Variational Models of Fracture

Blaise Bourdin (Louisiana State University),
Corrado Maurini (Université Pierre et Maris Curie),
Gilles A. Francfort (Université Paris Nord),
Christopher J. Larsen (Worcester Polytechnic Institute)

May 8 – May 13, 2016

Variational models of brittle fracture find their origin in Griffith's classical theory, revisited using modern mathematical tools. They preserve the essence of Griffith argument (that brittle fracture can be seen as competition between bulk and surface energies), and recast it as sequences of unilateral minimization principle for a free-discontinuity energy. In doing so, they largely address most of the shortcomings of the classical brittle fracture approach, such as the difficulty to handle changes in cracks topology, the required a priori knowledge of crack path, or the difficulty to handle three-dimensional problems or heterogeneous materials. In contrast, variational models of fracture require no a priori assumption on cracks geometry, can easily handle changes in fracture topology such as new cracks nucleation, branching or merging, and are formulated identically in two or three space dimensions. The numerical implementation of these models also requires special techniques, with the most popular ones being based on some form of regularized or "phase–field" models. In the last few years, the interest of the computing, engineering and industrial communities to this area has been mostly driven by their unprecedented capabilities to tackle problems out of reach of the classical approaches. This has resulted in many extensions of the original variational framework, most of which are currently beyond reach of rigorous mathematical analysis.

1 Scientific context

Since Francfort and Marigo initiated a revisiting of Griffith's classical theory of fracture of brittle fracture with modern mathematical tools [11, 6], variational models in fracture mechanics have been the subject of intense activity. The interest of the mathematical community for this class of problem can be measured by a large body of work in areas including mathematical analysis, mathematical modeling, numerical analysis and computational science, several workshop and mini-symposia including weeklong meetings in Oberwolfach in 2007 and 2011 and at BIRS in 2011, or G.A. Francfort's plenary lecture at the 2011 ICIAM congress. In recent years, the interest of the engineering community in this class of model has also rapidly increased, mainly motivated by the potential for rigorous, accurate and efficient numerical simulations. This interest can be measured for instance by the size of mini-symposia relating to "phase–field" approach in fracture mechanics congresses, or through M. Ortiz and C. Miehe's plenary lectures at the 2014 WCCM conference. The interest of various industries in this class of problems is also noticeable with ongoing projects at Corning, Chevron, Lafarge, and Airbus, amongst others.

Yet, as the mathematical understanding of this problem is reaching its maturity, we are reaching a tipping point where the mathematics and engineering or computational science communities are becoming increasingly divided, instead of mutually benefiting from each other's progress. The reasons of this growing schism are multiple: the mathematical literature can be very technical and deter engineering students. Conversely, mathematicians may have a hard time translating issues mentioned in the engineering and technical literature into well defined mathematical problems. This is especially true of recent extensions to dynamic or rate dependent problems or modeling of coupled problems in reservoir engineering, fracture of ferromagnetic materials or corrosion cracks in thin coatings, for which a rigorous mathematical understanding is lacking. Finally, graduate students and young researchers often lack a common culture with engineers being unfamiliar with modern mathematical tools such as geometric measure theory or Γ–convergence, and mathematicians lacking awareness of actual problems.

1.1 Specific objectives

The goal of this workshop was to bring together a group of mathematicians, mechanicians, engineers and computational scientist sharing an interest in variational models of fracture mechanics in order to achieve a breakthrough in the mathematical understanding of current topics, tools and issues, and in the scope of the numerical applications of the current theories. The specific objectives were:

- O_1*: To present the state of the art of the mathematical analysis of problems arising from variational models of fracture.* A better understanding of the mathematical issues arising in these problems is essential to reach a deep understanding of the numerical methods. Yet, there is a lack of concise and focussed literature at the graduate level. We will begin this workshop with a few introductory lectures on mathematical modeling and tools so as to give the more applied participants a (possibly critical) overview of the current state of the theory.

- O_2*: To gain a better understanding of the challenges facing this class of methods.* "Real life" problems can be quite at odds with those favored by mathematicians and are often beyond the reach of rigorous analysis. Engineers and industry partners will be invited to present current or potential applications and algorithms related to the variational models of fracture. The rationale is that a better theoretical understanding of this problem can lead to more efficient numerical tools, while exposure to a broader range of problems will stimulate new theoretical developments.

- O_3*: To devise a set of reference problems that can be analyzed rigorously, then used in order to assess the accuracy and efficiency of algorithms.* The popularity of benchmark problems in fracture mechanics is highly skewed by the strength and weaknesses of classical methods. The resulting tests are often inappropriate or even non sensical from the standpoint of the variational approach to fracture. Devising proper numerical experiments highlighting specific properties of a model or implementation is difficult and time consuming. In addition to the lack of common reference tests, comparing methods is difficult. We propose to come up with a small set of problems which will be used in the years to come.

1.2 Participants

In order to achieve these goals, the organizers seek to build a compact participant list with even representation from the mathematics, mechanics, numerical simulation, experimental, and industrial communities, as well as some experts from adjacent fields.

