Density of states and level statistics for 1-d Schrödinger operators

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Known Facts on RSO

$(\Omega, \mathcal{F}, \mathbf{P})$: probability space

$$(H_\omega \phi)(x) := \sum_{|y-x|=1} \phi(y) + V_\omega(x)\phi(x), \quad \omega \in \Omega, \quad \phi \in \ell^2(\mathbf{Z}^d)$$

$\{V_\omega(x)\}_{x \in \mathbf{Z}^d}$: i.i.d. with "good" distribution μ.

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(2) Anderson localization : $\exists I (\subset \Sigma)$ s.t. $\sigma(H_\omega) \cap I$ is a.s. pp with exponentially decaying e.f.'s.

IDS and Level Statistics

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$$\mathbf{P}\left(\sharp\left\{\text{ e.v.'s of } H_L \text{ in } E_0 + \frac{1}{L^d}[a_k, b_k]\right\} = n_k,\ k = 1, 2, \cdots, K\right)$$

$$\overset{L\to\infty}{\to} \prod_{k=1}^{K} \frac{(DS(E_0)(b_k - a_k))^{n_k}}{(n_k!)} e^{-DS(E_0)(b_k - a_k)}$$

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where $DS(E_0) := \frac{d}{dE} IDS(E_0)$. In other words,

$$\xi_n := \sum_k \delta_{L^d(E_k(L)-E_0)}(dE) \overset{d}{\to} Poisson(DS(E_0)dE)$$

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Decaying Potential Model

We consider

$$H := -\frac{d^2}{dt^2} + a(t)F(X_t) \quad \text{on} \quad L^2(\mathbf{R})$$

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$$a(t) = t^{-\alpha}(1 + o(1)), \quad t \to \infty, \quad \alpha > 0$$
$$F \in C^\infty(M), \quad M : torus, \quad \langle F \rangle := \int_M F(x)dx = 0,$$
$$\{X_t\}_{t\in\mathbf{R}} : \text{ BM. on } M.$$

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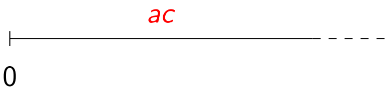

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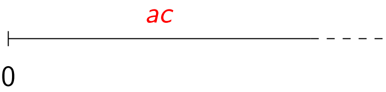

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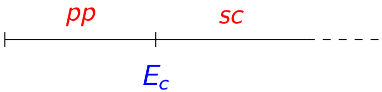

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$$\text{``}\kappa := \sqrt{E}\text{''}$$

$$N_L(\kappa_1, \kappa_2) := \sharp \left\{ \text{e.v.'s of } H_L \text{ in } (\kappa_1^2, \kappa_2^2) \right\}, \quad 0 < \kappa_1 < \kappa_2.$$

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Q : 2nd order (“CLT”) ?

Fluctuation of IDS (Notation)

Let

(1) $\{G(x)\}_{x>0}$: the Gaussian field with

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Further, let

$$G(\kappa_1, \kappa_2) := G(\kappa_2) - G(\kappa_1) + \left(\frac{1}{\kappa_1} - \frac{1}{\kappa_2} \right) G_0$$

Fluctuation of IDS (Results 1)

Theorem 0

(1) (AC case) $\alpha > \frac{1}{2}$:

$$N_L(\kappa_1, \kappa_2) = \frac{L}{\pi}(\kappa_2 - \kappa_1) + C_{random}(\kappa_1, \kappa_2) + M_\infty(\kappa_1, \kappa_2) + o(1)$$

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(3) (PP Case) $\frac{1}{2m} \le \alpha < \frac{1}{2(m-1)}$, $m = 2, 3, \cdots,$:

$$N_L(\kappa_1, \kappa_2) = \frac{L}{\pi}(\kappa_2 - \kappa_1) + C_2(\kappa_1, \kappa_2) L^{1-2\alpha} + C_3(\kappa_1, \kappa_2) L^{1-3\alpha} + \cdots + C_m(\kappa_1, \kappa_2) L^{1-m\alpha} + L^{\frac{1}{2}-\alpha} G(\kappa_1, \kappa_2) + \cdots$$

$C_j(\kappa_1, \kappa_2)$: non-random const.'s

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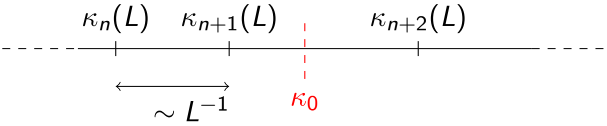

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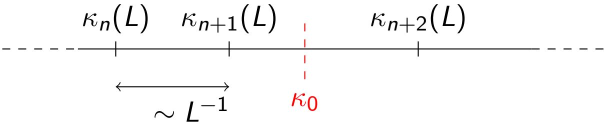

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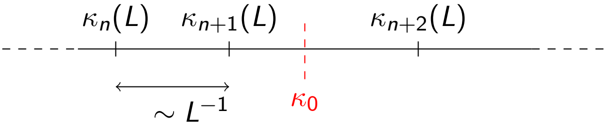

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Problem : $\xi_L \to ?$ as $L \to \infty$.

