Stability issues and size effects in gradient damage models

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Gradient damage model

- Isotropic damage model
- Scalar damage variable: $\alpha \in [0,1]$
- Strain energy density

$$W(\varepsilon, \alpha, \nabla\alpha) = \frac{1}{2}\mathsf{A}(\alpha)\varepsilon \cdot \varepsilon + \mathsf{w}(\alpha) + \frac{1}{2}\mathsf{w}_1 \ell^2 \nabla\alpha \cdot \nabla\alpha$$

Elastic energy

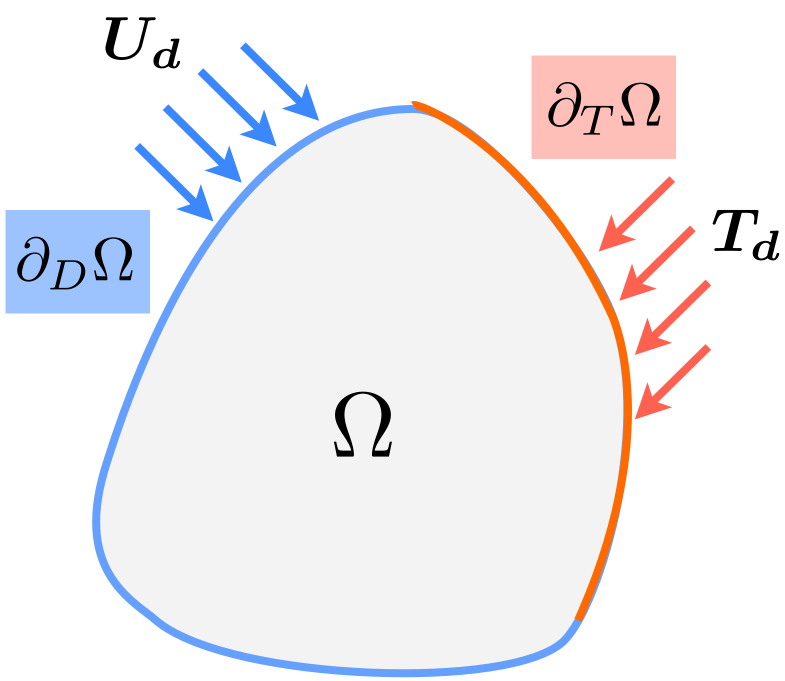
Dissipated energy in a homogeneous process

Regularizing term

- 2 material functions: $\alpha \mapsto \mathsf{A}(\alpha)$ $d \mapsto \mathsf{w}(d)$
- 1 material parameter: ℓ

How to identify the material functions?

Stability criterion



- Total energy for an admissible state (u, α)

$$\mathcal{P}(u,\alpha) = \int_\Omega W(\varepsilon(u)(x), \alpha(x), \nabla\alpha(x))\, dx - \int_{\partial_T\Omega} \boldsymbol{T_d}(x)\cdot u(x)\, dx$$

- Stability criterion

$$\forall \delta u \in \mathcal{C}_0, \forall \delta\alpha \geq 0, \quad \exists r > 0, \quad \forall h \in [0, r),$$
$$\mathcal{P}(u + h\delta u, \alpha + h\delta\alpha) \geq \mathcal{P}(u,\alpha)$$

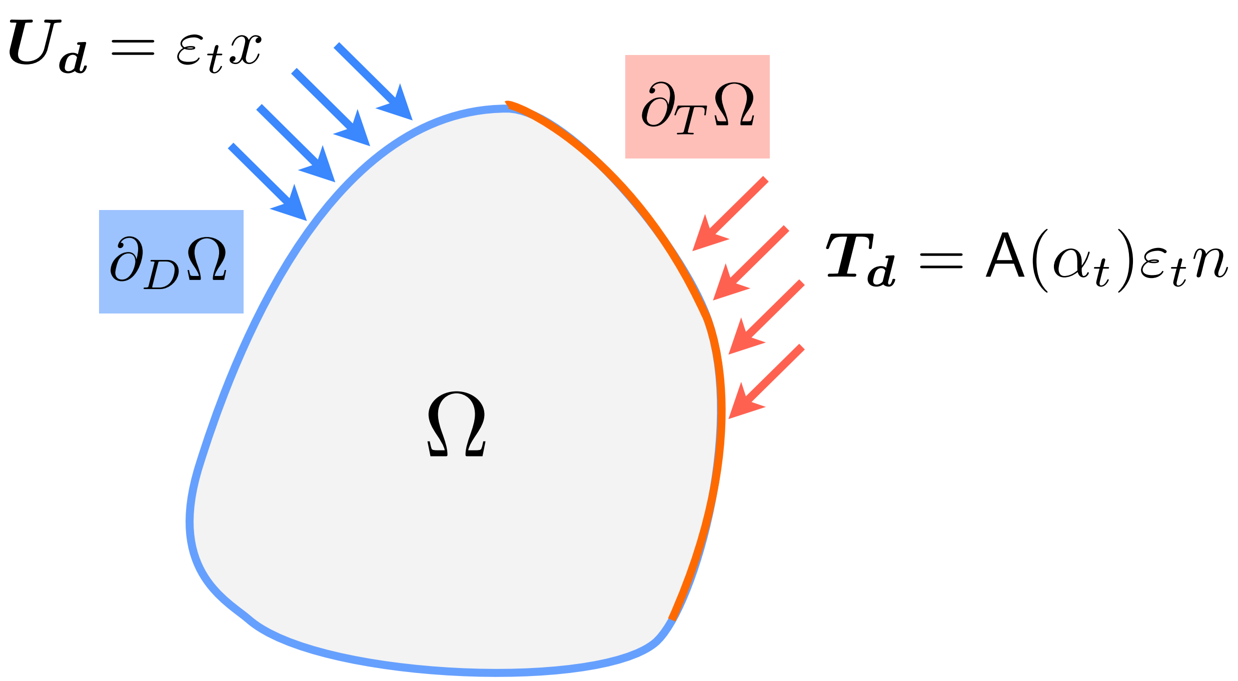
Taylor development up to the second order of the total energy

$$\mathcal{P}(u + h\delta u, \alpha + h\delta\alpha) = \mathcal{P}(u,\alpha) + h\mathcal{P}'(u,\alpha)(\delta u, \delta\alpha) + \frac{h^2}{2}\mathcal{P}''(u,\alpha)(\delta u,\delta\alpha) + o(h^2)$$

(u,α) stable if (resp. only if) for all $(\delta u, \delta\alpha)$ with $\delta\alpha \geq 0$

$$\begin{cases} \mathcal{P}'(u,\alpha)(\delta u,\delta\alpha) > (\text{resp.} \geq) 0 \\ \mathcal{P}''(u,\alpha)(\delta u,\delta\alpha) > (\text{resp.} \geq) 0 \quad \text{if} \quad \mathcal{P}'(u,\alpha)(\delta u,\delta\alpha) = 0 \end{cases}$$

Homogeneous states



$$u(x) = \varepsilon_t x$$

$$\alpha(x) = \alpha_t x \qquad \alpha_t < 1$$

Stability (at fixed loading)?