2 Presentation highlights

The participants were instructed to focus on the three specific objectives listed above in there presentation and during informal discussion sessions. We summarize some of the presentations that best addressed our objectives, as well as those bringing awareness to new problem, methods and challenges.

The first day was dedicated to four hour-long "keynote" lectures: K. Ravi-Chandar opened the workshop by presenting some and numerical results for mixed-mode fracture. In two dimensions, he proposed a modified compact-tension experiment for which stable mixed-mode I-II crack propagation along a curved crack path can be achieved. In three dimension, he focussed on mixed mode I-III propagation leading to echelon and factory crack propagation. Such problems have received a lot of attention lately including numerical simulations by participant A. Karma and experiment by participant A. Zehnder: “Transition from spiral to factory roof type fracture under tension”. Both are excellent benchmarks problems addressing objective O_3.

The following presentations focussed objective O_1 with recent results on variational and phase field models in the mathematics, mechanics and physics communities respectively. G. Dal Maso focussed on recent mathematical results dealing with existence theory for evolutions satisfying a maximum dissipation condition in the dynamic regime. J.-J. Marigo presented recent works on coupling variational models of damage and plasticity. A. Karma focussed on dynamic instabilities of fast moving cracks in brittle solids.

During the next few days several presentations addressed extension of variational and phase field models to plates and shells (B. Roman: “Fracture path in thin sheets”, M. Arroyo: “Phase-field modeling of fracture in thin shells”, J.-F. Babadjian: “Reduced models for linearly elastic thin films allowing for fracture, debonding or delamination”, and B. Schmidt: “On a quantitative piecewise rigidity result and Griffith-Kirchhoff functionals for thin brittle beams”). Applications to hydraulic fracturing also emerged as a critical application that requires a better understanding with presentations from C. Landis: “Phase-Field modeling of Hydraulic Fracture”, M.F. Wheeler: “Phase-field modeling of proppant-filled fractures in a poroelastic medium”, A. Pandolfi: “A variational model of poro-mechanical damaging material”, and K. Yoshioka: “variational fracture modeling applied to hydraulic fracturing (fracking)”. Extensions to plasticity and ductile fracture, were also addressed in presentations by G. Lancioni: “A variational approach to gradient plasticity”, and M. Ortiz: “Optimal scaling in ductile fracture”. These topics clearly emerged as areas of need, as stated in objective O_2.

Objective O_1 was again at the center of presentations focussing on analysis of models by A. Chambolle: “Some remarks on the energy release rates in planar linearized elasticity” and K. Pham: “Stability analysis of homogeneous states in gradient damage models”. Alternate numerical methods and extension were the focus of by M. Negri‘: “Convergence in time of discrete evolutions generated by alternate minimizing schemes”, and M. Kimura: “Unidirectional gradient flow and its application to a crack propagation model”.

Finally multiple participants presented alternative models of damage and fracture, and innovative numerical methods, including R. Haber: “Spacetime interfacial damage model for dynamic fracture in brittle materials”, G. Allaire “Damage and fracture evolution in brittle materials by shape optimization methods”, R. Lehoucq: “recent developments in peridynamic mechanics”, M. Sarkis: “Finite element methods on non-aligned meshes for interface problems”, A. Lew: “high-fidelity simulation of brittle fracture problems with Universal Meshes”, and M. Luskin: “Lattice stability of hybrid atomistic to continuum coupling methods for fracture modeling”.

It is also to be noted that feedback on challenges facing the industry was provided by K. Yoshioka (Chevron ETC) and V. Subramanian (Corning inc.).

2.1 Outline of presentations

Grégoire Allaire: *Damage and fracture evolution in brittle materials by shape optimization method.* This joint work with François Jouve and Nicolas Van Goethem is devoted to a numerical implementation of the Francfort-Marigo model of damage evolution in brittle materials. This quasi-static model is based, at each time step, on the minimization of a total energy which is the sum of an elastic energy and a Griffith-type dissipated energy. Such a minimization is carried over all geometric mixtures of the two, healthy and damaged, elastic phases, respecting an irreversibility constraint. Numerically, we consider a situation where two well-separated phases coexist, and model their interface by a level set function that is transported according to the shape derivative of the minimized total energy. In the context of interface variations (Hadamard method) and using a steepest descent algorithm, we compute local minimizers of this quasi-static damage model. Initially, the damaged zone is nucleated by using the so-called topological derivative. We show that, when the damaged phase is very weak, our numerical method is able to predict crack propagation, including kinking and branching. Several numerical examples in $2d$ and $3d$ are discussed.

Jean-François Babadjian: *Reduced models for linearly elastic thin films allowing for fracture, debonding or delamination.* This talk is devoted to highlighting the interplay between fracture and delamination in thin films. The usual scaling law on the elasticity parameters and the toughness of the medium with respect to its thickness gives rise to traditional cracks which are invariant in the transverse direction. We will show that, upon playing on this scaling law, it is also possible to observe debonding effects (delamination as well as decohesion) through the appearance of cracks which are orthogonal to the thin direction. Starting from a three-dimensional brittle elastic thin film, we will first present how both phenomena can be recovered independently through a Gamma-convergence analysis as the thickness tends to zero. Then, working on a ?toy model" for scalar anti-plane displacements, we will show how both phenomena can be obtained at the same time. Some partial results in the full three-dimensional case will be presented. These are joint works with Blaise Bourdin, Duvan Henao, Andrs Leon Baldelli. and Corrado Maurini.