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Assumption (A)

Subsequence $\{L_j\}$ satisfies $L_j \stackrel{j \to \infty}{\to} \infty$ and

$$\kappa_0 L_j = m_j \pi + \beta + o(1), \quad j \to \infty, \quad m_j \in \mathbf{N}, \quad \beta \in [0, \pi).$$

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Theorem 1 (AC-case $\Longrightarrow$ clock process)

Assume (A). Then we have

$$\lim_{j \to \infty} \mathbf{E}\left[e^{-\xi_{L_j}(f)} \right] = \int_0^{\pi} d\mu_\beta(\phi) \exp\left(- \sum_{n \in \mathbf{Z}} f(n\pi - \phi) \right)$$

for some probability measure μ_β on $[0, \pi]$.

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Remark.

(1) $\mu_\beta =$ distribution of $\{\beta + \lim_{t \to \infty}(\theta_t(\kappa_0) - \kappa_0 t)\}_{\pi \mathbf{Z}}$

$$\mu_\beta = \begin{cases} \text{not uniform on } [0, \pi] & \alpha > \frac{1}{2} \\ \text{uniform on } [0, \pi] & \alpha \le \frac{1}{2} \end{cases}$$

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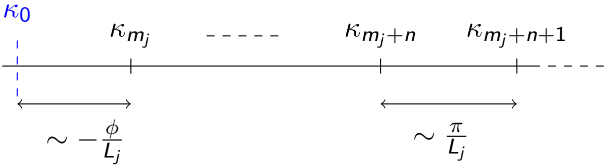
(2) μ_β : uniform on $[0, 2\pi]$, for CMV matrices (Killip-Stoiciu).

Eigenvalue spacing : 2nd order



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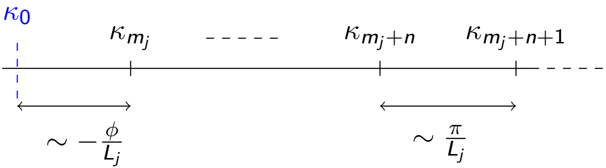

To study the eigenvalue spacing in the second order, we set

$$X_j(n) := \left\{ \left(\kappa_{m_j+n+1}(L_j) - \kappa_{m_j+n}(L_j) \right) L_j - \pi \right\} L_j^{\alpha - \frac{1}{2}},$$

where $n \in \mathbf{Z}$.

2nd Limit Theorem

Theorem 2 (2nd Limit Theorem)

$\{X_j(n)\}_n \overset{j\to\infty}{\to}$ Gaussian system with covariance

$$C(n,n') := \frac{C(\kappa_0)}{8\kappa_0^2} Re \int_0^1 s^{-2\alpha} e^{2i(n-n')\pi s} 2(1-\cos 2\pi s) ds$$

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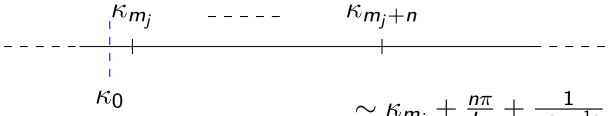
So, roughly speaking,



$$\sim \kappa_{m_j} + \frac{n\pi}{L_j} + \frac{1}{L_j^{(\alpha+\frac{1}{2})}} \quad (\text{Gaussian})$$

Two reference energies

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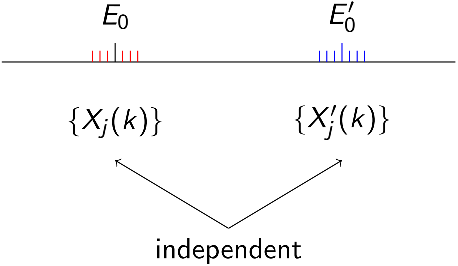
If we consider two reference energies $E_0 \neq E_0'$, then the corresponding $\{X_j(k)\}_j$, $\{X_j'(k)\}_{j'}$ jointly converge to two independent Gaussian systems.



