Cohomology and Support Theory for Quantum Groups

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References

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Notation (Classical Lie Theory)

- G: a simple, simply connected algebraic group over $\mathbb{C}$
- $\mathfrak{g} = \text{Lie}\, G$
- T: a maximal split torus in G
- Φ: the corresponding (irreducible) root system associated to (G, T)
- $\Phi^{\pm}$: the positive (respectively, negative) roots
- B: a Borel subgroup containing T corresponding to the negative roots
- $\Delta = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$: an ordering of the simple roots
- W: the Weyl group of Φ
- $W_\ell = W \ltimes \ell\mathbb{Z}\Phi$: the affine Weyl group of Φ
- ρ: the Weyl weight defined by $\rho = \frac{1}{2}\sum_{\alpha\in\Phi^+}\alpha$
- α_0 highest short root
- h: the Coxeter number of Φ, given by $h = \langle \rho, \alpha_0^\vee \rangle + 1$

Quantum Groups: Conventions

Assumptions

Let $\zeta \in \mathbb{C}$ be a primitive ℓth root of unity.

- *The integer ℓ is odd and greater than 1.*
- *If the root system Φ has type G_2, then 3 does not divide ℓ.*
- *The integer ℓ is* good *for Φ, that is ℓ is not divisible by a bad prime for Φ. If ℓ is not good it is called* bad.

Quantum Groups: Notation

- $\mathbb{U}_\zeta(\mathfrak{g})$: quantized enveloping algebra specialized at ζ
- $\mathbb{U}(\mathfrak{g})$: ordinary universal enveloping algebra
- $U_\zeta(\mathfrak{g})$: Lusztig $\mathcal{A}$-form specialized at ζ (distribution algebra)
- $u_\zeta(\mathfrak{g})$: small quantum group (f.d. Hopf algebra)

$$\mathbb{U}_\zeta(\mathfrak{g}) \twoheadrightarrow u_\zeta(\mathfrak{g}) \hookrightarrow U_\zeta(\mathfrak{g})$$

Quantum Groups: Representation Theory

- $X = \mathbb{Z}\omega_1 \oplus \cdots \oplus \mathbb{Z}\omega_n$: the weight lattice, where $\omega_i \in \mathbb{E}$ are the fundamental weights.
- $X_+ = \mathbb{N}\omega_1 + \cdots + \mathbb{N}\omega_n$: the dominant weights
- $L_\zeta(\lambda)$: simple module $U_\zeta(\mathfrak{g})$ of highest weight $\lambda \in X_+$
- $\nabla_\zeta(\lambda) = \text{ind}_{U_\zeta(\mathfrak{b})}^{U_\zeta(\mathfrak{g})}\lambda$: induced module where $\lambda \in X_+$
- $\Delta_\zeta(\lambda)$: Weyl module where $\lambda \in X_+$
- $T_\zeta(\lambda)$: tilting module of high weight $\lambda \in X_+$

$$\begin{array}{ccc} \Delta_\zeta(\lambda) & \xrightarrow{\text{surjective}} & L_\zeta(\lambda) \\ {\scriptstyle\text{injective}}\downarrow & & \downarrow{\scriptstyle\text{injective}} \\ T_\zeta(\lambda) & \xrightarrow{\text{surjective}} & \nabla_\zeta(\lambda). \end{array}$$

Connections to Complex Algebraic Geometry

- $u_\zeta(\mathfrak{g}) \lhd U_\zeta(\mathfrak{g})$ (normal subHopf algebra)
- $U_\zeta(\mathfrak{g})/u_\zeta(\mathfrak{g}) \cong \mathbb{U}(\mathfrak{g})$

- Let M be in $\text{mod}(U_\zeta(\mathfrak{g}))$. Then the cohomology $\text{H}^\bullet(u_\zeta(\mathfrak{g}), M)$ becomes a rational G-module. (i.e., cohomology takes you from the quantum world to the classical world)
- In fact we will see that if $\ell > h$ the cohomology yields a "naive functor" from $\text{mod}(U_\zeta(\mathfrak{g}))$-modules to $\text{Coh}(\mathcal{N})$ (i.e., f.g. graded $\mathbb{C}[\mathcal{N}]$-modules).
- One point to keep in mind is that unlike the case with algebraic groups in characteristic $p > 0$, in the quantum case you can only apply Frobenius map once.

