# Some Mathematical and Numerical Challenges in First-Principle and Multiscale Modeling of Nano-Systems 

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Outline of the talk ..... 1

1. Toward certified and optimized molecular simulations
2. Modeling and simulation of infinite aperiodic systems

## 1 - Toward certified and optimized molecular simulations

Molecular simulation is a huge consumer of CPU time.


In some applications, comparison with experiment is not possible.


## 1 - Toward certified and optimized molecular simulations



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$\mathbf{e}=\mathbf{e}_{\mathrm{m}}+\mathbf{e}_{\mathrm{d}}+\mathbf{e}_{\mathrm{a}}+\mathbf{e}_{\mathrm{i}}+\mathbf{e}_{\mathrm{c}}$


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- Between Jan. 1982 and Nov. 1983, Vancouver stock exchange dropped from 1,000 to 524 instead of going up to 1,098 due to truncation errors.
- Patriot missile failure (Feb. 1991) due to round-off errors: storage of 0.1 on 24 bits only in the internal clock ( 500 m drift after 100 h ).
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Computing error $\mathbf{e}_{\mathrm{c}}$
Resiliency will be one of the toughest challenges in future exascale systems. Memory errors contribute more than $40 \%$ of the total hardware-related failures and are projected to increase in future exascale systems. The use of error correction codes (ECC) and checkpointing are two effective approaches to fault tolerance. (Li et al. 2011)

Periodic Gross-Pitaevskii model

- used to simulate Bose-Einstein condensates;
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where $\Omega=(0,2 \pi)^{d}(d=1,2$ or 3$)$ and where

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E(v)=\int_{\Omega}|\nabla v|^{2}+\int_{\Omega} V|v|^{2}+\frac{\mu}{2} \int_{\Omega}|v|^{4}, \\
H_{\#}^{1}(\Omega)=\left\{v \in L_{\mathrm{loc}}^{2}\left(\mathbb{R}^{d}\right) \mid \nabla v \in\left(L_{\mathrm{loc}}^{2}\left(\mathbb{R}^{d}\right)\right)^{d}, v 2 \pi \mathbb{Z}^{d} \text {-periodic }\right\},
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$V$ being a $2 \pi \mathbb{Z}^{d}$-periodic real-valued continuous function and $\mu>0$.

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$V$ being a $2 \pi \mathbb{Z}^{d}$-periodic real-valued continuous function and $\mu>0$.

- (1) has exactly two minimizers $u$ (with $u>0$ in $\Omega$ ) and $-u$;
- $\exists!\lambda \in \mathbb{R}$ such that $(\lambda, u)$ satisfies the nonlinear Schrödinger equation

$$
-\Delta u+V u+\mu u^{3}=\lambda u, \quad\|u\|_{L_{\#}^{2}}=1
$$

- $\lambda$ is the lowest eigenvalue of the self-adjoint operator $-\Delta+V+\mu u^{2}$.

Approximation space: for all $\mathcal{N} \in \mathbb{N}$,
$X_{\mathcal{N}}=\left\{v_{\mathcal{N}}=\sum_{|\mathbf{k}| \leq \mathcal{N}} \widehat{v}_{\mathbf{k}} e_{\mathbf{k}}, v_{\mathcal{N}}\right.$ real valued $\}$
where $e_{\mathbf{k}}(\mathbf{r})=\frac{e^{i \mathbf{k} \cdot \mathbf{r}}}{(2 \pi)^{d / 2}}, \quad \mathbf{k} \in \mathbb{Z}^{d}$.

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Periodic Sobolev spaces: for all $s \in \mathbb{R}$,
$H_{\#}^{s}(\Omega):=\left\{v=\sum_{\mathbf{k} \in \mathbb{Z}^{d}} \widehat{v}_{\mathbf{k}} e_{\mathbf{k}}, v\right.$ real valued, $\left.\|v\|_{H_{\#}^{s}}^{2}:=\sum_{\mathbf{k} \in \mathbb{Z}^{d}}\left(1+|\mathbf{k}|^{2}\right)^{s}\left|\widehat{v}_{\mathbf{k}}\right|^{2}<\infty\right\}$.