- Hardening properties
- Size effects
- Boundary conditions

For homogeneous states:

$$\mathcal{P}'(u,\alpha)(\delta u,\delta\alpha) = \left(\frac{1}{2}\mathsf{A}'(\alpha_t)\varepsilon_t\cdot\varepsilon_t + \mathsf{w}'(\alpha_t)\right)\int_\Omega \delta\alpha\,dx$$

$$\mathcal{P}''(u,\alpha)(\delta u,\delta\alpha) = \int_\Omega \Bigg(\mathsf{A}(\alpha_t)\varepsilon(\delta u)\cdot\varepsilon(\delta u) + 2\mathsf{A}'(\alpha_t)\varepsilon_t\cdot\varepsilon(\delta u)\delta\alpha$$
$$+\left(\frac{1}{2}\mathsf{A}''(\alpha_t)\varepsilon_t\cdot\varepsilon_t + \mathsf{w}''(\alpha_t)\right)(\delta\alpha)^2 + \mathsf{w}_1\ell^2\nabla\delta\alpha\cdot\nabla\delta\alpha\Bigg)\,dx$$

Homogeneous states

$$\mathcal{P}'(u,\alpha)(\delta u,\delta\alpha) = \left(\frac{1}{2}\mathsf{A}'(\alpha_t)\varepsilon_t\cdot\varepsilon_t + \mathsf{w}'(\alpha_t)\right)\underbrace{\int_\Omega \delta\alpha\, dx}_{\geq 0}$$

First-order necessary stability condition:

$$\frac{1}{2}\mathsf{A}'(\alpha_t)\varepsilon_t\cdot\varepsilon_t + \mathsf{w}'(\alpha_t) \geq 0$$

damage criterion for homogeneous state

$$\frac{1}{2}\mathsf{A}'(\alpha_t)\varepsilon_t\cdot\varepsilon_t + \mathsf{w}'(\alpha_t) > 0$$

Elastic states

$$\mathcal{P}'(u,\alpha)(\delta u,\delta\alpha) > 0 \quad \text{if} \quad \delta\alpha \neq 0$$

$$\mathcal{P}''(u,\alpha)(\delta u,0) = \int_\Omega \mathsf{A}(\alpha_t)\varepsilon(\delta u)\cdot\varepsilon(\delta u)\,dx > 0$$

➡ **Stable**

$$\frac{1}{2}\mathsf{A}'(\alpha_t)\varepsilon_t\cdot\varepsilon_t + \mathsf{w}'(\alpha_t) = 0$$

Damaging states ➡ Second derivative required

$$\mathcal{P}'(u,\alpha)(\delta u,\delta\alpha) = 0$$

Hardening properties

<u>Elastic domain given by the local damage criterion at a material point</u>

In strain space $\mathcal{R}(\alpha) = \left\{ \varepsilon \in \mathbb{M}_s : \quad \frac{1}{2}\mathsf{A}'(\alpha)\varepsilon \cdot \varepsilon + \mathsf{w}'(\alpha) \geq 0 \right\}$

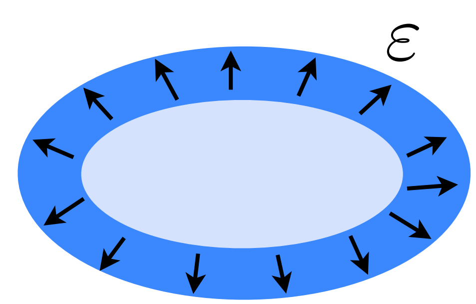
In stress space $\mathcal{R}^*(\alpha) = \left\{ \sigma \in \mathbb{M}_s : \quad -\frac{1}{2}\mathsf{S}'(\alpha)\sigma \cdot \sigma + \mathsf{w}'(\alpha) \geq 0 \right\}$

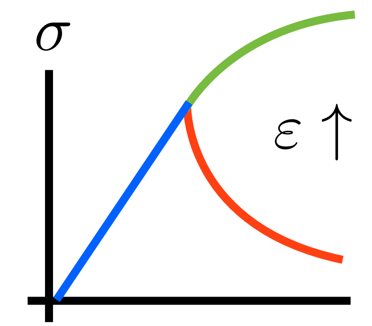
$\mathsf{S} = \mathsf{A}^{-1}$

No snap-back

<u>Strain-hardening</u>: elastic space is *increasing* in strain space

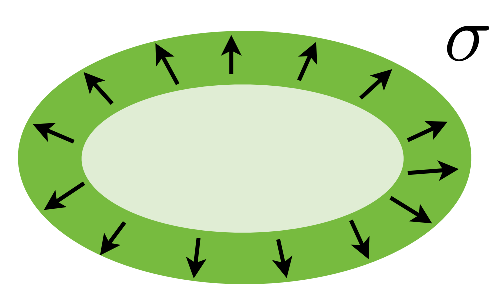
$$\mathsf{A}''(\alpha)\mathsf{w}'(\alpha) - \mathsf{A}'(\alpha)\mathsf{w}''(\alpha) > 0$$

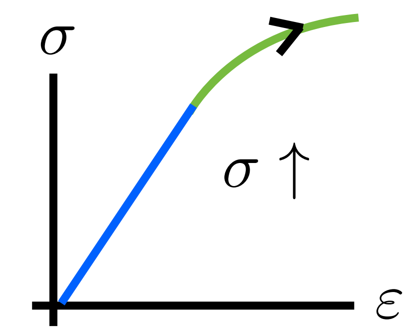




<u>Stress-hardening</u>: elastic space is *increasing* in stress space

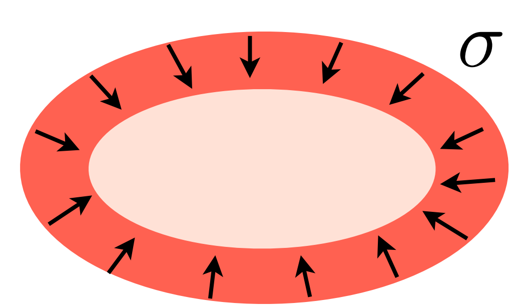
$$\mathsf{S}''(\alpha)\mathsf{w}'(\alpha) - \mathsf{S}'(\alpha)\mathsf{w}''(\alpha) < 0$$

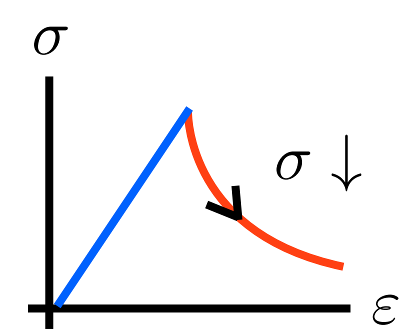




<u>Stress-softening</u>: elastic space is *decreasing* in stress space

$$\mathsf{S}''(\alpha)\mathsf{w}'(\alpha) - \mathsf{S}'(\alpha)\mathsf{w}''(\alpha) > 0$$





Role of hardening properties in stability

$$\mathcal{P}''(u,\alpha)(\delta u,\delta\alpha) = \int_\Omega \Big(\mathsf{A}(\alpha_t)\varepsilon(\delta u)\cdot\varepsilon(\delta u) + 2\mathsf{A}'(\alpha_t)\varepsilon_t\cdot\varepsilon(\delta u)\delta\alpha + \Big(\frac{1}{2}\mathsf{A}''(\alpha_t)\varepsilon_t\cdot\varepsilon_t + \mathsf{w}''(\alpha_t)\Big)(\delta\alpha)^2 + \mathsf{w}_1\ell^2\nabla\delta\alpha\cdot\nabla\delta\alpha\Big)\,dx$$