Calculation of the Cohomology Ring

Theorem (Ginzburg-Kumar, 1993)

Let $\mathfrak{g}$ be a complex simple Lie algebra and $\mathcal{N}$ be the nilpotent cone. If $l > h$ then

(a) $\mathrm{H}^{odd}(u_\zeta(\mathfrak{g}), \mathbb{C}) = 0$;

(b) $\mathrm{H}^{2\bullet}(u_\zeta(\mathfrak{g}), \mathbb{C}) = \mathbb{C}[\mathcal{N}]$.

In [BNPP], we calculated the cohomology ring when $l < h$ and showed that the odd degree cohomology vanishes and in most cases $\mathrm{H}^{2\bullet}(u_\zeta(\mathfrak{g}), \mathbb{C}) = \mathbb{C}[G \cdot \mathfrak{u}_J]$ where $G \cdot \mathfrak{u}_J$ is the closure of some Richardson orbit. Our calculations heavily used the fact that

$$R^n \mathrm{ind}_{P_J}^G S^\bullet(\mathfrak{u}_J^*) = 0$$

for $n > 0$ (Grauert-Riemenschneider Vanishing Theorem).

Support Theory

Theorem (BNPP)

Let $M \in \mathrm{mod}(u_\zeta(\mathfrak{g}))$ *and* $R = \mathrm{H}^{2\bullet}(u_\zeta(\mathfrak{g}), \mathbb{C})$. *Then* $\mathrm{Ext}^\bullet_{u_\zeta(\mathfrak{g})}(\mathbb{C}, M)$ *is a finitely generated* R*-module.*

Definition

Let $M \in \mathrm{mod}(u_\zeta(\mathfrak{g}))$. Let J_M be the annihilator of the action of R on $\mathrm{Ext}^\bullet_{u_\zeta(\mathfrak{g})}(\mathbb{C}, M^* \otimes M)$. The *support variety* of M is defined as

$$\mathcal{V}_{u_\zeta(\mathfrak{g})}(M) = \mathrm{MaxSpec}(R/J_M).$$

ABG Equivalence of Categories

Theorem (Arkhipov-Bezrukavnikov-Ginzburg)

Let $l > h$. There exists the following equivalences of derived categories

$$D^b(U_\zeta(\mathfrak{g})_0^{\text{mix}}) \cong D^b(Coh^{G\times\mathbb{C}^*}(G \times_B \mathfrak{n})) \cong D^b(Perv^{\text{mix}}(\text{Gr})).$$

The ABG equivalence establishes connections between

- principal block of representations of the quantum at a root of 1;
- G-equivariant coherent sheaves on the Springer resolution;
- perverse sheaves on the loop Grassmannian for the Langlands dual group.

A version of the "naive functor" is built into the first equivalence

$$D^b(U_\zeta(\mathfrak{g})_0^{\text{mix}}) \cong D^b(Coh^{G\times\mathbb{C}^*}(G \times_B \mathfrak{n}))$$

Consequences of the ABG Equivalence Theorem

- The ABG equivalence yields a proof of Lusztig's Character Formula (LCF) for simple $U_\zeta(\mathfrak{g})$-modules for $\ell > h$. The LCF for quantum groups was proved in the mid 1990s by work of Kazhdan-Lusztig and Kashiwara-Tanisaki.
- The validity of the LCF completely determines, through parity considerations, the groups $\text{Ext}^{\bullet}_{U_\zeta(\mathfrak{g})}(L_\zeta(\lambda), L_\zeta(\mu))$ when λ, μ are regular high weights. In fact, the dimensions of these cohomology groups are given in terms of Kazhdan-Lusztig polynomials.
- Bezrukanikov used the ABG equivalence to compute the support varieties of tilting modules.