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Truncation operator $\Pi_{\mathcal{N}}$ : for all $\mathcal{N} \in \mathbb{N}, \Pi_{\mathcal{N}}$ is defined on $\mathcal{S}_{\#}^{\prime}(\Omega)$ by

$$
\Pi_{\mathcal{N}}\left(\sum_{\mathbf{k} \in \mathbb{Z}^{d}} \widehat{v}_{\mathbf{k}} e_{\mathbf{k}}\right)=\sum_{|\mathbf{k}| \leq \mathcal{N}} \widehat{v}_{\mathbf{k}} e_{\mathbf{k}}
$$

$\Pi_{\mathcal{N}}$ is the orthogonal projector from $H_{\#}^{s}(\Omega)$ onto $X_{\mathcal{N}}$ for all $s \in \mathbb{R}$.

Galerkin approximation
Find a minimizer $u_{\mathcal{N}}$ to the finite-dimensional variational problem

$$
I_{\mathcal{N}}=\inf \left\{E\left(v_{\mathcal{N}}\right), v_{\mathcal{N}} \in X_{\mathcal{N}}, \int_{\Omega}\left|v_{\mathcal{N}}\right|^{2}=1\right\} \quad\left(u_{\mathcal{N}}, 1\right)_{L_{\#}^{2}} \geq 0
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If $V \in H_{\#}^{\sigma}(\Omega)$ for some $\sigma \geq 0$, then $u \in H_{\#}^{\sigma+2}(\Omega)$ and

$$
\forall s<\sigma+2, \quad\left\|u-\Pi_{\mathcal{N}} u\right\|_{H_{\#}^{s}}=\left(\sum_{|\mathbf{k}|>\mathcal{N}}\left(1+|\mathbf{k}|^{2}\right)^{s}\left|\widehat{u}_{\mathbf{k}}\right|^{2}\right)^{1 / 2} \leq \frac{\|u\|_{H_{\#}^{\sigma+2}}}{\mathcal{N}^{\sigma+2-s}}
$$

## Theorem (EC, Chakir, Maday '10)

There exists $0<c \leq C<\infty$ such that for all $\mathcal{N} \in \mathbb{N}$,

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\begin{gathered}
\left\|u-\Pi_{\mathcal{N}} u\right\|_{H_{\#}^{1}} \leq\left\|u-u_{\mathcal{N}}\right\|_{H_{\#}^{1}} \leq C\left\|u-\Pi_{\mathcal{N}} u\right\|_{H_{\#}^{1}} \underset{\mathcal{N} \rightarrow 0}{\longrightarrow} 0 \\
c\left\|u-u_{\mathcal{N}}\right\|_{H_{\#}^{1}}^{2} \leq I_{N}-I=E\left(u_{\mathcal{N}}\right)-E(u) \leq C\left\|u-u_{\mathcal{N}}\right\|_{H_{\#}^{1}}^{2} .
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Assume that $V \in H_{\#}^{\sigma}(\Omega)$ for some $\sigma>d / 2$. Then

- $\left(u_{\mathcal{N}}\right)_{\mathcal{N} \in \mathbb{N}}$ converges to $u$ in $H_{\#}^{\sigma+2}(\Omega)$;
- there exists positive constants $C$ and $C_{s}$ such that

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\forall-\sigma \leq s<\sigma+2, \quad\left\|u-u_{\mathcal{N}}\right\|_{H_{\#}^{s}} \leq \frac{C_{s}}{\mathcal{N}^{\sigma+2-s}}, \quad\left|\lambda-\lambda_{\mathcal{N}}\right| \leq \frac{C}{\mathcal{N}^{2(\sigma+1)}}
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Remark: these a priori convergence rates are optimal.