⬇

$$\mathcal{P}''(u,\alpha)(\delta u,\delta\alpha) = \int_\Omega \mathsf{A}(\alpha_t)\big(\varepsilon(\delta u) - \boldsymbol{e}_t\delta\alpha\big)\cdot\big(\varepsilon(\delta u) - \boldsymbol{e}_t\delta\alpha\big)\,dx + \mathsf{w}_1\ell^2\int_\Omega \nabla\delta\alpha\cdot\nabla\delta\alpha\,dx - \underline{\Big(\frac{1}{2}\mathsf{S}''(\alpha_t)\sigma_t\cdot\sigma_t - \mathsf{w}''(\alpha_t)\Big)}\int_\Omega(\delta\alpha)^2\,dx$$

with $\boldsymbol{e}_t = \mathsf{S}'(\alpha_t)\sigma_t$

Stress-hardening

↳ $\mathsf{S}''(\alpha)\mathsf{w}'(\alpha) - \mathsf{S}'(\alpha)\mathsf{w}''(\alpha) < 0$ ➡ $\frac{1}{2}\mathsf{S}''(\alpha_t)\sigma_t\cdot\sigma_t - \mathsf{w}''(\alpha_t) < 0$ ➡ **Stable**

Stress-softening

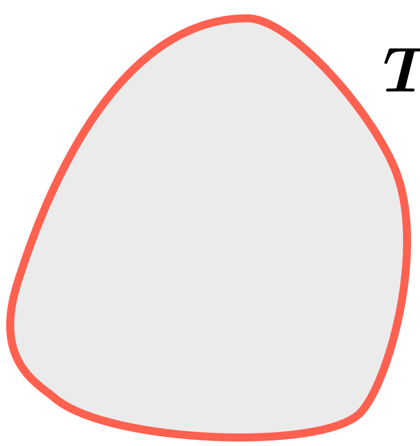
↳ $\mathsf{S}''(\alpha)\mathsf{w}'(\alpha) - \mathsf{S}'(\alpha)\mathsf{w}''(\alpha) > 0$ ➡ $\frac{1}{2}\mathsf{S}''(\alpha_t)\sigma_t\cdot\sigma_t - \mathsf{w}''(\alpha_t) > 0$ ➡ Requires the study of a Rayleigh ratio

Rayleigh ratio

$$R(\delta u, \delta\alpha) = \frac{\int_\Omega \mathsf{A}(\alpha_t)\big(\varepsilon(\delta u) - \boldsymbol{e}_t\delta\alpha\big)\cdot\big(\varepsilon(\delta u) - \boldsymbol{e}_t\delta\alpha\big)\,dx + \mathsf{w}_1\ell^2\int_\Omega \nabla\delta\alpha\cdot\nabla\delta\alpha\,dx}{\left(\frac{1}{2}\mathsf{S}''(\alpha_t)\sigma_t\cdot\sigma_t - \mathsf{w}''(\alpha_t)\right)\int_\Omega(\delta\alpha)^2\,dx}$$

Stable if (only if) $\inf_{\delta u\in\mathcal{C}_0,\delta\alpha\in\mathcal{D}} R(\delta u, \delta\alpha) \quad > (resp. \geq) \quad 1$

- Under fully prescribed forces



$\boldsymbol{T_d} = \mathsf{A}(\alpha_t)\varepsilon_t n$

Particular choice: $\delta u = \boldsymbol{e}_t x, \quad \delta\alpha = 1$

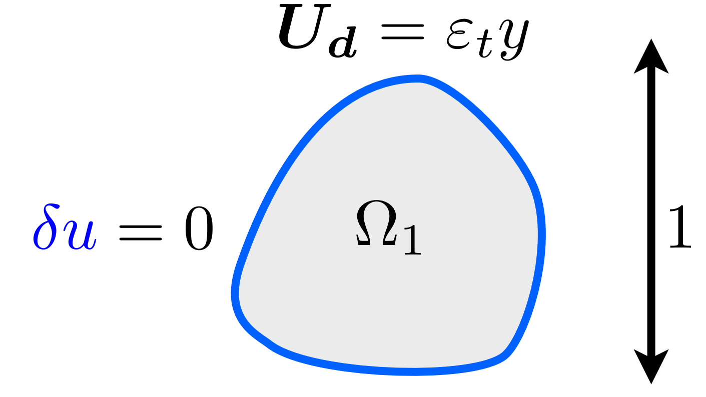
$R(\boldsymbol{e}_t x, 1) = 0$ ➡ $\inf_{\delta u\in\mathcal{C}_0,\delta\alpha\in\mathcal{D}} R(\delta u, \delta\alpha) = 0$

Unstable

Case of fully prescribed displacement

<u>Change of variable</u> $y = \dfrac{x}{L}$

L characteristic size of the body



$$R_L(\delta u, \delta\alpha) = \frac{\int_{\Omega_1} \mathsf{A}(\alpha_t)\big(\varepsilon(\delta u) - \boldsymbol{e}_t\delta\alpha\big)\cdot\big(\varepsilon(\delta u) - \boldsymbol{e}_t\delta\alpha\big)\,dy + \mathsf{w}_1\frac{\ell^2}{L^2}\int_{\Omega_1}\nabla\delta\alpha\cdot\nabla\delta\alpha\,dy}{\Big(\frac{1}{2}\mathsf{S}''(\alpha_t)\sigma_t\cdot\sigma_t - \mathsf{w}''(\alpha_t)\Big)\int_{\Omega_1}(\delta\alpha)^2\,dy}$$

<u>Case of small domains under prescribed displacement:</u> $L \to 0$

$$\rho_L \to \rho_0 = \frac{\mathsf{A}(\alpha_t)\boldsymbol{e}_t\cdot\boldsymbol{e}_t}{\frac{1}{2}\mathsf{S}''(\alpha_t)\sigma_t\cdot\sigma_t - \mathsf{w}''(\alpha_t)} > 1$$

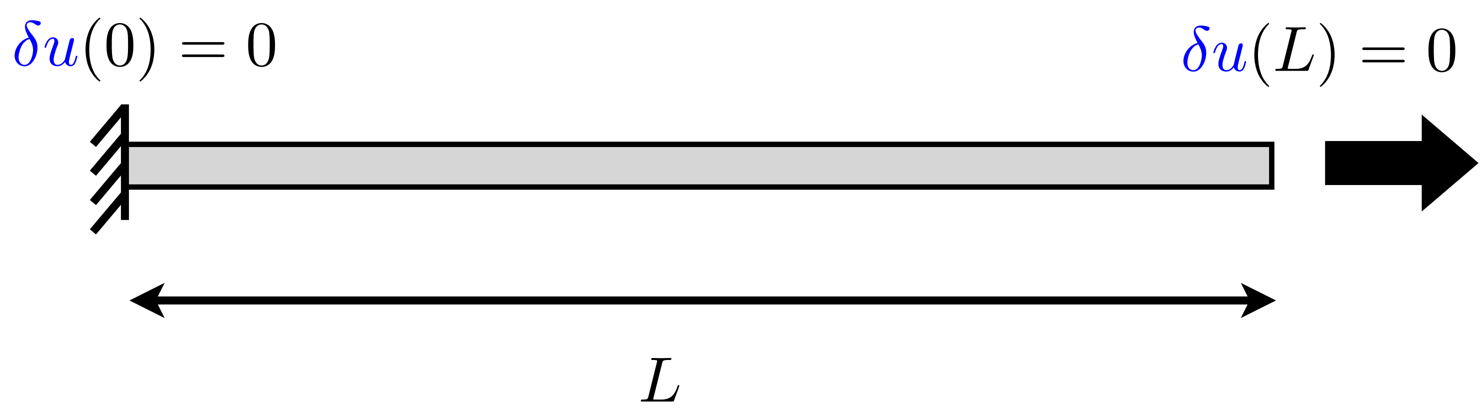
<u>Stable</u> (provided strain-hardening)

