Conformal embeddings and realizations of certain simple W-algebras

Dražen Adamović
University of Zagreb, Croatia
Supported by CSF, grant. no. 2634

Conference: Vertex algebras and quantum groups

Banff, Canada
February 7 - 12, 2016

Plan of talk

- Vertex algebras, affine vertex algebras, affine W–algebras
- Some affine fusion rules
- Conformal embeddings of affine vertex algebras into W–algebras: (joint work with V. Kac, P. Moseneder-Frajria , P. Papi and O. Perše.)
- Explicit realization of certain affine and superconformal vertex algebras and their modules.
  (D. Adamović, Transform. Groups (2015))
- Connection with W-algebras in LCFT

Notations and terminology

- $\mathfrak{g}$ simple Lie (super)algebra over $\mathbb{C}$
- $V^k(\mathfrak{g})$ universal affine VOA of level k (k is not critical)
- $V_k(\mathfrak{g})$ simple quotient of $V^k(\mathfrak{g})$
- ω_{sug} Sugawara Virasoro vector in $V_k(\mathfrak{g})$ of central charge

$$c(sug) = \frac{k \mathrm{sdim} \mathfrak{g}}{k + h^\vee}.$$

- Let V are VOA with conformal vector ω_V, U subVOA with conformal vector ω_U. U is conformally embedded into V if

$$\omega_U = \omega_V.$$

Notations, terminology and history

- Let KL^k be the subcategory of $\mathcal{O}_k$ consisting of modules M on which $\mathfrak{g}$–acts locally finite.
- Modules from KL^k are $V^k(\mathfrak{g})$–modules.
- Catagory KL_k: $V_k(\mathfrak{g})$–modules which are in KL^k.
- Important problem: Classify irreducible modules in KL_k.
- For generic k: $KL^k = KL_k$ (Kazhdan-Lusztig, Lepowsky-Huang-Zhang)
- Classified for k admissible by T. Arakawa (2015) (conjectured by D.A, A.Milas 20 years ago)
- Classified for certain non-admissible, non-generic k for special cases (D.A; O. Perše, T. Arakawa, Anne Moreau in several papers)
- Investigate fusion rules and associated fusion algebras for modules from KL_k. Tensor category of KL_k modules ?

Notations and terminology

- $V_k(\mathfrak{g})$–module M_1 is a simple-current in the category KL_k if for every irreducible $V_k(\mathfrak{g})$–module M_2 from the category KL_k , there is a unique irreducible $V_k(\mathfrak{g})$–module M_3 in the category KL_k such that the vector spaces of the intertwining operators $I\binom{M_3}{M_1 \ M_2}$ is 1–dimensional and $I\binom{N}{M_1 \ M_2} = 0$ for any other irreducible $V_k(\mathfrak{g})$–module N which is not isomorphic to M_3.
- "Much easier definition"
- $M_1 \times M_2$ is irreducible module in KL_k for every irreducible module M_2 in KL_k.

Simple current $V_k(sl(n))$–modules at non-admssible levels

For $i \in \mathbb{Z}$ we define $M_{k,i} = L_{sl(n)}(\lambda_{k,i})$ where

$$\lambda_{k,i} = (k-i)\Lambda_0 + i\Lambda_1 \ (i \geq 0), \quad \lambda_{k,i} = (k+i)\Lambda_0 - i\Lambda_n \ (i < 0),$$

Theorem (D.A, O. Perše (2014))

Let $k = -1$ and $n \geq 3$. In the category KL_k of $V_{-1}(sl(n))$–modules, the following fusion rules holds:

$$M_{k,i} \times M_{k,j} = M_{k,i+j}.$$

Remark.

This results implies simplicity of Feingold-Frenkel realization of $V_{-1}(sl(n))$.

"Cloning" $V_{-1}(sl(n))$

Conjecture

Let $k = -\frac{n+1}{2}$ $(n \geq 4)$. In the category KL_k of $V_k(sl(n))$–modules, the following fusion rules holds:

$$M_{k,i} \times M_{k,j} = M_{k,i+j}.$$

There are more similar affine VOAs.

Conformal embedding $V_k(gl(n))$ into $V_k(sl(n+1))$.