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Remark: these a priori convergence rates are optimal. They are useful

- to estimate the convergence rate of various quantities of interest;
$\mathbf{E x}(d=3, \sigma=5 / 2): \quad 0 \leq E\left(u_{\mathcal{N}}\right)-E(u) \leq \frac{C}{\mathcal{N}^{7}}, \quad\left|u(0)-u_{\mathcal{N}}(0)\right| \leq \frac{C}{\mathcal{N}^{3}}$.


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Remark: these a priori convergence rates are optimal. They are useful

- to estimate the convergence rate of various quantities of interest;
- to design more efficient numerical methods.

CPU time reduction by post-treatment


Solving the pb in a fine approximation space $X_{\mathcal{N}}$ is $\operatorname{costly}\left(\sim \mathcal{K} \mathcal{N}^{d} \ln \mathcal{N} \mathbf{o p}.\right)$.

CPU time reduction by post-treatment


Solving the pb in a fine approximation space $X_{\mathcal{N}}$ is costly ( $\sim \mathcal{K} \mathcal{N}^{d} \ln \mathcal{N}$ op.). Alternative:

1. solve the pb on a coarser approximation space $X_{n}, n \ll \mathcal{N}\left(\sim \mathcal{K} n^{d} \ln n\right.$ op. $)$;

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2. construct from $u_{n}$ an approx. $u_{n, \mathcal{N}}$ of $u$ in $X_{\mathcal{N}}\left(\sim \kappa \mathcal{N}^{d} \ln \mathcal{N}\right.$ op., $\left.\kappa \ll \mathcal{K}\right)$.

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1. solve the nonlinear eigenvalue problem in the coarse approx. space $X_{n}$;
2. freeze the self-consistent potential $V+\mu u_{n}^{2}$ and/or the eigenvalue $\lambda_{n}$;
3. solve the linear eigenpb $\left(-\Delta+V+\mu u_{n}^{2}\right) \widetilde{u}=\widetilde{\lambda} \widetilde{u}$ in the fine approx. space $X_{\mathcal{N}}$.

CPU time reduction by post-treatment: two grid methods (introduced by $\mathbf{X u}$ and $\mathbf{Z h o u}$ in $\mathbf{2 0 0 0}$ for solving nonlinear elliptic pbs)

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Theorem (EC, Chakir, He, Maday '14)
Assume that $V \in H_{\#}^{\sigma}(\Omega)$ for some $\sigma>d / 2$. There exists $C \in \mathbb{R}_{+}$s.t.

$$
\begin{aligned}
\left\|u_{n, \mathcal{N}}-u\right\|_{H_{\#}^{1}} & \leq C\left(n^{-\sigma-3}+\mathcal{N}^{-\sigma-1}\right) \\
0 \leq E\left(u_{n, \mathcal{N}}\right)-E(u) & \leq C\left(n^{-\sigma-3}+\mathcal{N}^{-\sigma-1}\right)^{2}
\end{aligned}
$$

$\longrightarrow \quad$ For $n \sim \mathcal{N}^{\frac{\sigma+1}{\sigma+3}}$, significant gain in CPU time with no loss of accuracy.

Self-consistent algorithm (simple fixed point - not optimal!)

$$
\left\{\begin{array}{l}
-\Delta v_{k}+V v_{k}+\mu v_{k-1}^{2} v_{k}=\lambda_{k} v_{k}, \quad v_{k} \in H_{\#}^{1}(\Omega), \quad\left\|v_{k}\right\|_{L^{2}}=1 \\
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$\longrightarrow$ convergence ( $\mu$ small) or oscillation between two states ( $\mu$ large).

Analysis in EC - Le Bris '00, Levitt '12 (for the Hartree-Fock model).

A posteriori error estimators and error balancing: theory

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Error estimator

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Theorem (EC, Dusson, Maday, Stamm, Vohralik '14).

$$
0 \leq J\left(v_{k}\right) \leq \eta_{\mathrm{d}, k}+\eta_{\mathrm{a}, k}
$$

$\eta_{\mathrm{d}, k}=\frac{1}{2}\left(\lambda_{u_{k}, n_{k}}-\lambda_{v_{k}, \infty}\right) \geq 0, \quad \eta_{\mathrm{a}, k}=\frac{1}{2}\left(\mu \int_{\Omega}\left(v_{k}^{2}-v_{k-1}^{2}\right) v_{k}^{2}+\lambda_{k}-\lambda_{v_{k}, n_{k}}\right) \geq 0$.