<u>Case of large domains under prescribed displacement:</u> $L \to +\infty$

$$\rho_L \to \rho_\infty = \inf_{\delta u\in\mathcal{C}_0,\delta\alpha\in\mathcal{D}} \frac{\int_{\Omega_1} \mathsf{A}(\alpha_t)\big(\varepsilon(\delta u) - \boldsymbol{e}_t\delta\alpha\big)\cdot\big(\varepsilon(\delta u) - \boldsymbol{e}_t\delta\alpha\big)\,dy}{\Big(\frac{1}{2}\mathsf{S}''(\alpha_t)\sigma_t\cdot\sigma_t - \mathsf{w}''(\alpha_t)\Big)\int_{\Omega_1}(\delta\alpha)^2\,dy}$$

<u>?</u>

Size effects: the 1D case



$$R_L(\delta u, \delta\alpha) = \frac{\int_0^1 E(\alpha_t)\big(\delta u' - e_0\delta\alpha\big)\cdot\big(\delta u' - e_0\delta\alpha\big)\,dx + \mathsf{w}_1\frac{l^2}{L^2}\int_0^1(\delta\alpha')^2\,dx}{\left(\frac{1}{2}\mathsf{S}''(\alpha_t)\sigma_t\cdot\sigma_t - \mathsf{w}''(\alpha_t)\right)\int_0^1(\delta\alpha)^2\,dx}$$

Calculation of is $\rho_L = \inf_{\delta u\in\mathcal{C}_0,\delta\alpha\in\mathcal{D}} R_L(\delta u, \delta\alpha)$ explicit in 1D.

Homogeneous damaging state is stable if (resp. only if)

$$L < (\text{resp. } \leq)\sqrt{\frac{\pi^2\mathsf{w}_1 E(\alpha_t)S'(\alpha_t)^4\sigma_t^4}{(\frac{1}{2}S''(\alpha_t)\sigma_t^2 - \mathsf{w}''(\alpha_t))^3}}\,\ell$$