Antonin Chambolle: *Some remarks on the energy release rates in planar linearized elasticity* The talks reviewed a series of results obtained in the past ten years with J.-F. Babadjian, G. Francfort, A. Lemenant and J.-J. Marigo on the singularities at the tip of cracks in planar elasticity with minimal regularity assumptions, and the appropriate definition of a generalised Energy Release Rate in such a setting.

In [7, 8], we have shown that in the variational approach to brittle fracture (which generalises the classical Griffith theory, see [11, 6]), if one considers that a crack may grow in an arbitrary way which is not necessarily smooth at infinitesimal scales, then one should arrive to the conclusion that a straight fracture under a mode II loading should become unstable for lower loads than classically predicted (by criteria such as the "principle of local symmetry" or "maximum ERR") and kink "brutally" rather than in a continuous way. Obviously, the "correct" scale cannot be prescribed by Griffith's model and the statement, as such, is unphysical: however it should give an idea of how the macroscopic behaviour of brittle materials should appear in such situations

In [9] (antiplane case) and [4] (planar elasticity), we have partially extended these results to situations where the initial crack itself is not even smooth, showing that also in such situations, the singularity at the tip of an arbitrary crack which looks like a half-line at infinitesimal scales is (as expected) the same as for a smooth crack ending in a point, and similar results should be expected. A natural conclusion (which is mathematically out of hand by now) should be that quasi-static fractures whose length is growing in a continuous way (as a function of time) should be at least C^1 curves.

Gianni Dal Maso: *A maximal dissipation condition for dynamic fracture: an existence result in a constrained case.* The study of dynamic fracture is based on the dynamic energy-dissipation balance. It is easy to see that this condition is always satisfied by a stationary crack together with a displacement satisfying the system of elastodynamics. Therefore to predict crack growth a further principle is needed. In this talk we introduce a maximal dissipation condition that, together with elastodynamics and energy balance, provides a model for dynamic fracture, at least within a certain class of possible crack evolutions. In particular, we prove the existence of dynamic fracture evolutions satisfying this condition, subject to smoothness constraints, and exhibit an example to show that maximal dissipation can indeed rule out stationary cracks.

Robert Haber: *Spacetime Interfacial Damage Model for Dynamic Fracture in Brittle Materials.* While most of fundamental physical questions in dynamic fracture mechanics are settled science, there remains a significant gap between this fundamental understanding and our ability to apply it in computational models of failure in the complex systems and materials that arise in geophysics, biology, and contemporary engineering design. This talk describes recent progress at the University of Illinois and the University of Tennessee Space Institute in developing new numerical methods and models intended to close at least some aspects of this gap. A spacetime discontinuous Galerkin (SDG) method is the numerical foundation for our fracture model. This particular SDG method [2] is tailored to the requirements of hyperbolic systems, and differs from most others in that it is asynchronous, locally implicit, embarrassingly parallel, and supports fine-grain adaptive meshing. It enforces jump conditions with respect to Riemann solutions on element boundaries to preserve the characteristic structure of the underlying system. As with other DG methods, conservation fields balance to within machine precision on every (spacetime) element. We model crack opening and closure with specialized Riemann solutions for the various modes of frictional dynamic contact [1]. We weakly enforce these Riemann solutions using the same framework that enforces jump conditions and boundary conditions at inter-element and domain boundaries elsewhere in the SDG formulation. This approach produces contact conditions that are distinct from those that arise from simple constraints against inter-element penetration. We can implement cohesive fracture models in this SDG framework by incorporating traction? separation laws in the Riemann solutions [3]. However, in this presentation we focus on interfacial damage as an alternative means to model fracture along sharp interfaces. Time-delay evolution equations determine the damage rate as functions of the tractions and velocity jumps across fracture surfaces. A probabilistic model for microscopic flaws provides a mechanism for nucleating new fracture surfaces and is sufficient to captures crack branching. Adaptive refinement ensures that the solution fields are well resolved at multiple crack tips and along wave fronts. The same adaptive procedures continuously reconfigure the mesh so that it follows the crack paths determined by our physical model. We discuss some of the open challenges in modeling fracture with interfacial damage and present several numerical examples to demonstrate existing capabilities.