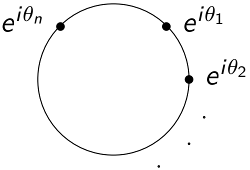
Same for the critical case $(\alpha = \frac{1}{2})$.

Circular β-ensemble

Definition

(1) The circular β-ensemble with n-points is given by

$$\mathbf{P}\left(\quad \right) \propto \quad |\triangle(e^{i\theta_1}, \cdots, e^{i\theta_n})|^{\beta}$$



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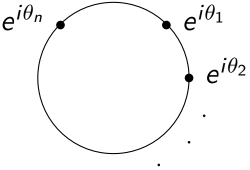

$\triangle$: Vandermonde determinant, $\beta > 0$.

(2) The scaling limit ζ_β^C of the circular β-ensemble is defined by

$$\zeta_\beta^C := \lim_{n \to \infty} \sum_{j=1}^{n} \delta_{n\theta_j}.$$

Characterization of ζ_β^C

Theorem (Killip-Stoiciu (2009))

$$\mathbf{E}[e^{-\zeta_\beta^C(f)}] = \mathbf{E}\left[\int_0^{2\pi} \frac{d\theta}{2\pi} \exp\left(-\sum_{n\in\mathbf{Z}} f\left(\Psi_1^{-1}(2n\pi + \theta)\right)\right)\right]$$

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$$d\Psi_t(\lambda) = \lambda\, dt + \frac{2}{\sqrt{\beta t}} Re\left\{(e^{i\Psi_t(\lambda)} - 1)dZ_t\right\},$$

$$\Psi_0(\lambda) = 0$$

Z_t : complex B.M.

Gaussian β-ensemble

Definition

(1) The Gaussian β-ensemble with n-points is given by

$$\mathbf{P}\left(\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n\right) \propto \exp\left(-\frac{\beta}{4}\sum_{k=1}^{n}\lambda_k^2\right)|\triangle(\{\lambda_j\})|^{\beta}$$

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(2) The scaling limit ζ_β^G of the Gaussian β-ensemble is defined by

$$\zeta_\beta^G := \lim_{n\to\infty}\sum_{j=1}^{n}\delta_{\sqrt{4n}\lambda_j}$$

which is called the Sine$_\beta$-process.

Characterization of ζ_β^G

Theorem (Valko-Virag 2009)

Let $\alpha_t(\lambda)$ be the solution to the following SDE.

$$d\alpha_t(\lambda) = \lambda \cdot \frac{\beta}{4} e^{-\frac{\beta}{4}t} dt + Re\left[\left(e^{i\alpha_t} - 1\right) dZ_t\right],$$
$$\alpha_0(\lambda) = 0.$$

Then for $\lambda > 0$, $t \mapsto \lfloor \alpha_t(\lambda)/2\pi \rfloor$ is non-decreasing and $\alpha_\infty(\lambda) := \exists \lim_{t\to\infty} \alpha_t(\lambda) \in 2\pi \mathbf{Z}$, a.s. Then Sine$_\beta$-process on each interval is given by

$$\zeta_\beta^G[\lambda_1, \lambda_2] \stackrel{d}{=} \frac{\alpha_\infty(\lambda_2) - \alpha_\infty(\lambda_1)}{2\pi}.$$

Remark

(1)(Valko-Virag (2009) Universality in the bulk)
Let μ_n (reference energy) is away from the Tracy-Widom region : $n^{\frac{1}{6}}(2\sqrt{n} - |\mu_n|) \to \infty$.

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(ii) Valko-Virag : SDE has no singularity, but $\Psi_{t-}^{VV} \in 2\pi \mathbf{Z}$

Critical Case

Go back to our model and let $\alpha = \frac{1}{2}$: $a(t) = t^{-\frac{1}{2}}(1 + o(1))$.