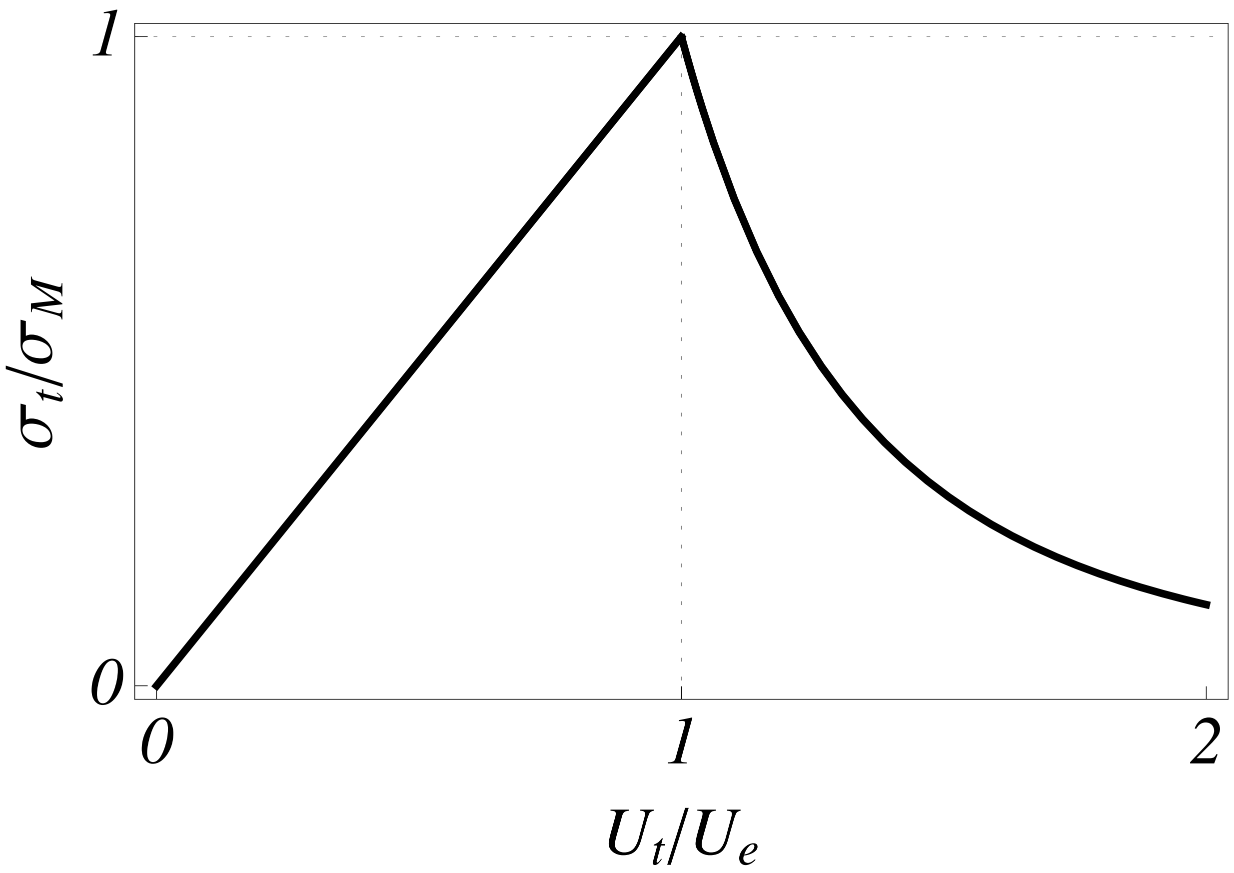
KP, Marigo, Maurini, JMPS 2011

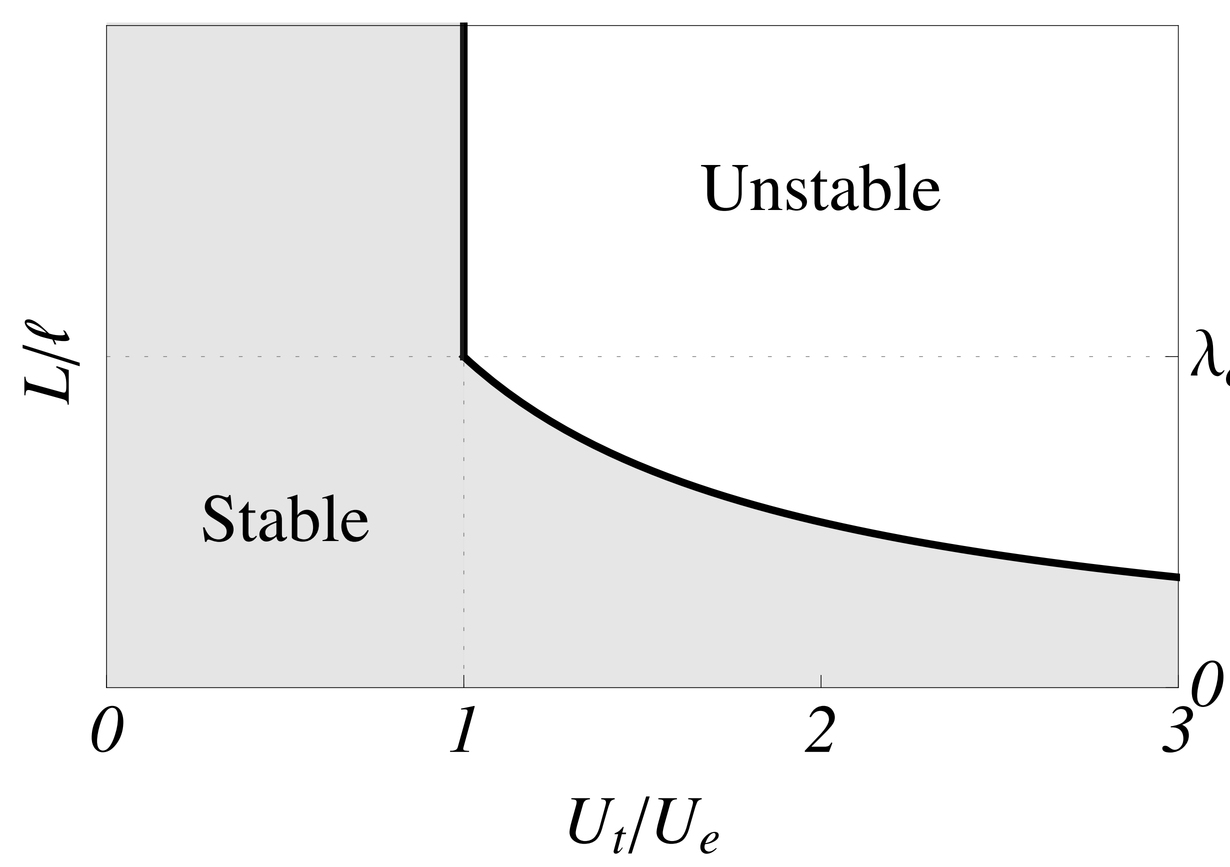
Example in 1D case

Classical damage law with elastic phase

$$E(\alpha) = E_0(1-\alpha)^2, \qquad \mathrm{w}(\alpha) = \frac{\sigma_e^2}{E_0}\alpha$$

Stability analysis: $\displaystyle \frac{L}{\ell} \leq \lambda_c \frac{U_e}{U_t} \quad \text{for} \quad U_t \geq U_e \quad \text{with} \quad \lambda_c = \frac{4\pi}{3\sqrt{3}}$





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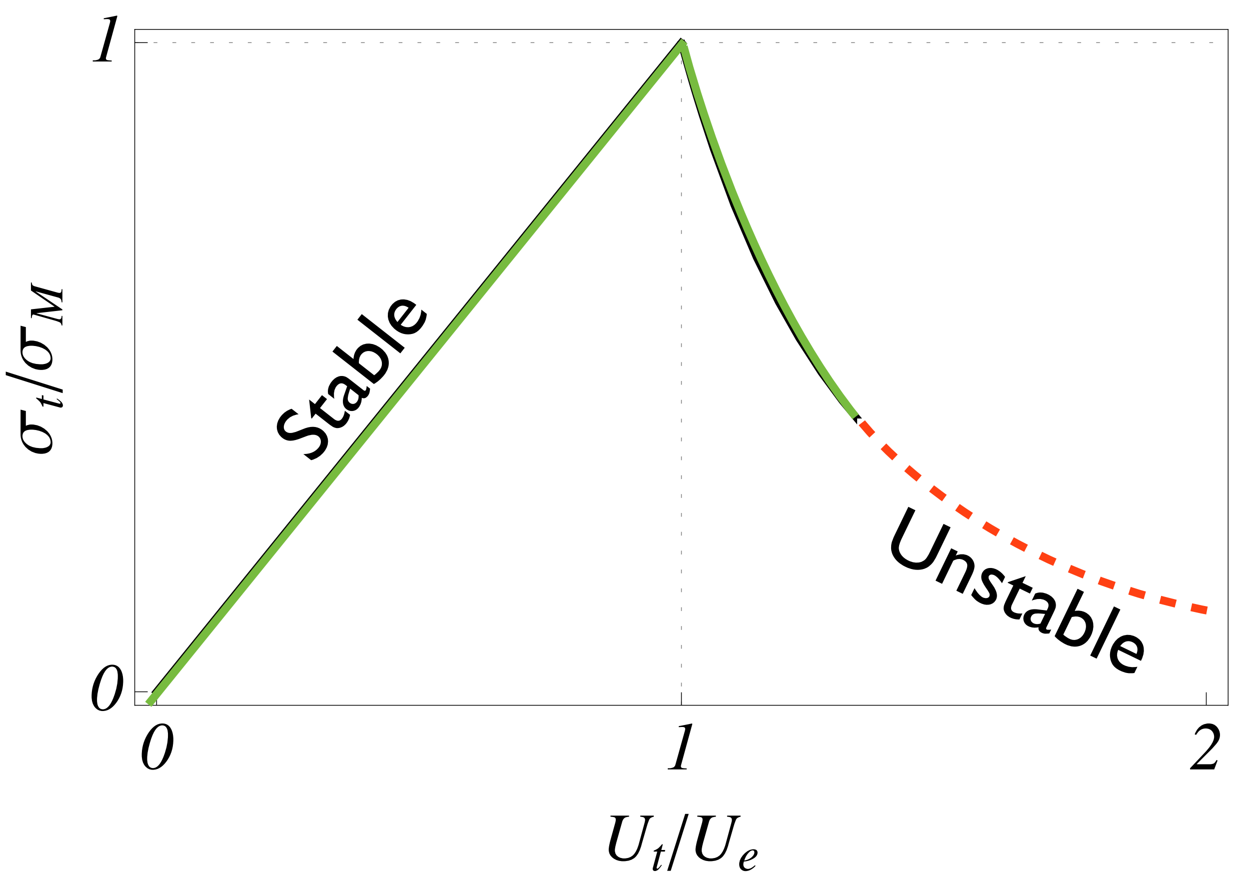
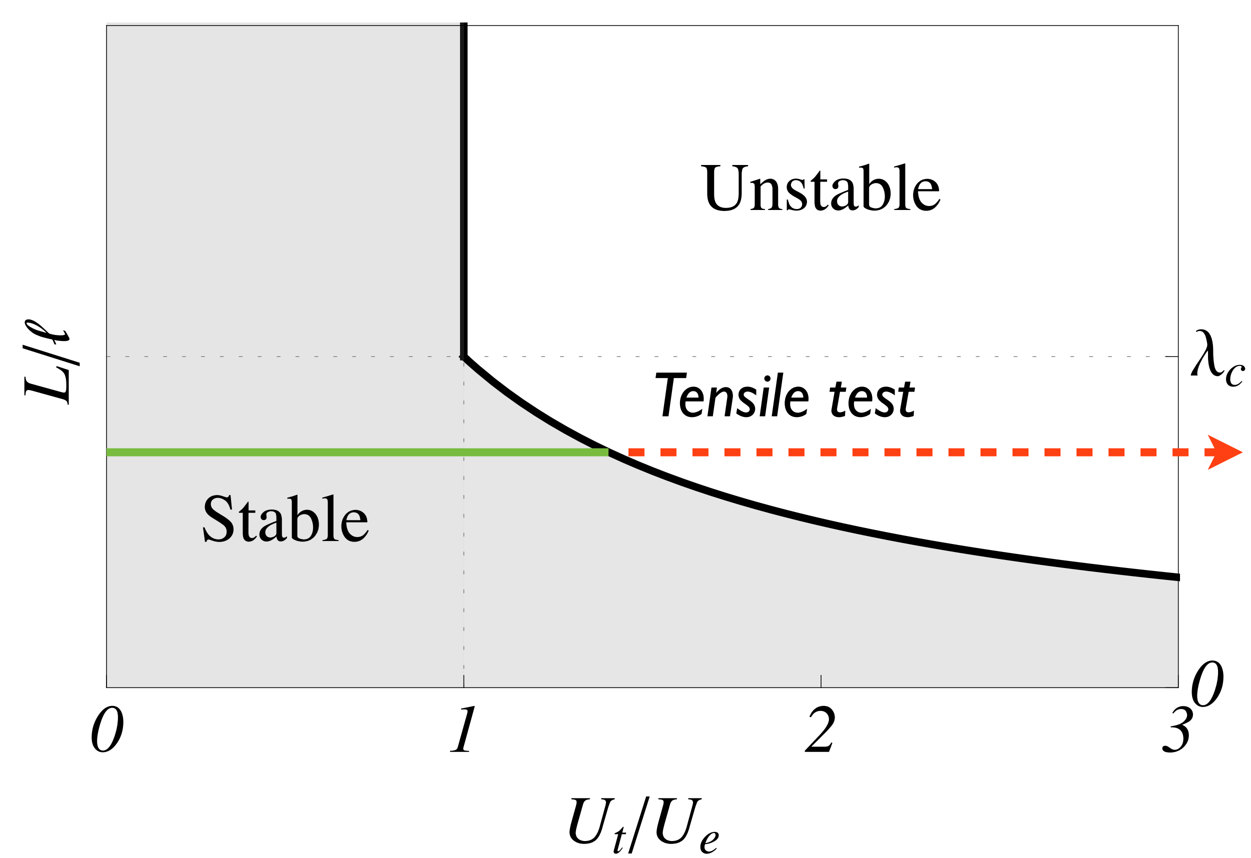