Work in collaboration with Reza Abedi, Mechanical, Aerospace & Biomedical Engineering, University of Tennessee Space Institute (UTSI) / Knoxville (UTK), 411 B. H. Goethert Parkway, Tullahoma, TN 37388

Alain Karma: *Phase-field modeling of rapid fracture in linear and nonlinear elastic solids.* This talk will discuss phase-field modeling of dynamic instabilities of fast moving cracks in brittle solids. Experiments in thin gels have shown that cracks can attain extreme speeds approaching the shear wave speed when micro branching, which limits propagation to smaller speeds in thick samples, is suppressed. Furthermore, they have revealed the existence of an oscillatory instability with an intrinsic system-size-independent wavelength above a threshold speed. In apparent contradiction with experimental observations, the commonly used phase-field formulation of dynamic fracture yields crack that branch by tip splitting at roughly half the shear wave speed. A phenomenologically based phase-field formulation is proposed that can model crack propagation at extreme speeds by maintaining the wave speed constant inside the microscopic process zone. Simulations of this model for linear elasticity outside the process zone produce crack that tip split above a high threshold speed but no oscillations. In contrast, simulations for nonlinear neo-Hookean elasticity yield crack oscillations above a ultra-high speed threshold. Those oscillations have an intrinsic wavelength that scale with the size of the nonlinear zone surrounding the crack tip, which can be much larger than the process zone scale, and bear striking similarity with observed oscillations in thin gels.

This work was carried out in collaboration with Chih-Hung Chen and Eran Bouchbinder and his supported by a grant from the US-Israel Binational Science Foundation.

Masato Kimura: *Unidirectional gradient flow and its application to a crack propagation model.* We consider a nonlinear diffusion equation with irreversible property and construct a unique strong solution by using implicit time discretization. A new regularity estimate for the classical obstacle problem is established and is used in the construction of the strong solution.

As an application, we consider a quasi-static fracture model of brittle material using the idea of the phase field model. The Francfort-Marigo energy which is based on the classical Griffith theory is introduced, where the sharp crack profile is approximated by a smooth damage function using the idea of the Ambrosio-Tortorelli regularization. The crack propagation model is derived as a gradient flow of the energy of the damage variable with an irreversible constraint. Some numerical examples in various settings computed by finite element method are also presented in the talk.

The contents is based on the joint works with Goro Akagi (Kobe Univ.) and with Takeshi Takaishi (Hiroshima Kokusai Gakuin Univ.).

Giovanni Lancioni: *A variational approach to gradient plasticity.* In this talk, a variational model for gradient plasticity is proposed, which is based on an energy functional sum of a stored elastic bulk energy, a non-convex dissipative plastic energy, and a quadratic non-local term, depending on the gradient of the plastic strain. The basic modelling ingredients are presented in a simple one-dimensional setting, where the key physical aspects of the phenomena can easily be extracted. The evolution laws are deduced by using the mathematical tool of incremental energy minimization, and they are commented, highlighting the main differences and similarities with variational damage models. The typical assumptions of classical plasticity, such as yield condition, hardening rule, consistency condition, and elastic unloading, are obtained as necessary conditions for a minimum. Then, analytical solutions are determined, and attention is focused on the correlations between the convex-concave properties of the plastic energy and the distribution of the deformation field. The issue of solution stability is also addressed. Finally, some numerical results are discussed. First, tensile tests on steel bars and concrete samples are reproduced, and, then, a more complex two-dimensional crystal plasticity is proposed, and the process of microstructures evolution in metals is described by assuming a double-well plastic potential.

Chad Landis: *Phase-field Modeling of Hydraulic Fracture.* In this talk a theoretical framework implementing the phase-field approach to fracture is used to couple the physics of flow through porous media and cracks with the mechanics of fracture. The main modeling challenge addressed in this work, which is a challenge for all diffuse crack representations, is on how to allow for the flow of fluid and the action of fluid pressure on the aggregate within the diffuse damage zone of the cracks. The theory is constructed by presenting the general physical balance laws, postulating a kinematic ansatz for an effective porosity, and conducting a consistent thermodynamic analysis to constrain the constitutive relationships. Constitutive equations that reproduce the desired responses at the various limits of the effective porosity are proposed in order to capture Darcy-type flow in the intact porous medium and Stokes-type flow within open cracks. A finite element formulation for the solution of the governing model equations is presented and discussed. Finally, the theoretical and numerical model is shown to compare favorably to several important analytical solutions. More complex and interesting calculations are also presented to illustrate some of the advantageous features of the approach.

Richard Lehoucq: *Recent developments in peridynamic mechanics.* My presentation introduces the peridynamic model for predicting the initiation and evolution of complex fracture patterns. The model, a continuum variant of Newton's second law, uses integral rather than partial differential operators where the region of integration is over a domain. The force interaction is derived from a novel nonconvex strain energy density function, resulting in a nonmonotonic material model. The resulting equation of motion is proved to be mathematically well-posed. The model has the capacity to simulate nucleation and growth of multiple, mutually interacting dynamic fractures. In the limit of zero region of integration, the model reproduces the classic Griffith model of brittle fracture. The simplicity of the formulation avoids the need for supplemental kinetic relations that dictate crack growth or the need for an explicit damage evolution law.