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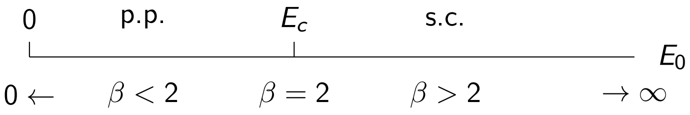
"Non-Universality"



cf. Breuer, Forrester, Smilansky (2006)

Coincidence of two β-ensembles

Corollary 4

The limits of C_β-ensemble and G_β-ensemble are equal :

$$\zeta_\beta^C \stackrel{d}{=} \zeta_\beta^G .$$

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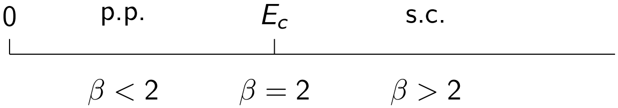
(2) Valko-Virag have "direct" proof of this fact (Valko, private communication)

Remarks



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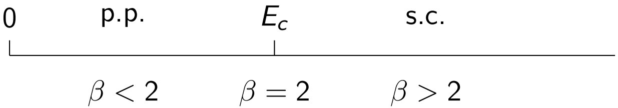

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Sine$_\beta$-process has a "phase transition" between at $\beta = 2$.

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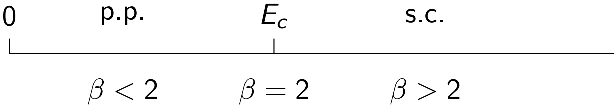

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(1)(Valko-Virag (2009))

(i) $\beta < 2$: $\Psi_t(\lambda)$ approaches to $2\pi\mathbf{Z}$ from below a.s.

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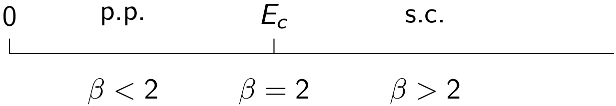

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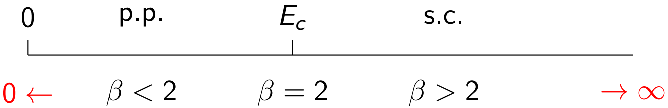
(2)(Valko, private communication)

$\exists H_{Dirac}$ on s.t. $\sigma(H_{Dirac}) \stackrel{d}{=} Sine_\beta$.

$\beta \le 2 \Longrightarrow H_{Dirac}$: limit point

$\beta > 2 \Longrightarrow H_{Dirac}$: limit circle

Remarks(Continued)



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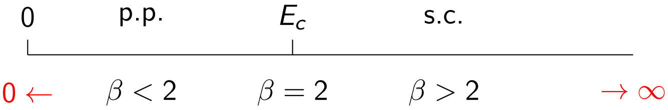

(1) As $\beta \uparrow \infty$, $\mathrm{Sine}_\beta \xrightarrow{d}$ Clock process (μ uniform on $[0, 2\pi]$)

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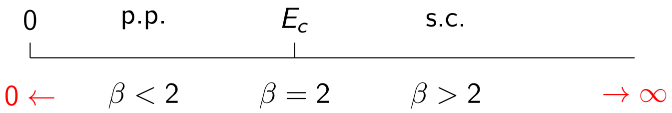

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(2) (Allez - Dumaz (2014))
As $\beta \downarrow 0$, $\mathrm{Sine}_\beta \xrightarrow{d}$ Poisson process with intensity $(2\pi)^{-1} d\lambda$.

PP case ($\alpha < \frac{1}{2}$)

Theorem 5 (PP case $\Longrightarrow$ Poisson process)

$$\xi_L(dx) \stackrel{d}{\to} \text{Poisson}\left(\frac{1}{\pi}dx\right)$$

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Summary

(1) $\alpha > \frac{1}{2}$: $\xi_L(dx) \xrightarrow{d}$ Clock process

(2) $\alpha = \frac{1}{2}$: $\xi_L(dx) \xrightarrow{d} \text{Sine}_\beta$

(3) $\alpha < \frac{1}{2}$: $\xi_L(dx) \xrightarrow{d} \text{Poisson}\left(\frac{1}{\pi} dx\right)$

Outline of proof 1

Let x_t be the solution to $H_L x_t = \kappa^2 x_t$ which we write in the Prüfer coordinate.

$$\begin{pmatrix} x_t \\ x_t'/\kappa \end{pmatrix} = r_t \begin{pmatrix} \sin\theta_t \\ \cos\theta_t \end{pmatrix}, \quad \theta_0 = 0.$$

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be the relative Prüfer phase. Then we have

$$\mathbf{E}[e^{-\xi_L(f)}] = \mathbf{E}\left[\exp\left(-\sum_{n \geq n(L)-m(\kappa_0,L)} f\left(\Psi_L^{-1}(n\pi - \phi(\kappa_0, L))\right)\right)\right]$$

where $\theta_L(\kappa_0, L) = m(\kappa_0, L)\pi + \phi(\kappa_0, L)$,
$m(\kappa_0, L) \in \mathbf{Z}$, $\phi(\kappa_0, L) \in [0, \pi)$.