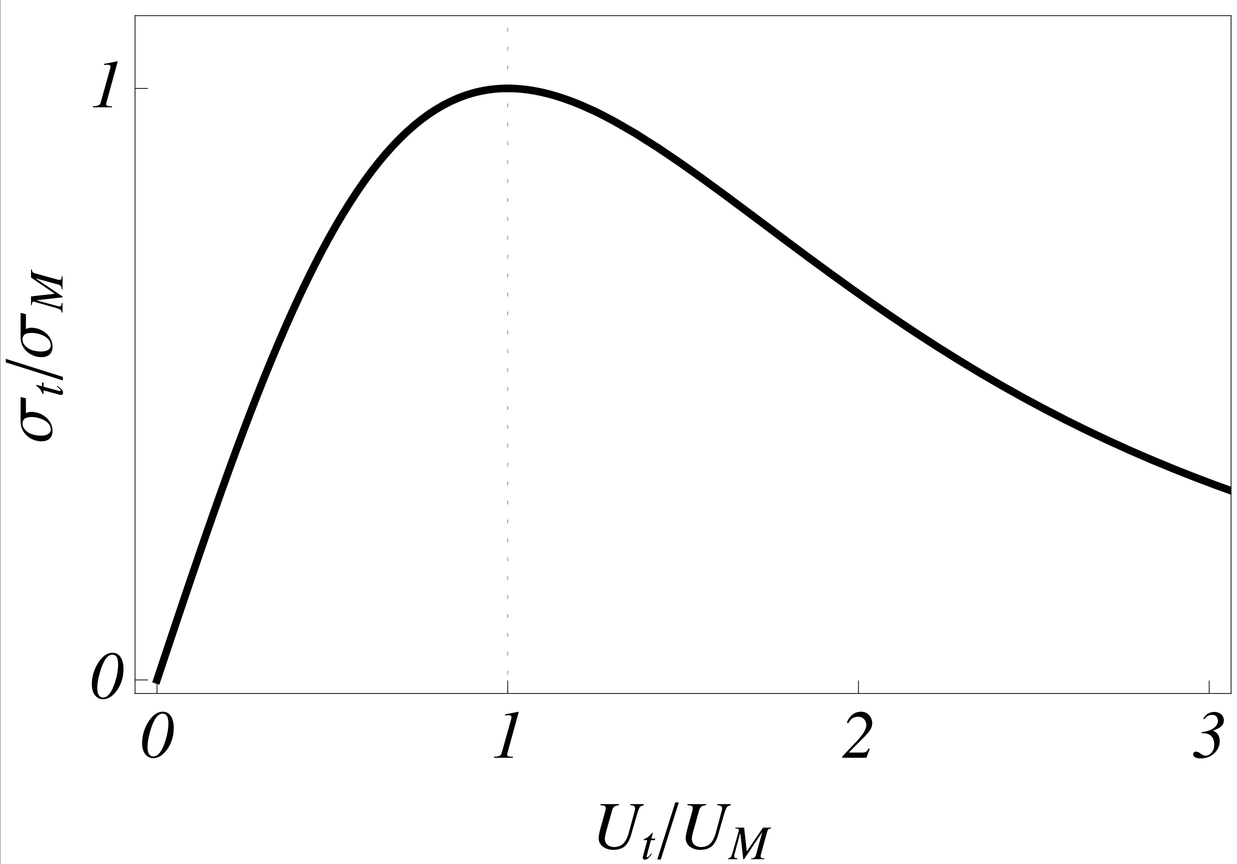
Size effects: illustration in the 1D case