Adrian Lew: *High-fidelity Simulation Of Brittle Fracture Problems With Universal Meshes* We describe our approach to simulating curvilinear brittle fractures in two-dimensions based on the use of Universal Meshes [10, 13, 16, 17, 18]. A Universal Mesh is one that can be used to mesh a class of geometries by slightly perturbing some nodes in the mesh, and hence the name universal. In this way, as the crack evolves, the Universal Mesh is always deformed so as to exactly mesh the crack surface. The advantages of such an approach are: (a) no elements are cut by the crack, (b) new meshes are automatically obtained as the crack evolves, (c) the crack faces are exactly meshed with a conforming mesh at all times, and the quality of the surface mesh is guaranteed to be good, and (d) apart from duplicating degrees of freedom when the crack grows, the connectivity of the mesh and the sparsity of the associated stiffness matrix remains unaltered. In addition to the mesh, we are now able to compute stress intensity factors with any order of convergence, which gives us unprecedented accuracy in computing the crack evolution. As a result, we observe first order convergence of the crack path as well as the tangent to the crack path in a number of different examples. In the presentation I will introduce the notion of a Universal Mesh, illustrate the progress we have made so far with some examples, and then focus on the simulation of curvilinear fractures, and on the tools we created to compute stress intensity factors. In particular, show examples in which the computed crack path converge to the exact crack path, regardless of the mesh. If time permits, simulation of thermally induced fracture and hydraulic fractures will be discussed.

Mitchell Luskin: *Lattice Stability of Hybrid Atomistic to Continuum Coupling Methods for Fracture Modeling.* Hybrid atomistic to continuum coupling methods have been developed to obtain the accuracy of atomistic modeling in the neighborhood of crack tips, while using continuum modeling to include long range elastic effects. We will present a survey of recent work to analyze the lattice stability error introduced by atomistic to continuum coupling methods. These error estimates are then utilized to develop accurate blended coupling methods with controllable error.

Joint work with Christoph Ortner, Mathew Dobson, Xingjie Helen Li, Derek Olson, and Brian Van Koten.

Matteo Negri: *Convergence in time of discrete evolutions generated by alternate minimizing schemes* We consider two quasi-static evolutions, of BV-type, for the Ambrosio-Tortorelli energy. Both are obtained by time discretization and by alternate minimization schemes, thus taking full advantage of the separate quadratic structure of the energy.

For the first we will employ a constrained alternate minimization scheme in which the time-update configuration is found by an iterative procedure, either finite or infinite. In the latter case the updated configuration is an equilibrium point for the energy. This algorithm can be recast both as a separate gradient flow, with respect to a suitable family of intrinsic norms, and also by a "quasi-Newton" method. Both the representations highlight the underlying family of intrinsic norms for the evolution. After re-parametrizing the evolution by means of an arc-length parameter we can conveniently pass to the limit, characterized in terms of a (parametrized) BV-evolution. In particular we show that in the regime of stable (or steady state) propagation the limit evolution satisfies equilibrium for the displacement variable and a suitable form of Griffith's criterion for the phase-field variable, written in terms of a phase-field energy release rate. We further show that the irreversibility constraint, given by the monotonicity of the phase-field variable, is thermodynamically consistent, since the associated dissipated energy is non-decreasing in time. Further we characterize the unstable regime of propagation in terms of gradient flows (in the parametrization variable) with respect to the intrinsic norms. The fact that the limit evolution is "simultaneous" in the two variable, even if the algorithm is not, is justified by continuous dependence.

In the second case we consider an irreversible L^2-gradient flow for the phase field variable combined with the equilibrium equation for the displacement field. We obtain again the time-continuous evolution as the limit of a time-discrete approximation, in which the incremental problem is a one-step alternate minimization. We can characterize the limit both in terms of energy balance and PDEs without relying on chain rule and thus on compactness in time-Sobolev spaces. Moreover, we study the limit as the viscosity vanishes. To this end, as in the previous case, we first parametrize the evolutions by arc-length and prove that their lengths are uniformly finite. This delicate technical step is obtained by means of a suitable discrete Gronwall argument, which in turns provides also the local regularity in time-Sobolev spaces. Then we can pass to the limit, which is characterized again in terms of a quasi-static BV-evolution. In this case we first show the energy balance and then deduce, by means of the chain rule, the corresponding PDEs. We can show that in the regime of stable propagation the limit evolution is again in equilibrium, in both the variables, and that Griffith's criterion holds, again for a phase-field energy release. Finally, unstable regimes of propagation are characterized by a system of PDEs which is nothing but the original time-continuous system re-written in the parametrization variable.

We briefly discuss the fact that different underlying norms can lead to qualitatively similar evolutions, in particular as far as steady state propagation is concerned. Such evolutions do not necessarily coincide since solutions to this systems are in general non-unique. We finally suggest some alternative 'norm' for the evolution in terms of the phase-field length.

Michael Ortiz: *Optimal scaling in ductile fracture.* Abstract: This work is concerned with the derivation of optimal scaling laws, in the sense of matching lower and upper bounds on the energy, for a solid undergoing ductile fracture. The specific problem considered concerns a material sample in the form of an infinite slab of finite thickness subjected to prescribed opening displacements on its two surfaces. The solid is assumed to obey deformation-theory of plasticity, in the case of metals, and classical rubber elasticity for polymers. When hardening exponents for metals are given values consistent with observation, or when chain failure is accounted for in polymers, the energy is found to exhibit sublinear growth. We regularize the energy through the addition of nonlocal energy terms of the strain-gradient plasticity type for metals and fractional strain-gradient elasticity for polymers. This nonlocal regularization has the effect of introducing an intrinsic length scale into the energy. Under these assumptions, ductile fracture emerges as the net result of two competing effects: whereas the sublinear growth of the local energy promotes localization of deformation to failure planes, the nonlocal regularization stabilizes this process, thus resulting in an orderly progression towards failure and a well-defined specific fracture energy. The optimal scaling laws derived here show that ductile fracture results from localization of deformations to void sheets in metals and to crazes in polymers, and that it requires a well-defined energy per unit fracture area. The optimal scaling laws show that ductile fracture is cohesive in nature, i.e., it obeys a well-defined relation between tractions and opening displacements. Finally, the scaling laws supply a link between micromechanical properties and macroscopic fracture properties. In particular, they reveal the relative roles that microplasticity and surface energy play as contributors to the specific fracture energy of the material.