Theorem (AKMPP, 2015)

Let $k = -\frac{n+1}{2}$, $n \geq 4$. *Then we have conformal embedding* $V_k(gl(n)) = V_k(sl(n)) \otimes M(1)$ *in* $V_k(sl(n+1))$.
Assume that $n \geq 4$. *Then*

$$V_k(sl(n+1)) = \bigoplus_{i \in \mathbb{Z}} V_k(sl(n+1))^{(i)} \qquad (1)$$

and each $V_k(sl(n+1))^{(i)}$*– is irreducible* $V_k(gl(n))$*–module.*

Remark.

In the case $n = 2$, *embedding can be described using explicit realization from D.A., Transform. Groups 2015. Connected with LCFT.*
The case $n = 3$ *is open.*

Conformal embedding $V_k(gl(n))$ into $V_k(sl(n+1))$.

Remark.

Note that $k = -\frac{n+1}{2}$ (for n even) is admissible level for $V_k(sl(n+1))$, but it is not admissible for $V_k(sl(n))$.
Consequence: Characters of non-admissible representations can be described using characters of admissible representations. Connections with MMF(?)

Remark.

$\widehat{sl(n)}$ highest weights of $V_k(sl(n+1))^{(i)}$ are

$$(k-i)\Lambda_0 + i\Lambda_1 \quad (i \geq 0), \quad (k+i)\Lambda_0 - i\Lambda_n \quad (i < 0).$$

Affine W algebra $W^k(\mathfrak{g}, f_\theta)$

- Choose root vectors e_θ and f_θ such that

$$[e_\theta, f_\theta] = x, \ [x, e_\theta] = e_\theta, \ [x, f_\theta] = -f_\theta.$$

- $\mathrm{ad}(x)$ defines minimal $\frac{1}{2}\mathbb{Z}$–gradation:

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-1/2} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{1/2} \oplus \mathfrak{g}_1.$$

- Let $\mathfrak{g}^\natural = \{a \in \mathfrak{g}_0 \mid (a|x) = 0\}$.
- $W^k(\mathfrak{g}, f_\theta)$ is strongly generated by vectors
- $G^{\{u\}}$, $u \in \mathfrak{g}_{-1/2}$, of conformal weight $3/2$;
- $J^{\{a\}}$, $u \in \mathfrak{g}^\natural$ of conformal weight 1;
- ω conformal vector of central charge

$$c(\mathfrak{g}, k) = \frac{k \mathrm{sdim} \mathfrak{g}}{k + h^\vee} - 6k + h^\vee - 4.$$

Affine vertex subalgebra of $W_k(\mathfrak{g}, f_\theta)$

- Assume that $\mathfrak{g}^\natural = \bigoplus_{i\in I} \mathfrak{g}_i^\natural$; and that $\mathfrak{g}_i^\natural$ is either simple or 1–dimensional abelien
- Let $\mathcal{V}^k(\mathfrak{g}^\natural)$ be the vertex subalgebra of $W^k(\mathfrak{g}, f_\theta)$ generated by $\{J^{\{a\}} \mid u \in \mathfrak{g}^\natural\}$. Then:

$$\mathcal{V}^k(\mathfrak{g}^\natural) = \bigotimes_{i\in I} V^{k_i}(\mathfrak{g}_i^\natural).$$

- Let $\mathcal{V}_k(\mathfrak{g}^\natural)$ be the image of $\mathcal{V}^k(\mathfrak{g}^\natural)$ in $W_k(\mathfrak{g}, f_\theta)$.
- Let ω_{sug} be the Sugawara Virasoro vector in $\mathcal{V}_k(\mathfrak{g}^\natural)$.
- Embedding $\mathcal{V}_k(\mathfrak{g}^\natural)$ in $W_k(\mathfrak{g}, f_\theta)$ is called conformal if

$$\omega_{sug} = \omega.$$

A numerical criterion for conformal embedding

Theorem (D.A, Kac, Moseneder-Frajria, Papi, Perše (2016))

Embedding $\mathcal{V}_k(\mathfrak{g}^\natural)$ in $W_k(\mathfrak{g}, f_\theta)$ is conformal if and only if

$$c(sug) = c(\mathfrak{g}, k).$$

Theorem (D.A, Kac, Moseneder-Frajria, Papi, Perše (2016))