Quantum Dimension

Following Parshall-Wang, define the height function $\mathrm{wht}: X \to \mathbb{Z}[\frac{1}{2}]$. For $\alpha \in \Phi$, let $d_\alpha = \langle \alpha, \alpha \rangle / 2 = \langle \alpha, \alpha \rangle / \langle \alpha_0, \alpha_0 \rangle \in \{1, 2, 3\}$. Given $\lambda = \sum_{\alpha \in \Pi} r_\alpha \alpha \in X$ ($r_\alpha \in \mathbb{Q}$), put

$$\mathrm{wht}(\lambda) := \sum_{\alpha \in \Delta} r_\alpha d_\alpha = \frac{2\langle \lambda, \rho \rangle}{\langle \alpha_0, \alpha_0 \rangle} = \frac{1}{2} \sum_{\alpha \in \Phi^+} d_\alpha \langle \lambda, \alpha^\vee \rangle. \tag{1}$$

Given a finite-dimensional X-graded vector space $V = \bigoplus_{\lambda \in X} V_\lambda$, its *generic dimension* is the Laurent polynomial

$$\dim_t V := \sum_{\lambda \in X} (\dim V_\lambda) t^{-2\,\mathrm{wht}(\lambda)} \in \mathbb{Z}[t, t^{-1}] \tag{2}$$

We also put $\mathrm{ch}(V) = \sum_{\lambda \in X} (\dim V_\lambda) e(\lambda)$ for the character of V.

For $\lambda \in X$, set

$$D_\lambda(t) = \prod_{\alpha \in \Phi^+} (t^{d_\alpha \langle \lambda+\rho, \alpha^\vee \rangle} - t^{-d_\alpha \langle \lambda+\rho, \alpha^\vee \rangle}) \in \mathbb{Z}[t, t^{-1}]. \tag{3}$$

Lemma (Parshall-Wang)

Suppose that V is a finite-dimensional X-graded vector space such that $\mathrm{ch}(V) = \chi(\lambda)$ *for some* $\lambda \in X^+$. *Then*

$$\dim_t V = D_\lambda(t)/D_0(t). \tag{4}$$

We call (4) the Weyl generic dimension formula. Its value at $t = 1$ gives Weyl's classical dimension formula for the f.d. irreducible $\mathbb{U}(\mathfrak{g})$-module of high weight λ.

Lower Bound for Support Varieties via the Quantum Dimension

For the root system Φ, let

$$d(\Phi^+, \ell) = |\{\alpha \in \Phi^+ \mid \ d_\alpha \langle \rho, \alpha^\vee \rangle = \mathrm{wht}(\alpha) \in \ell\mathbb{Z}\}|.$$

If $\ell \geq h$, one can check that $d(\Phi^+, \ell) = 0$.

Theorem (N-Parshall-Vella, 2002)

Let $M \in \mathrm{mod}(U_\zeta(\mathfrak{g}))$. Then

$$\dim \mathcal{V}_{u_\zeta(\mathfrak{g})}(M) \geq |\Phi| - d(\Phi, \ell) - 2s + 2$$

where s is a positive integer such that $\Phi_\ell(t)^s \nmid \dim_t M$. Here $\Phi_\ell(t)$ is the ℓth cyclotomic polynomial.

Support Varieties: $\nabla_\zeta(\lambda)$

Set $\Phi_\lambda = \{\alpha \in \Phi : \langle \lambda + \rho, \alpha^\vee \rangle \in \ell\mathbb{Z}\}$

Theorem

Let $\lambda \in X^+$, and choose $J \subseteq \Delta$ such that $w(\Phi_\lambda) = \Phi_J$ for some $w \in W$. Then

$$\mathcal{V}_{u_\zeta(\mathfrak{g})}(\nabla_\zeta(\lambda)) = G \cdot \mathfrak{u}_J.$$

- This theorem was proved by Ostrik (1997) for $\ell > h$.
- Bendel-Nakano-Parshall-Pillen proved the theorem for ℓ good, and have made calculations for when ℓ is bad. For bad ℓ, the orbits that arise need not be Richardson.