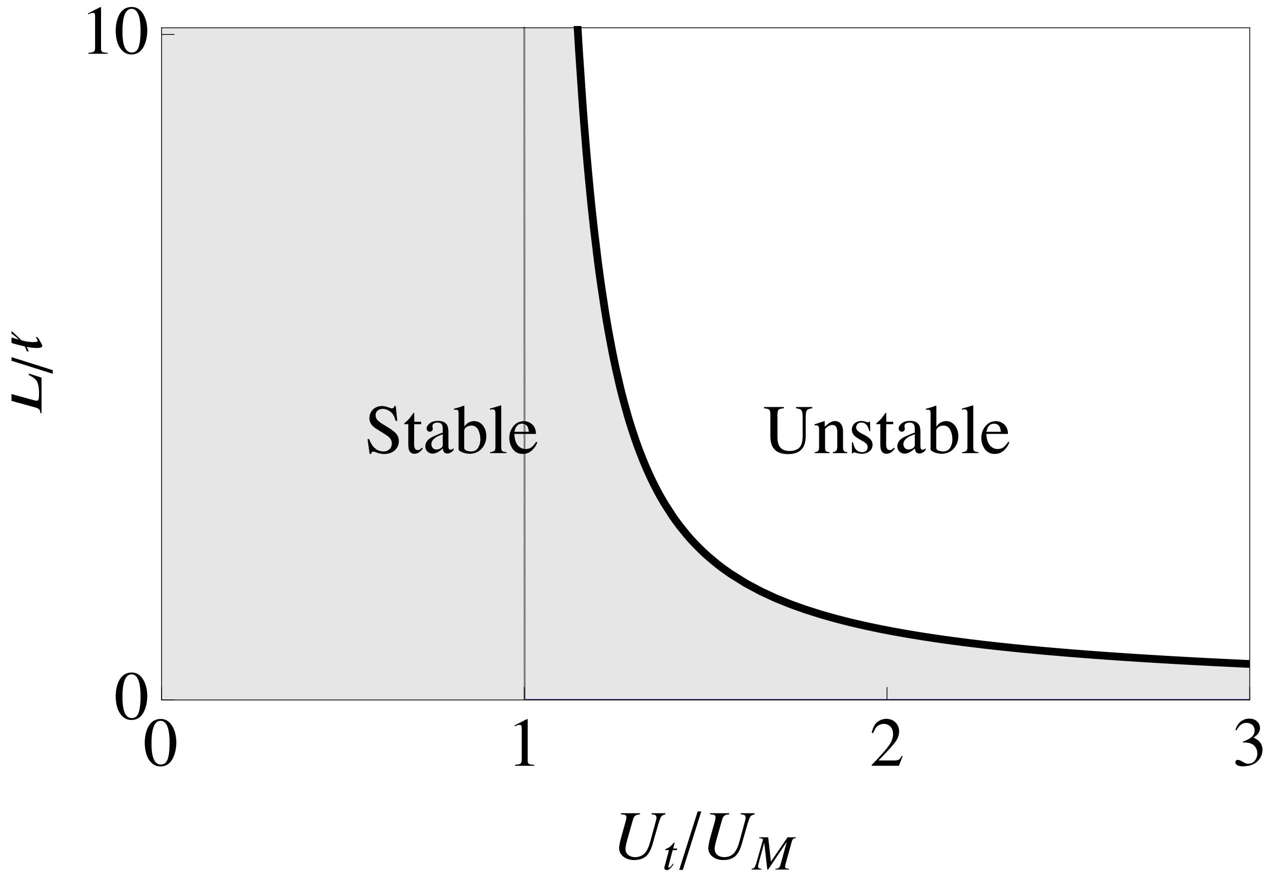
Ambrosio-Tortorelli law with no elastic range

$$E(\alpha) = E_0(1-\alpha)^2, \qquad \mathsf{w}(\alpha) = \frac{128}{27}\frac{\sigma_M^2}{E_0}\alpha^2$$

Stability analysis:

$$\frac{L}{\ell} \leq \frac{\pi\sqrt{3}}{4}\frac{U_t^2/U_M^2}{(U_t^2/U_M^2-1)^{3/2}}$$



