm{sgn}(D)\eta) \partial_x \xi |D| \xi \, dx
\end{aligned}$$

- With an identity for Hardy space functions $f + ig$

$$i\mathrm{sgn}(D)\big(fg\big) = \tfrac{1}{2}\big(f^2 - g^2\big)$$

where $g = -i\mathrm{sgn}(D)f$ is the Hilbert transform, this expression gives

$$\{H^{(2)}, K^{(3)}\} = \tfrac{1}{2} \int \eta \big((\partial_x \xi)^2 - (|D|\xi)^2 \big) \, dx = H^{(3)}$$

proof of Theorem 5

- With surface tension the dispersion relation is

$$\omega^2(k) = (g + \sigma k^2)k \tanh(hk)$$

  There can be resonant triples $(\omega(k_1), \omega(k_2), \omega(k_3))$
  If $h < +\infty$ these lie in a compact set in k-space.

- Aside from these resonances, solve the cohomological equation for $K^{(3)}$

$$\{H^{(2)}, K^{(3)}\} = H^{(3)} - [H^{(3)}]$$

- The Hamiltonian vector field $X^{K^{(3)}}(\eta, \xi)$ satisfies energy estimates on neighborhoods $B_R(0)$ in the function space $H_\eta^{r+1} \oplus H_\xi^{r+1/2}$ for $r > 1$