Assume that k is conformal level and that $W_k(\mathfrak{g}, f_\theta)$ does not collapse on its affine part. Then

$$k = -\frac{2}{3}h^\vee \quad or \quad k = -\frac{h^\vee - 1}{2}.$$

Example: $\mathfrak{g} = sl(m|n)$

- Assume: $m \geq n$; $m \neq n+2$.
- Set of conformal levels

$$\{-1, -\frac{h^\vee}{2}, -\frac{h^\vee - 1}{2}, -\frac{2}{3}h^\vee\}.$$

- $W_k(\mathfrak{g}, f_\theta) = M(1)$ if and only if $k = -1$.
- $W_k(\mathfrak{g}, \theta) = \mathcal{V}_k(sl(n-2|m))$ if and only if $k = -\frac{h^\vee}{2}$.

Application to the extension theory of affine VOA

- When V is a regular VOA, theory of SCE was developed by H. Li (several papers) Dong-Li-Mason (1997), Huang-Lepowsky-Kirilov,
- When V is not rational, not C_2 cofinite, and M is a SC V–module satisfying certain condition, it is non-trivial to show that $V \oplus M$ is again a VOA (recent work of Creutzig, Kanade, Linshaw).
- " Elegant solution". Realize:

$$W_k(\mathfrak{g}, f_\theta) = V \bigoplus M$$

- In [AKMPP(2016)] for every simple Lie superalgebra $\mathfrak{g}$ such that $\mathfrak{g}^\natural$ is a semi–simple Lie algebra, we take suitable conformal k and show that

$$W_k(\mathfrak{g}, f_\theta) = \mathcal{V}_k(\mathfrak{g}^\natural) \bigoplus M$$

where M is a simple $V_k(\mathfrak{g}^\natural)$–module.

Conformal embedding $V_{k+1}(gl(n))$ in $W_k(sl(n+2), f_\theta)$:

Theorem (AKMPP, 2016)

Let $k = -\frac{2}{3}(n+2)$*. Then we have conformal embedding* $V_{k+1}(gl(n))$ *in* $W_k = W_k(sl(n+2), f_\theta)$*.*
Assume that $n \geq 3$*. Then*

$$W_k = \bigoplus_{i \in \mathbb{Z}} W_k^{(i)} \tag{2}$$

and each $W_k^{(i)}$*– is irreducible* $V_{k+1}(gl(n))$*–module.*

Remark.

In all cases, $n \geq 3$ $W_k^{(i)}$ *are not admissible* $V_{k+1}(gl(n))$*–modules. We believe that they are simple currents in the category* KL_{k+1}*.*

Conformal emneddings with infinite decomposition property

- Conformal embeddings with infinite decomposition property are
- $V_{-3/2}(gl(2))$ in $V_{-3/2}(sl(3))$,
- $V_{-5/3}(gl(2))$ in $W_{-8/3}$.

- Each $V_{-3/2}(sl(3))^{(i)}$ and $W_{-8/3}^{(i)}$ are direct sum of infinitely many irreducible $\widehat{gl}_2$–modules.

- Analysis of these embedding uses explicit realization of certain vertex algebras from D.A, Transform. Groups (2015).

Wakimoto modules

- Let $p \in \mathbb{Z}_{\geq 0}$, $p \geq 2$. Assume now that $k+2 = \frac{1}{p}$. Let $L^{(p)}$ be the following lattice $L^{(p)} = \mathbb{Z}\alpha + \mathbb{Z}\beta + \mathbb{Z}\delta + \mathbb{Z}\varphi$
- with the $\mathbb{Q}$–valued bilinear form $\langle \cdot, \cdot \rangle$ and non-trival products that $\langle \alpha, \alpha \rangle = 1, \quad \langle \beta, \beta \rangle = -1, \quad \langle \delta, \delta \rangle = -\langle \varphi, \varphi \rangle = \frac{2}{p}$.
- Let $V_{L^{(p)}}$ be the associated generalized vertex algebra. If $p = 2$ then $V_{L^{(2)}}$ is a vertex superalgebra.
- There is a injective homomorphism of vertex algebras $V^k(sl_2) \to V_{L^{(p)}}$ given by

$$
\begin{aligned}
e &= e^{\alpha+\beta}, \quad h = -2\beta(-1) + \delta(-1), \\
f &= ((k+1)(\alpha(-1)^2 - \alpha(-2)) - \alpha(-1)\delta(-1) + \\
&\quad (k+2)\alpha(-1)\beta(-1))e^{-\alpha-\beta}.
\end{aligned}
$$