Support Varieties: $\nabla_\zeta(\lambda)$ (Main Ideas in the Proof)

1) In order to prove the inclusion $\mathcal{V}_{u_\zeta(\mathfrak{g})}(\nabla_\zeta(\lambda)) \subseteq G \cdot \mathfrak{u}_J$, one needs to use

- Relative support varieties and the "baby Verma modules"
- Spectral sequence techniques and localization of the cohomology

2) Since $G \cdot \mathfrak{u}_J$ is an irreducible variety, it suffices to show that

$$\dim \mathcal{V}_{u_\zeta(\mathfrak{g})}(\nabla_\zeta(\lambda)) = \dim G \cdot \mathfrak{u}_J = |\Phi| - |\Phi_J|.$$

This is accomplished by using the NPV-theorem, and proving that $\Phi_\ell(t)^s \nmid \dim_t \nabla_\zeta(\lambda)$ where $-d(\Phi,\ell) + 2s - 2 = |\Phi_J|$. Or equivalently, $s = |\Phi_J^+| - d(\Phi^+,\ell) + 1$.

Now observe that $|\Phi_J^+| - d(\Phi^+, \ell) = |\Phi_\lambda^+| - d(\Phi^+, \ell)$ is the multiplicity of $\Phi_\ell(t)$ as a divisor of $\dim_t \nabla_\zeta(\lambda)$.

$$\dim_t \nabla_\zeta(\lambda) = \prod_{\alpha \in \Phi^+} \frac{(t^{d_\alpha \langle \lambda+\rho, \alpha^\vee \rangle} - t^{-d_\alpha \langle \lambda+\rho, \alpha^\vee \rangle})}{(t^{d_\alpha \langle \rho, \alpha^\vee \rangle} - t^{-d_\alpha \langle \rho, \alpha^\vee \rangle})}.$$

Therefore, for $s = |\Phi_J^+| - d(\Phi^+, \ell) + 1$, it follows that $\Phi_\ell(t)^s \nmid \dim_t \nabla_\zeta(\lambda)$.

Lusztig Character Formula

The affine Weyl group W_ℓ is generated as a group by the fundamental system $S_\ell \subset W_\ell$. Given $I \subseteq S_\ell$, set $W_{\ell,I} = \langle I \rangle \leq W_\ell$, and set $W_\ell^I = \{w \in W_\ell : l(w) \leq l(ws) \text{ for all } s \in W_{\ell,I}\}$. Let $\leqslant$ denote the Chevalley–Bruhat partial ordering on W_ℓ.

Given $y \leqslant w$ in W_ℓ, $P_{y,w}(q)$ is the Kazhdan–Lusztig polynomial associated to the pair (y, w).

Deodhar introduced two generalizations of the $P_{y,w}$'s, called parabolic Kazhdan–Lusztig polynomials, which depend on a choice of subset $I \subseteq S_\ell$, and a choice of a root u of the equation $u^2 = q + (q-1)u$, i.e., $u = -1$ or $u = q$.

Given $I \subseteq S_\ell$, and given $(y, w) \in W_\ell^I \times W_\ell^I$ with $y \leqslant w$, the parabolic Kazhdan–Lusztig polynomial $P_{y,w}^{I,-1}$ associated to the root $u = q$ is related to the usual Kazhdan–Lusztig polynomials by the following equation.[1]

$$P_{y,w}^{I,-1} = \sum_{x \in W_I, yx \leqslant w} (-1)^{l(x)} P_{yx,w}. \tag{5}$$

We have several basic properties:

- If $y \nleqslant w$, then $P_{y,w}^{I,-1} = 0$.
- The coefficients of the $P_{y,w}^{I,-1}$ are non-negative integers.