Anna Pandolfi: *A variational model of poro-mechanical damaging material.* Deterioration of mechanical and hydraulic properties of rock masses and subsequent problems are closely related to changes in the stress state, formation of new cracks, and increase of permeability in porous media saturated with freely moving fluids. In fully saturated rocks, fluid and solid phases are interconnected and the interaction between fluid and rock is characterized by coupled diffusion-deformation mechanisms that convey an apparent time-dependent character to the mechanical properties of the rock. The two governing equations of the coupled problem are the linear momentum balance and the continuity equation (mass conservation). The kinematic quantities that characterize this picture are the porous solid displacement and the rate of fluid volume per unit area. Hydro-mechanical coupling arises from the influence of the mechanical variables (stress, strain and displacement) on the continuity equation, where the primary variable is the fluid pressure, and from the influence of the hydraulic variables (pore pressure and seepage velocity) on the equilibrium equations, where the primary variables are the displacements. We describe a coupled approach to model damage induced by hydro-mechanical processes in low permeability solids. We describe the solid as an anisotropic brittle continuum where the damage is characterized by the formation of nested micro-structures in the form of equi-distant parallel faults, characterized by distinct orientation and spacing. The particular geometry of the faults allows for the analytical derivation of the porosity and of the anisotropic permeability of the solid. The fractured medium can be regarded as an anisotropic porous material. Classic methods can be applied to describe the porous-mechanical behavior of the solid to estimate the flow of fluids across the medium according to the presence of a fluid pressure gradient. The approach can be used for a wide range of engineering problems, ranging from the prevention of water or gas outburst into underground mines to the prediction of the integrity of reservoirs for underground CO2 sequestration or hazardous waste storage. The work is done in collaboration with M.L. De Bellis, G. Della Vecchia and M. Ortiz.

Kim Pham: *Stability analysis of homogeneous states in gradient damage models.* In this talk I will talk about the stability of homogeneous states for gradient damage models. We will show how to exploit the second order derivative of the total energy to discriminate stable and unstable homogeneous states depending on the hardening properties, the loading and the size of the sample.

This is based on joint work with J-J Marigo and Corrado Maurini.

Benoit Roman: *Tearing : fracture path in brittle thin sheets.* In this presentation i gave a short review of several tearing experiments, and tried to explain the reasons why i think that the determination of fracture path in thin sheets is a good problem for the variational approach to fracture,

A first reason is that experiments show that in many configurations fracture in thin sheets follow very robust, reproducible trajectories. This gives clear experimental features that can be used to test a variational theory.

A second reason is that the description of these tearing experiments fall out of the classical linear elastic approach to fracture. A first difficulty is that plate equations are based on averaging mechanical load through the thickness. This is incompatible with fracture mechanics, which studies the stress field at the tip of the crack, requiring a detailed description in the thickness. One approach would therefore be to generalize the notion of stress intensity factor to linear plate equations, but to our knowledge this line of work is not very much advanced, and many tools need to be defined before the crack path can be defined. A second problem is that because large out of plane bending is involved, the elastic problem is most likely to be strongly non-linear.

If linear fracture mechanics fails, a variational approach to fracture including the complete non-linear plate description is completely possible. We have used a first simplified model, which consists in assuming that the sheet is inextensible (conserves its length) but infinitely bendable. In this framework, the elastic energy of the plate is zero, so that the work of the operator is entirely dissipated into fracture energy during propagation. The usual notions of modes (corresponding to stress intensity decomposition) has no meaning in this framework since we have neglected the elasticity of the system. However this approach leads to simple prediction for the crack path which found to be often close to the experimental observations.

The first successes of this simplified approach are encouraging, but some features of the crack path are not correctly described in the inextensible model. We are also lacking estimates of the error in the path due to these strong assumptions. A real description of the system should include both bending and stretching energies within a complete variational model based on non-linear plate mechanics.

Marcus Sarkis: *Finite Elements Methods on Non-Aligned Meshes for Interface Problems.* We define two finite element methods for elliptic problems with possibly discontinuous diffusion coefficients and divergence constraints where the meshes are not aligned with the interface. The first method is based on Immersed Interface Methods while the second one on Nitsche's methods. In order to obtain apriori error estimates totally independent of the contrast between diffusion coefficients and independent on how the interface crosses the mesh, we consider stabilizations based on the jump of the flux as well as the jump of solution across elements.