- Screening operators are

$$Q = \text{Res}_z Y(e^{\alpha+\beta-p\delta}, z), \quad \widetilde{Q} = \text{Res}_z Y(e^{-\frac{\alpha+\beta-p\delta}{p}}, z).$$

- The Weyl vertex algebra M is isomorphic to a subalgebra of $V_{L^{(p)}}$ generated by

$$a = e^{\alpha+\beta}, \ a^* = -\alpha(-1)e^{-\alpha-\beta}.$$

- Let $M_\delta(1)$ be the Heisenberg vertex algebra generated by δ.

$$F_{p/2} = M_\delta(1) \otimes \mathbb{C}[\mathbb{Z}\frac{p}{2}\delta] \quad \text{and} \quad M \otimes F_{p/2}$$

We have the following (generalized) vertex algebra

$$\mathcal{V}^{(p)} = \text{Ker}_{M\otimes F_{p/2}} \widetilde{Q}.$$

N=4 superconformal vertex algebra $V_c^{N=4}$

$V_c^{N=4} = W^k(psl(2,2), f_\theta)$, $c = -6(k+1)$; is generated by the Virasoro field L, three primary fields of conformal weight 1, J^0, J^+ and J^- (even part) and four primary fields of conformal weight $\frac{3}{2}$, $G^\pm$ and $\overline{G}^\pm$ (odd part).
The remaining (non-vanishing) λ–brackets are

$$
\begin{array}{cc}
[J^0_\lambda, J^\pm] = \pm 2 J^\pm & [J^0_\lambda J^0] = \frac{c}{3} I \\
[J^+_\lambda J^-] = J^0 + \frac{c}{6}\lambda & [J^0_\lambda G^\pm] = \pm G^\pm \\
[J^0_\lambda \overline{G}^\pm] = \pm \overline{G}^\pm & [J^+_\lambda G^-] = G^+ \\
[J^-_\lambda G^+] = G^- & [J^+_\lambda \overline{G}^-] = -\overline{G}^+ \\
[J^-_\lambda \overline{G}^+] = -\overline{G}^- & [G^\pm_\lambda \overline{G}^\pm] = (T + 2\lambda) J^\pm \\
[G^\pm_\lambda \overline{G}^\mp] = & L \pm \frac{1}{2} T J^0 \pm \lambda J^0 + \frac{c}{6}\lambda^2
\end{array}
$$

Let $L_c^{N=4} = W_k(psl(2,2), f_\theta)$.

N=4 superconformal vertex algebra $L_c^{N=4}$ with $c = -9$

We shall present some results from D.Adamović, Transformation Groups (2015)

Theorem

(i) $V_k(sl_2)$ *with* $k = -3/2$ *is conformally embedded into* $L_c^{N=4}$ *with* $c = -9$.
(ii)

$$L_c^{N=4} \cong \mathcal{V}^{(2)} = (M \otimes F)^{int}$$

where $M \otimes F$ *is a maximal* sl_2*–integrable submodule of the Weyl-Clifford vertex algebra* $M \otimes F$.

$L_c^{N=4}$ with $c=-9$ as an $\widehat{sl_2}$–module

$L_c^{N=4}$ with $c=-9$ is completely reducible $\widehat{sl_2}$–module and the following decomposition holds:

$$L_c^{N=4} \cong \bigoplus_{m=0}^{\infty} (m+1) L_{A_1}(-(\frac{3}{2}+m)\Lambda_0 + m\Lambda_1).$$

$L_c^{N=4}$ is a completely reducible $sl_2 \times \widehat{sl_2}$–modules. sl_2 action is obtained using screening operators for Wakimoto realization of $\widehat{sl_2}$–modules at level $-3/2$.

The affine vertex algebra $V_k(sl_3)$ with $k = -3/2$.