[1] We are following the notational convention used by Kashiwara and Tanisaki, so the superscript in $P_{y,w}^{I,a}$ indicates the opposite root of the equation $u^2 = q + (q-1)u$

Fix $\lambda^- \in \overline{C}_{\mathbb{Z}}^-$. The stabilizer in W_ℓ of λ^- is defined by $W_{\ell,\lambda^-} = \{w \in W_\ell \mid w \cdot \lambda^- = \lambda^-\}$; it is generated as a group by the set $I := W_{\ell,\lambda^-} \cap S_\ell$. Then $W_{\ell,\lambda^-} = W_{\ell,I} := \langle I \rangle \leq W_\ell$ is a parabolic subgroup of W_ℓ. If $w \in W_\ell$ is minimal dominant for λ^-, then $w \in W_\ell^I$.

Theorem

Let $w \in W_\ell$ be minimal dominant for λ^-, and write $\lambda = w \cdot \lambda^-$. Let $I \subseteq S_\ell$ be such that $W_{\ell,\lambda^-} = W_{\ell,I}$. Then

$$\operatorname{ch} L_\zeta(\lambda) = \sum_{y \in W_\ell^I} (-1)^{l(w)-l(y)} P_{y,w}^{I,-1}(1) \operatorname{ch} \Delta_\zeta(y \cdot \lambda^-). \tag{6}$$

$$\dim_t L_\zeta(\lambda) = \sum_{y \in W_\ell^I} (-1)^{l(w)-l(y)} P_{y,w}^{I,-1}(1) D_{y\cdot\lambda^-}(t)/D_0(t),$$

Support Varieties: $L_\zeta(\lambda)$

Theorem (DNP, 2014)

Let $\ell > h$ and $\lambda \in X^+$. Choose $J \subseteq \Delta$ such that $w(\Phi_\lambda) = \Phi_J$ for some $w \in W$. Then

$$\mathcal{V}_{u_\zeta(\mathfrak{g})}(L_\zeta(\lambda)) = G \cdot \mathfrak{u}_J.$$

This theorem provides the most extensive calculation for the support varieties for irreducible representations for an extensive class of finite-dimensional Hopf algebras.

Support Varieties: $L_\zeta(\lambda)$ (Main Ideas in the Proof)

1) Using the results on the supports of $\nabla_\zeta(\lambda)$, one can use induction on the weight ordering to prove that

$$\mathcal{V}_{u_\zeta(\mathfrak{g})}(L_\zeta(\lambda)) \subseteq G \cdot \mathfrak{u}_J.$$

2) As in the prior calculation, we need to show that $\Phi_\ell(t)^s \nmid \dim_t L_\zeta(\lambda)$ where $2s - 2 = |\Phi_J|$ (i.e., $s - 1 = |\Phi_J^+|$ or $s = |\Phi_J^+| + 1$). That is, ζ is not a root of $\dim_t L_\zeta(\lambda)$ of multiplicity $s = |\Phi_J^+| + 1$. But, $\dim L_\zeta(\lambda)$ is given via Kazhdan-Lusztig polynomials!

Set $f(t) = D_0(t) \cdot \dim_t L_\zeta(\lambda)$. Recall that

$$\dim_t L_\zeta(\lambda) = \sum_{y \in W_\ell^I} (-1)^{l(w)-l(y)} P_{y,w}^{I,-1}(1) D_{y\cdot\lambda^-}(t)/D_0(t),$$

If $f^{(i)}(\zeta) = 0$ for all $0 \leq i < n$, but $f^{(n)}(\zeta) \neq 0$, then ζ occurs as a root of f with multiplicity exactly equal to n.

Set $n = |\Phi_{\lambda^-}^+| = |\Phi_J^+|$. Then $n = |\Phi_{y\cdot\lambda^-}^+|$ for any $y \in W_\ell$.

We know that $\Phi_\ell(t)$ occurs as a factor of $D_{y\cdot\lambda^-}(t)$ precisely n times. It remains to show that $f^{(n)}(\zeta) \neq 0$.