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$\lambda_{v_{k}, n_{k}}$ and $\lambda_{v_{k}, \infty}$ can be estimated accurately at a low computational cost by means of Rayleigh-Schrödinger perturbation method.

A posteriori error estimators and error balancing: in practice,

- the inequalities

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E\left(v_{k}\right)-E(u) \leq J\left(v_{k}\right) \leq \eta_{\mathrm{d}, k}+\eta_{\mathrm{a}, k}
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are 'almost" equalities
(observed in numerical simulation, justified by a priori error estimates);

- $\eta_{\mathrm{d}, k}$ and $\eta_{\mathrm{a}, k}$ provide relatively cheap and sharp estimators of $\mathbf{e}_{\mathrm{d}}$ and $\mathbf{e}_{\mathrm{a}}$ resp. at iteration $k$ if the quantity of interest is the energy.

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Numerical strategy:

- refine if $\eta_{\mathrm{d}, k} \gg \eta_{\mathrm{a}, k}$,
- iterate otherwise,
until $\eta_{\mathrm{d}, k}+\eta_{\mathrm{a}, k} \leq \varepsilon$ (desired accuracy).


## Extension to Kohn-Sham models:

- Optimal a priori error estimators for Kohn-Sham are available (EC, Chakir, Maday M2AN '12)
- Numerical quadrature errors can be estimated (EC, Chakir, Maday, J. Sci. Comput. '10)
- A posteriori error estimators and computational load reduction strategies are under development: Lin, Yang et al. (Berkeley), Schneider et al. (Berlin), Zhou et al. (Beijing), our (now delocalized) team (EC, Dusson, Maday, Stamm, Vohralík, J. Comp. Phys. '16)
- k-point sampling on the Brillouin zone for metals (EC, Ehrlacher, Gontier, Levitt, Lombardi, in prep.)
- Construction of optimized pseudopotentials (EC, Mourad CMS '16, Blanc, EC, Dupuy, in prep)
- Error estimators on energy differences (with G. Dusson and Y. Maday).


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- Quantum Monte Carlo methods: maybe

Educated choice of the exchange-correlation potential in DFT

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\left\{\begin{array}{l}
\text { ground state density } \rho_{0}^{\left\{\mathbf{R}_{k}\right\}}(\mathbf{r})=\rho_{0}(\mathbf{r})=\sum_{i=1}^{N}\left|\phi_{i}(\mathbf{r})\right|^{2} \\
-\frac{1}{2} \Delta \phi_{i}+V_{\rho_{0}}^{\mathrm{KS}} \phi_{i}=\lambda_{i} \phi_{i}, \quad \lambda_{1}<\lambda_{2} \leq \lambda_{3} \leq \cdots \\
\int_{\mathbb{R}^{3}} \phi_{i} \phi_{j}=\delta_{i j} \\
V_{\rho_{0}}^{\mathrm{KS}}=V_{\rho_{0}}^{\mathrm{H}}+v_{\rho_{0}}^{\mathrm{xc}} \quad v_{\rho_{0}}^{\mathrm{xc}}: \text { exchange-correlation potential } \\
-\Delta V_{\rho_{0}}^{\mathrm{H}}=4 \pi\left(\rho_{0}-\sum_{k=1}^{M} z_{k} \delta_{\mathbf{R}_{k}}\right)
\end{array}\right.
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\begin{aligned}
& \text { (ground state density } \rho_{0}^{\left\{\mathbf{R}_{k}\right\}}(\mathbf{r})=\rho_{0}(\mathbf{r})=\sum_{i=1}^{N}\left|\phi_{i}(\mathbf{r})\right|^{2} \\
& -\frac{1}{2} \Delta \phi_{i}+V_{\rho_{0}}^{\mathrm{KS}} \phi_{i}=\lambda_{i} \phi_{i}, \quad \lambda_{1}<\lambda_{2} \leq \lambda_{3} \leq \cdots \\
& \int_{\mathbb{R}^{3}} \phi_{i} \phi_{j}=\delta_{i j} \\
& V_{\rho_{0}}^{\mathrm{KS}}=V_{\rho_{0}}^{\mathrm{H}}+v_{\rho_{0}}^{\mathrm{xc}} \\
& -\Delta V_{\rho_{0}}^{\mathrm{H}}=4 \pi\left(\rho_{0}-\sum_{k=1}^{M} z_{k} \delta_{\mathbf{R}_{k}}\right) \\
& \text { Perspective on } \\
& \text { Meron Burke }
\end{aligned}
$$