Theorem

(i) The simple affine vertex algebra $V_k(sl_3)$ with $k = -3/2$ is realized as a subalgebra of $L_c^{N=4} \otimes F_{-1}$ with $c = -9$. In particular $V_k(sl_3)$ can be realized as subalgebra of

$$M \otimes F \otimes F_{-1}.$$

(ii) $L_c^{N=4} \otimes F_{-1}$ is a completely reducible $A_2^{(1)}$–module at level $k = -3/2$.

On representation theory of $L_c^{N=4}$ with $c = -9$

- $L_c^{N=4}$ has only one irreducible module in the category of strong modules. Every $\mathbf{Z}_{>0}$–graded $L_c^{N=4}$–module with finite-dimensional weight spaces (with respect to $L(0)$) is semisimple ("Rationality in the category of strong modules")
- $L_c^{N=4}$ has two irreducible module in the category $\mathcal{O}$. There are non-semisimple $L_c^{N=4}$–modules from the category $\mathcal{O}$.
- $L_c^{N=4}$ has infinitely many irreducible modules in the category of weight modules.
- $L_c^{N=4}$ admits logarithmic modules on which $L(0)$ does not act semi-simply.

Theorem (D.A, 2015)

Assume that U is an irreducible $L_c^{N=4}$–module with $c = -9$ such that $U = \bigoplus_{j \in \mathbb{Z}} U^j$ is $\mathbb{Z}$–graded (in a suitable sense).
Let F_{-1} be the vertex superalgebra associated to lattice $\mathbb{Z}\sqrt{-1}$. Then

$$U \otimes F_{-1} = \bigoplus_{s \in \mathbb{Z}} \mathcal{L}_s(U), \quad \text{where} \quad \mathcal{L}_s(U) := \bigoplus_{i \in \mathbb{Z}} U^i \otimes F_{-1}^{-s+i}$$

and for every $s \in \mathbb{Z}$ $\mathcal{L}_s(U)$ is an irreducible $A_2^{(1)}$–module at level $-3/2$.

Correspondence between W–algebras and $\widehat{sl(n)}$

Higher rang generalization gives a realization of $V_k(sl(n))$:

Theorem (AKMPP, 2016)

(1) *There is a conformal embedding of* $V_k(gl(n))$ *into the simple* W*-algebra* $\mathcal{W}_{k'} = \mathcal{W}_{k'}(sl(2|n), f_\theta)$ *for* $k = -(n+1)/2$, $k' = (n-1)/2$.

(2) $\mathcal{W}_{k'}$ *is a semisimple* $V_k(gl_n)$*–module.*

(3) *The admissible affine vertex algebra* $V_k(sl(n+1))$ *can be embedded in vertex algebra* $\mathcal{W}_{k'} \otimes F_{-1}$.

Connection with C_2–cofinite vertex algebras appearing in LCFT:

Vacuum space of $V_k(sl_3)$ with $k = -3/2$ contains the vertex algebra $\mathcal{W}_{A_2}(p)$ with $p = 2$ (which is conjecturally C_2–cofinite).

Affine vertex algebra $V_k(sl_2)$ for $k + 2 = \frac{1}{p}$, $p \geq 2$ can be conformally embedded into the vertex algebra $\mathcal{V}^{(p)}$ generated by $L_k(sl_2)$ and 4 primary vectors $\tau^{\pm}_{(p)}, \overline{\tau}^{\pm}_{(p)}$.

$\mathcal{V}^{(p)} \cong L_c^{N=4}$ for $p = 2$.

Drinfeld-Sokolov reduction maps $\mathcal{V}^{(p)}$ to the doublet vertex algebra $\mathcal{A}(p)$ and even part $(\mathcal{V}^{(p)})^{even}$ to the triplet vertex algebra $\mathcal{W}(p)$. (C_2–cofiniteness and RT of these vertex algebras were obtain in a work of D.A and A. Milas)

The Vertex algebra $\mathcal{W}_{A_2}(p)$: Definition

We consider the lattice

$$\sqrt{p}A_2 = \mathbb{Z}\gamma_1 + \mathbb{Z}\gamma_2, \quad \langle \gamma_1, \gamma_1 \rangle = \langle \gamma_2, \gamma_2 \rangle = 2p, \ \langle \gamma_1, \gamma_2 \rangle = -p.$$