One can differentiate $f(t)$ n times,

$$\begin{aligned} f^{(n)}(\zeta) &= \sum_{y \in W_\ell^I} (-1)^{l(w)-l(y)} P_{y,w}^{I,-1}(1) D_{y\cdot\lambda^-}^{(n)}(\zeta) \\ &= \sum_{y \in W_\ell^I} (-1)^{l(w)-(a_{\lambda^-})} P_{y,w}^{I,-1}(1)(n!) \left(\prod_{\alpha \in \Phi_\lambda^+} 2d_\alpha \langle \lambda + \rho, \alpha^\vee \rangle \right) \zeta^{-n} E_{\lambda^-}(\zeta) \\ &= \left((-1)^{l(w)-a_{\lambda^-}} (n!) \zeta^{-n} E_{\lambda^-}(\zeta) \right) \left(\sum_{y \in W_\ell^I} P_{y,w}^{I,-1}(1) \left(\prod_{\alpha \in \Phi_\lambda^+} 2d_\alpha \langle \lambda + \rho, \alpha^\vee \rangle \right) \right) \end{aligned}$$

The first term in the product is non-zero. The second term in the product is a sum of non-negative integers (by the positivity property for the parabolic Kazhdan-Lusztig polynomials). Since $P_{w,w}^{I,-1}(1) = 1$, we conclude that the second term in the product is a strictly positive integer, hence that $f^{(n)}(\zeta) \neq 0$.

Support Varieties: Irreducible Representations, char p

Theorem (NPV, DNP)

Assume that G is a simple, simply-connected algebraic group over an algebraically closed field k of characteristic $p > h$. Assume that the Lusztig character formula holds for all restricted dominant weights. Then for $\lambda \in X^+$ and $J \subseteq \Delta$ with $w(\Phi_\lambda) = \Phi_J$,

$$\mathcal{V}_{u(\mathfrak{g})}(L(\lambda)) = G \cdot \mathfrak{u}_J.$$

- Williamson has shown that the Lusztig Character Formula (LCF) can fail to hold even when $p > h$. It is known to hold for very large primes by Andersen-Jantzen-Soergel (1994) and work of Fiebig (2012).
- It would be interesting to know in cases when the LCF fails to hold if the supports for irreducibles are given by the aforementioned formula. When the LCF fails to hold the Ext groups are different.

Epilogue: Open Problems

1) [Calculation of Ext-groups]

Calculate $\mathrm{Ext}^n_{U_\zeta(\mathfrak{g})}(L(\lambda), L(\mu))$ for all $\lambda, \mu \in X_+$.

From the proof of the Lusztig Character Formula, this is known when λ and μ are regular weights. What about singular weights?

2) [Explicit Description of Support Varieties]

Is there a "rank variety" description of $\mathcal{V}_{u_\zeta(\mathfrak{g})}(M)$?

Epilogue: Open Problems, con't

3) [Connection to Affine Lie Algebras]

Let $\hat{\mathfrak{g}} = \mathbb{C}[t, t^{-1}] \otimes \mathfrak{g} \oplus \mathbb{C}c \oplus \mathbb{C}d$ be the corresponding untwisted affine Lie algebra, and $\tilde{\mathfrak{g}}$ be the subalgebra $\tilde{\mathfrak{g}} = \mathbb{C}[t, t^{-1}] \otimes \mathfrak{g} \oplus \mathbb{C}c \subseteq \hat{\mathfrak{g}}$. For $\kappa \in \mathbb{C}$ we let $\mathcal{O}_\kappa$ be the full subcategory of all $\tilde{\mathfrak{g}}$-modules, M, for which the central element c acts by κ and M satisfies certain category $\mathcal{O}$-type finiteness conditions. For a certain κ, there exists an equivalence of tensor categories

$$F_\ell : \mathcal{O}_\kappa \to \text{mod}(U_\zeta(\mathfrak{g}))$$

a) Is there any new information about support varieties and cohomology that can be gained by using this equivalence of categories?

b) How does the twisting of modules under the Frobenius morphism behave under this equivalence?

Thank you for your attention.