Outline of Proof 2

We replace L by n, and consider

$$\Psi_t^{(n)}(\lambda) := \theta_{nt}(\kappa_\lambda) - \theta_{nt}(\kappa_0),$$

$$\sim \lambda t + \frac{1}{2\kappa_0} Re \int_0^{nt} a(s) \left(e^{2i\theta_s(\kappa_\lambda)} - e^{2i\theta_s(\kappa_0)} \right) F(X_s) ds$$

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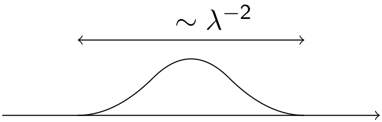
Moreover in PP, $\Psi_t^{(n)}(\lambda) \stackrel{d}{\to} \pi Poisson_{\mathbf{R}^2}([0,t] \times [0,\lambda])$, with intensity $\pi^{-1} 1_{[0,1]}(s) ds d\lambda$.

Decaying Coupling Model

In 1-dim, $H = -\triangle + \lambda V$ generically has localization length $\sim \lambda^{-2}$.



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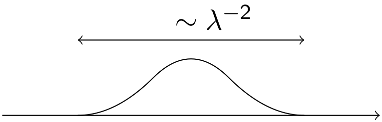

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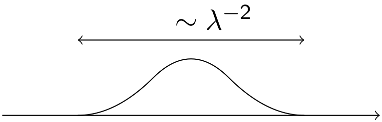

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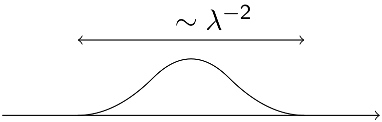

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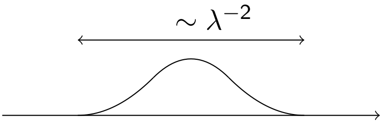
Decaying Coupling Model

In 1-dim, $H = -\triangle + \lambda V$ generically has localization length $\sim \lambda^{-2}$.



So, for $H_L := H|_{[0,L]}$, we expect

(1) $L \ll \frac{1}{\lambda^2} (\Leftrightarrow \lambda \ll \frac{1}{\sqrt{L}}) \Longrightarrow$ "extended" $\Longrightarrow \xi_L \to$ clock

(2) $L \gg \frac{1}{\lambda^2} (\Leftrightarrow \lambda \gg \frac{1}{\sqrt{L}}) \Longrightarrow$ "localized" $\Longrightarrow \xi_L \to$ Poisson

(3) $L \sim \frac{1}{\lambda^2} (\Leftrightarrow \lambda \sim \frac{1}{\sqrt{L}}) \Longrightarrow$ "critical" $\Longrightarrow \xi_L \to \beta$-ensemble ?

Hamiltonian

$$H_\lambda := -\frac{d^2}{dt^2} + \lambda F(X_t)$$

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In this section, we always assume :

Assumption Subseq. $\{L_j\}$ satisfies $L_j \overset{j\to\infty}{\to} \infty$ and

$$\kappa_0 L_j = m_j \pi + \beta + o(1), \quad j \to \infty.$$

for some $m_j \in \mathbf{N}$, $\beta \in [0, \pi)$.

Results

Theorem

(1) (Extended) $\alpha > \frac{1}{2} \Longrightarrow \xi_L \to$ (deterministic) clock process with Gaussian 2nd order

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Results

Theorem

(1) (Extended) $\alpha > \frac{1}{2} \Longrightarrow \xi_L \to$ (deterministic) clock process with Gaussian 2nd order

(2) (Critical) $\alpha = \frac{1}{2} \Longrightarrow \xi_L \to \mathrm{Sch}_\tau$-process

(3) (Localized) $\alpha < \frac{1}{2} \Longrightarrow \xi_L \to$ Poisson process

References

[1] S. Kotani and F. Nakano
Level statistics for the one-dimensional Schrödinger operators with random decaying potential, Interdisciplinary Mathematical Sciences, **17**(2014), 343-373.

[2] F. Nakano
Level statistics for one-dimensional Schrödinger operators and Gaussian beta ensemble, J. Stat. Phys. **156**(2014), 66-93.

[3] S. Kotani and F. Nakano,
Poisson statistics for 1d Schrödinger operators with random decaying potentials, preprint.

[4] F. Nakano,
Fluctuation of density of states for 1d Schrödinger operators, in preparation.