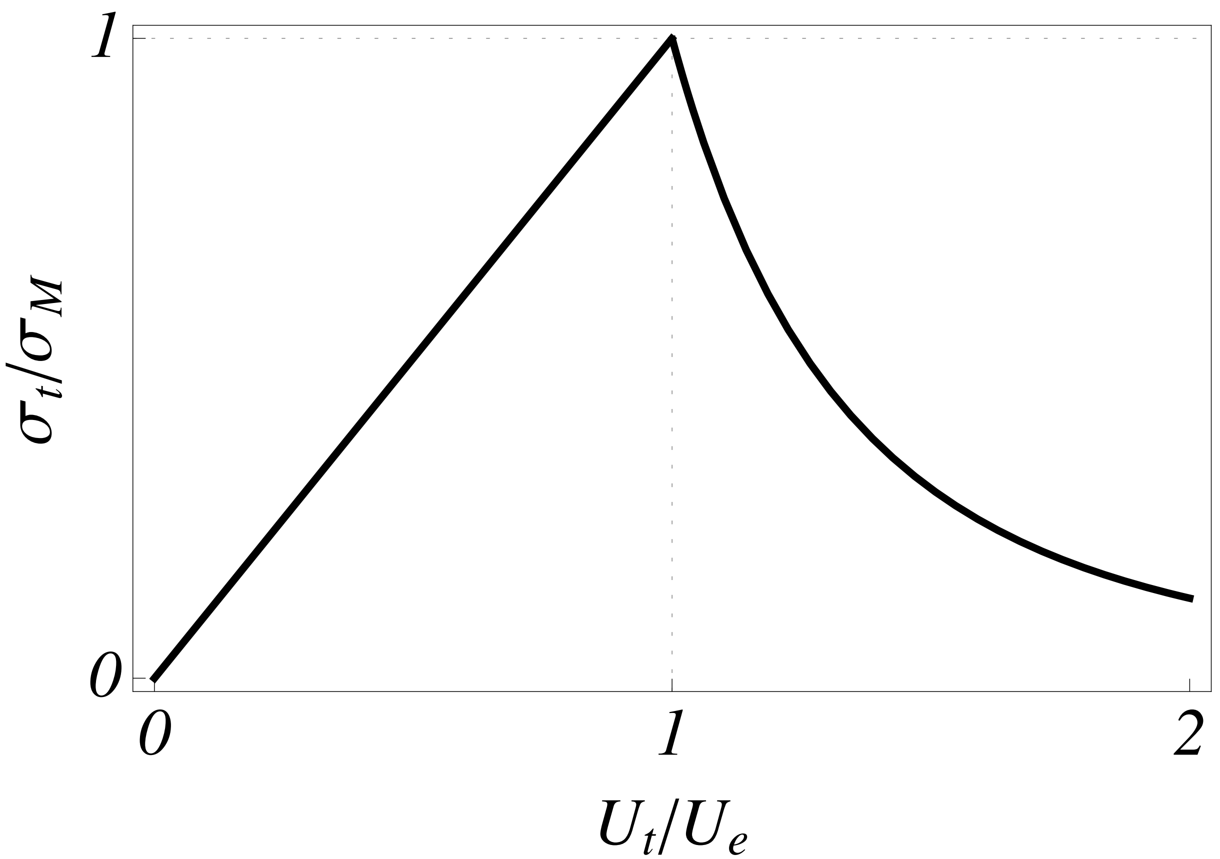


Size effects: illustration in the 1D case

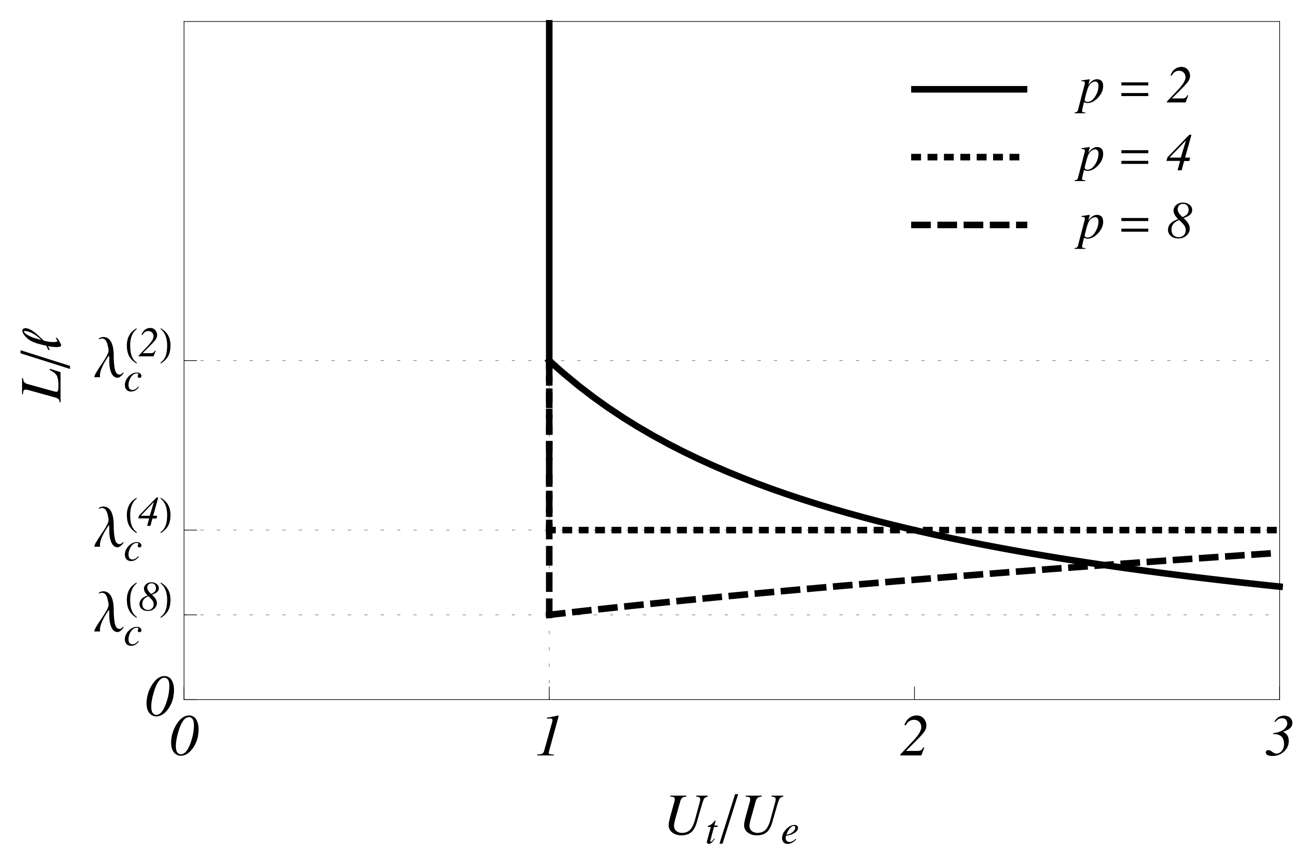
Damage laws with same uniaxial stress-strain response

$$E(\alpha) = E_0(1-\alpha)^p, \qquad \mathrm{w}(\alpha) = \frac{\sigma_M^2}{E_0}(1-(1-\alpha)^{p/2})$$

Same 1D homogeneous response for any p



but different stability diagrams!



Effects of the boundary conditions

small domains $\rho_0 > 1$

Stable

ρ_L ↘

?

ρ_∞ large domains

$\rho_\infty < 1$: Stability depends on size effect (ℓ / L)

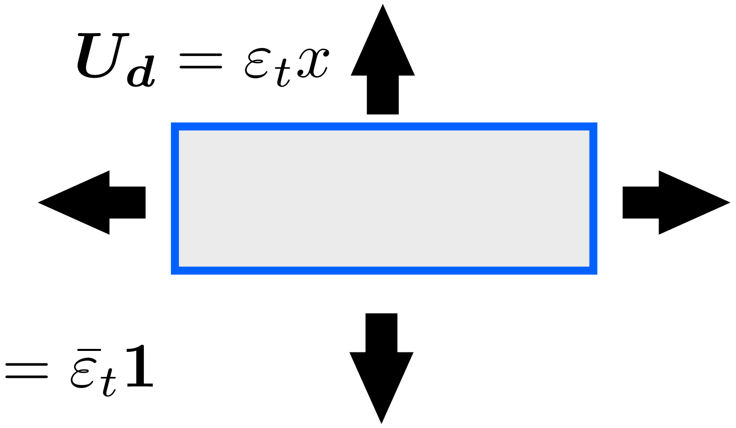
$\rho_\infty > 1$: Unconditionally stable

Uniaxial states:

$\rho_\infty = 0$ Always unstable for large domains

Spherical states: $\alpha \mapsto \mathsf{A}(\alpha) = (E(\alpha), \nu)$ KP, Marigo, JE 2013

$\boldsymbol{U_d} = \varepsilon_t x$



$\varepsilon_t = \bar{\varepsilon}_t \mathbf{1}$

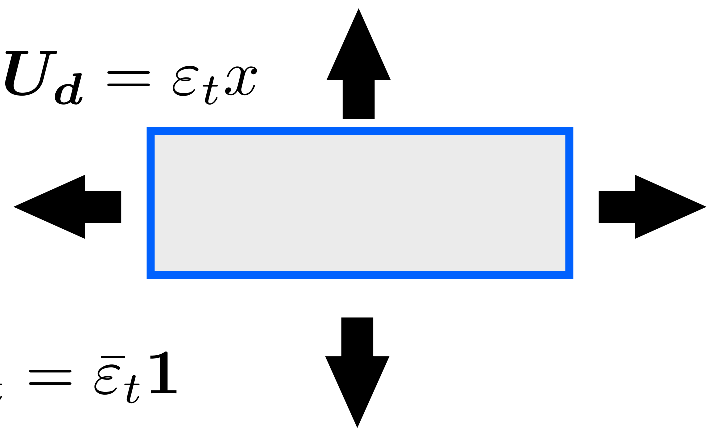
$$\rho_\infty = \frac{2(1-2\nu)}{3(1-\nu)} \frac{2(s'(\alpha_t))^2}{s(\alpha_t)s''(\alpha_t)} \qquad s = E^{-1}$$

Unconditional stability depends on the damage laws

Effects of the boundary conditions

Spherical states $\quad \alpha \mapsto \mathsf{A}(\alpha) = (E(\alpha), \nu)$

$\boldsymbol{U_d} = \varepsilon_t x$



$\varepsilon_t = \bar{\varepsilon}_t \mathbf{1}$

$$\rho_\infty = \frac{2(1-2\nu)}{3(1-\nu)} \frac{2(s'(\alpha_t))^2}{s(\alpha_t)s''(\alpha_t)} \qquad s = E^{-1}$$

$$E(\alpha) = E_0(1-\alpha)^p, \qquad \mathsf{w}(\alpha) = \mathsf{w}_1 \alpha$$

$$\rho_\infty = \frac{4(1-2\nu)q}{3(1-\nu)(p+1)}$$