Bernd Schmidt: *On a quantitative piecewise rigidity result and Griffith-Euler-Bernoulli functionals for thin brittle beams.* We study a planar thin brittle beam subject to elastic deformations and cracks described in terms of a nonlinear Griffith energy functional acting on SBV deformations of the beam. In particular we consider the case in which elastic bulk contributions due to finite bending of the beam are comparable to the surface energy which is necessary to completely break the beam into several large pieces.

In the limit of vanishing aspect ratio we rigorously derive an effective Griffith-Euler-Bernoulli functional which acts on piecewise $W^{2,2}$ regular curves representing the midline of the beam. The elastic part of this functional is the classical Euler-Bernoulli functional for thin beams in the bending dominated regime in terms of the curve's curvature. In addition there also emerges a fracture term proportional to the number of discontinuities of the curve and its first derivative [19].

A key ingredient in the Γ-convergence proof is a novel quantitative geometric rigidity estimate for special functions of bounded deformation in a planar setting, recently obtained in joint work with M. Friedrich. It generalizes a result by Friesecke, James and Müller for Sobolev functions obtained in nonlinear elasticity theory and a qualitative piecewise rigidity result by Chambolle, Giacomini and Ponsiglione for brittle materials which do not store elastic energy. We show that for each deformation there is an associated triple consisting of a partition of the domain, a corresponding piecewise rigid motion being constant on each connected component of the cracked body and a displacement field measuring the distance of the deformation from the piecewise rigid motion [12].

Vijay Subramanian: *Variational fracture modeling at Corning Incorporated.* The variational approach to fracture has been in development for the past two decades. Over two years ago Corning began a collaboration with Prof. Blaise Bourdin at Louisiana State University. The presentation summarizes some of the developments that have occurred as part of the collaboration as well as summarize some of the internal efforts within Corning Incorporated.

Some of the businesses at Corning Incorporated for which fracture and damage plays an important role on performance are: Specialty Materials, Display Technologies, Environmental Technologies, and Pharmaceutical Technologies to name a few. The corresponding applications in these businesses would be strong glass for portable handheld devices (Gorilla Glass), Display glass substrates for televisions and hand held devices, catalytic converters and diesel particular filters, and pharmaceutical tubes respectively. The typical sources of field failures for these glasses and ceramics may come from sharp contact damage (impact, scratch, indentation), thermal shock, and/or mechanical loading. Some of the key areas of interest at Corning where the field of variational fracture can provide differentiation is in the following: fracture and strength analysis of thin-glass sheets (strengthened and non-strengthened), crack nucleation and growth in ceramic substrates due to thermo-mechanical loading, and finite elasto-plastic fracture. Many of these areas would also require large deformation capabilities as well as the ability to handle crack growth evolution under crack-face contact. A suitable reference problem for crack-face contact could be the wing-crack problem which involves an inclined crack subject to compression. Solving the wing-crack problem using the variational fracture approach may require additional development and it is hoped that future efforts will focus on devising suitable unilateral contact laws that would allow solution of wing-cracks in a robust manner.

Mary F. Wheeler: *Phase-field modeling of proppant-filled fractures in a poroelastic medium* This work presents proppant and fluid-filled fracture with quasi-Newtonian fluid in a poroelastic medium [14, 15]. Lower-dimensional fracture surface is approximated by using the phase field function. The two-field displacement phase-field system solves fully-coupled constrained minimization problem due to the crack irreversibility. This constrained optimization problem is handled by using active set strategy. The pressure is obtained by using a diffraction equation where the phase-field variable serves as an indicator function that distinguishes between the fracture and the reservoir. Then the above system is coupled via a fixed-stress iteration. The transport of the proppant in the fracture is modeled by using a power-law fluid system. The numerical discretization in space is based on Galerkin finite elements for displacements and phase-field, and an Enriched Galerkin method is applied for the pressure equation in order to obtain local mass conservation. The concentration is solved with cell-centered finite elements. Nonlinear equations are treated with Newton's method. Predictor-corrector dynamic mesh refinement allows to capture more accurate interface of the fractures with reasonable number for degree of freedoms.

Keita Yoshioka: *Variational Fracture Modeling Applied to Hydraulic Fracturing (Fracking).* Despite many applications of hydraulic fracturing, coupling of reservoir fluid flow, heat transfer, and fracture(s) propagation, especially in the presence of other pre-existing fractures, remains a major challenge in predicting the well stimulation and the evolution of well injectivity in the petroleum industry. To date simulation has focused upon the problem of a single planar in mode-I driven by a pressurized fluid using classical fracture mechanics models, or superposition of single planar fractures whilst neglecting the interaction between fractures. In contrast, realistic applications involve multiple fractures propagating along complex and unknown paths. Recently, the variational approach to fracture, which was originally proposed by Francfort and Marigo [11] and numerically implemented by Bourdin et al. [5, 6], has been increasingly applied to simulation of hydraulic fracturing because of its ability to track any number of arbitrary fractures in an efficient manner. In this talk, we present a variational fracture model extended to hydraulic fracturing by accounting for pressure force within the fracture and in-situ stresses. We then show illustrative examples to demonstrate that the model responses are closely matched with existing analytical solutions of fluid-driven fracture propagation. Finally we will present several applications to practical problems.