Let $M_{\gamma_1,\gamma_2}(1)$ be the s Heisenberg vertex subalgebra of $V_{\sqrt{p}A_2}$ generated by the Heisenberg fields $\gamma_1(z)$ and $\gamma_2(z)$.

$$\mathcal{W}_{A_2}(p) = \text{Ker}_{V_{\sqrt{p}A_2}} e_0^{-\gamma_1/p} \bigcap \text{Ker}_{V_{\sqrt{p}A_2}} e_0^{-\gamma_2/p}.$$

We also have its subalgebra:

$$\mathcal{W}^0_{A_2}(p) = \text{Ker}_{M_{\gamma_1,\gamma_2}(1)} e_0^{-\gamma_1/p} \bigcap \text{Ker}_{M_{\gamma_1,\gamma_2}(1)} e_0^{-\gamma_2/p}$$

$\mathcal{W}_{A_2}(p)$ and $\mathcal{W}^0_{A_2}(p)$ have vertex subalgebra isomorphic to the simple $\mathcal{W}(2,3)$–algebra with central charge $c_p = 2 - 24\frac{(p-1)^2}{p}$.

The Vertex algebra $\mathcal{W}_{A_2}(p)$: Conjecture

(i) $\mathcal{W}_{A_2}(p)$ is a C_2–cofinite vertex algebra for $p \geq 2$ and that it is a completely reducible $\mathcal{W}(2,3) \times sl_3$–module.

(ii) $\mathcal{W}_{A_2}(p)$ is strongly generated by $\mathcal{W}(2,3)$ generators and by $sl_3.e^{-\gamma_1-\gamma_2}$, so by 8 primary fields for the $\mathcal{W}(2,3)$–algebra.

Note that $\mathcal{W}_{A_2}(p)$ is a generalization of the triplet vertex algebra $\mathcal{W}(p)$ and $\mathcal{W}^0_{A_2}(p)$ is a generalization of the singlet vertex subalgebra of $\mathcal{W}(p)$.

Relation with parafermionic vertex algebras for $p = 2$

(i) Let $K(sl_3, k)$ be the parafermion vertex subalgebra of $V_k(sl_3)$.

(iii) For $k = -3/2$ we have

$$K(sl_3, k) = \mathcal{W}^0_{A_2}(p).$$

Vertex algebras $\mathcal{R}^{(p)}$

Let $F_{-p/2}$ denotes the generalized lattice vertex algebra associated to the lattice $\mathbb{Z}(\frac{p}{2}\varphi)$ such that

$$\langle \varphi, \varphi \rangle = -\frac{2}{p}.$$

Let $\mathcal{R}^{(p)}$ by the subalgebra of $\mathcal{V}^{(p)} \otimes F_{-p/2}$ generated by $x = x(-1)\mathbf{1} \otimes 1$, $x \in \{e, f, h\}$, $1 \otimes \varphi(-1)\mathbf{1}$ and

$$e_{\alpha_1,p} := \frac{1}{\sqrt{2}} \, e^{\frac{p}{2}(\delta+\varphi)} \quad (3)$$

$$f_{\alpha_1,p} := \frac{1}{\sqrt{2}} f(0) e_{\alpha_1,p} \quad (4)$$

$$e_{\alpha_2,p} := \frac{1}{\sqrt{2}} Q e^{\frac{p}{2}(\delta-\varphi)} \quad (5)$$

$$f_{\alpha_2,p} := \frac{1}{\sqrt{2}} f(0) e_{\alpha_2,p} \quad (6)$$

Realization of simple W–algebra $W_k(sl(4), f_\theta)$

Recall that $\mathcal{R}^{(2)} \cong L_{A_2}(-\frac{3}{2}\Lambda_0)$.

Theorem

- *$\mathcal{R}^{(3)} \cong W_k(sl_4, f_\theta)$ with $k = -8/3$.*
- *$W_k(sl(4), f_\theta) = \bigoplus_{i \in \mathbb{Z}} W_k(sl(4), f_\theta)^{(i)}$.*
- *Each $W_k(sl(4), f_\theta)^{(i)}$ is a direct sum of infinitely many $V_{-5/3}(gl(2))$–modules.*

Thank you!