Alan Zehnder: *Fracture Surface Transition for Notched Bars in Torsion.* Keywords: mixed-mode, factory roof, spiral fracture, PMMA, brittle fracture It is well known that a brittle rod loaded in torsion will fail on a spiral surface angled 45 degrees to the axis of the rod so that local Mode-I conditions are maintained. It is also known that under Mode-III loading, pre-cracked brittle materials will fail with a surface that starts out planar to the original fracture surface but with a linked set of fractures at 45 degrees to the surface, forming what prior authors have deemed the ?factory roof? fracture pattern. We explore the transition from the globally flat, but locally faceted, factory roof fracture to the spiral fracture surface by performing torsion experiments on circumferentially notched rods of Polymethylmethacrylate (PMMA). Varying the notch depth we observe a transition from the factory roof to spiral fracture. Micro-CT scanning is used to image the fracture surfaces and to reveal the pattern of internal cracks that form during failure. Results of this work are posed as a challenge to the computational simulation of crack surface evolution.

Joint work with Natasha Zella.

3 Scientific Progress Made

4 Outcome of the Meeting

References

[1] R. Abedi and R.B. Haber. Riemann solutions and spacetime discontinuous Galerkin method for linear elastodynamic contact. *Comp. Meth. Appl. Mech. Engng.*, 270:150–177, 2014.

[2] R. Abedi, R.B. Haber, and B. Petracovici. A spacetime discontinuous Galerkin method for elastodynamics with element-level balance of linear momentum. *Comp. Meth. Appl. Mech. Engng.*, 195:3247–3273, 2006.

[3] R. Abedi, M.A. Hawker, R.B. Haber, and K. Matous. An adaptive spacetime discontinuous galerkin method for cohesive models of elastodynamic fracture. *Int. J. Num. Meth. Engng.*, 1(1–42), 2009.

[4] J.-F. Babadjian, A. Chambolle, and A. Lemenant. Energy release rate for non-smooth cracks in planar elasticity. *J. Éc. polytech. Math.*, 2:117–152, 2015.

[5] B. Bourdin, G. A. Francfort, and J.-J. Marigo. Numerical experiments in revisited brittle fracture. *J. Mech. Phys. Solids*, 48(4):797–826, 2000.

[6] B. Bourdin, G. A. Francfort, and J.-J. Marigo. The variational approach to fracture. *J. Elasticity*, 91(1-3):1–148, 2008.

[7] A. Chambolle, G. A. Francfort, and J.-J. Marigo. When and how do cracks propagate? *J. Mech. Phys. Solids*, 57(9):1614–1622, 2009.

[8] A. Chambolle, G. A. Francfort, and J.-J. Marigo. Revisiting energy release rates in brittle fracture. *J. Nonlinear Sci.*, 20(4):395–424, 2010.

[9] A. Chambolle and A. Lemenant. The stress intensity factor for non-smooth fractures in antiplane elasticity. *Calc. Var. Partial Differential Equations*, 47(3-4):589–610, 2013.

[10] M.M. Chiaramonte, Y. Shen, L.M. Keer, and A.J. Lew. Computing stress intensity factors for curvilinear cracks. *Int. J. Num. Meth. Engng.*, 2015.

[11] G. A. Francfort and J.-J. Marigo. Revisiting brittle fracture as an energy minimization problem. *J. Mech. Phys. Solids.*, 46(8):1319–1342, 1998.

[12] M. Friedrich and B. Schmidt. A quantitative geometric rigidity result in sbd. Preprint available at www.arxiv.org/abs/1503.06821., 2015.

[13] M.J. Hunsweck, Y. Shen, and A.J. Lew. A finite element approach to the simulation of hydraulic fractures with lag. *Int. J. Numer. Anal. Meth. Geomech.*, 37(9):993–1015, 2013.

[14] S. Lee, M.F. Wheeler, and T. Wick. Pressure and fluid-driven fracture propagation in porous media using an adaptive finite element phase field model. to appear, 2016.

[15] A. Mikelic, M. F. Wheeler, and T. Wick. Phase-field modeling of a fluid-driven fracture in a poroelastic medium. In press, 2015.

[16] R. Rangarajan, M.M. Chiaramonte, M.J. Hunsweck, Y. Shen, and A.J. Lew. Simulating curvilinear crack propagation in two dimensions with universal meshes. *Int. J. Num. Meth. Engng.*, 2014.

[17] R. Rangarajan and A.J. Lew. Analysis of a method to parameterize planar curves immersed in triangulations. *SIAM J. Numer. Anal.*, 51(3):1392–1420, 2013.

[18] R. Rangaran and A.J. Lew. Universal Meshes: A method for triangulating planar curved domains immersed in nonconforming triangulations. *Int. J. Num. Meth. Engng.*, 98(4):236–264, 2014.

[19] B. Schmidt. A Griffith-Euler-Bernoulli theory for thin brittle beams derived from nonlinear models in variational fracture mechanics. preprint available at www.arxiv.org/abs/1602.07594, 2016.