Educated choice of the exchange-correlation potential in DFT

$$
\left\{\begin{array}{l}
\text { ground state density } \rho_{0}^{\left\{\mathbf{R}_{k}\right\}}(\mathbf{r})=\rho_{0}(\mathbf{r})=\sum_{i=1}^{N}\left|\phi_{i}(\mathbf{r})\right|^{2} \\
-\frac{1}{2} \Delta \phi_{i}+V_{\rho_{0}}^{\mathrm{KS}} \phi_{i}=\lambda_{i} \phi_{i}, \quad \lambda_{1}<\lambda_{2} \leq \lambda_{3} \leq \ldots \\
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V_{\rho_{0}}^{\mathrm{KS}}=V_{\rho_{0}}^{\mathrm{H}}+v_{\rho_{0}}^{\mathrm{xc}} \\
-\Delta V_{\rho_{0}}^{\mathrm{H}}=4 \pi\left(\rho_{0}-\sum_{k=1}^{M} z_{k} \delta_{\mathbf{R}_{k}}\right)
\end{array}\right.
$$

\# citations of the B3LYP paper (Google Scholar, Aug. 2016): 66,338

Educated choice of the exchange-correlation potential in DFT

$$
\left\{\begin{array}{l}
\text { ground state density } \rho_{0}^{\left\{\mathbf{R}_{k}\right\}}(\mathbf{r})=\rho_{0}(\mathbf{r})=\sum_{i=1}^{N}\left|\phi_{i}(\mathbf{r})\right|^{2} \\
-\frac{1}{2} \Delta \phi_{i}+V_{\rho_{0}}^{\mathrm{KS}} \phi_{i}=\lambda_{i} \phi_{i}, \quad \lambda_{1}<\lambda_{2} \leq \lambda_{3} \leq \ldots \\
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V_{\rho_{0}}^{\mathrm{KS}}=V_{\rho_{0}}^{\mathrm{H}}+v_{\rho_{0}}^{\mathrm{xc}} \\
-\Delta V_{\rho_{0}}^{\mathrm{H}}=4 \pi\left(\rho_{0}-\sum_{k=1}^{M} z_{k} \delta_{\mathbf{R}_{k}}\right)
\end{array}\right.
$$

Can machine learning help?

## Machine learning in molecular simulation: model selection



Machine learning in molecular simulation: direct computation of QOI


DFT calculations are used to compute the QOI in training and validation sets (Ceder et al '10, Rupp, Tkatchenko, Müller, von Lilienfeld '12,
Burke et al. '12, Csányi et al. '13, Mallat et al. '14, ..., IPAM program '16)

2 - Modeling and simulation of infinite aperiodic systems

Homogeneous vs heterogeneous aperiodic systems


Homogenous aperiodic systems

- general theory of one-particle linear models developed in the '90s by Bellissard and co-workers
$\longrightarrow \quad$ application to incommensurate bilayer systems (talk by P. Cazeaux)
- Kohn-Sham models for homogeneous aperiodic systems are not fully understood from a mathematical point of view (the problem originates from long-range Coulomb interactions)
$\longrightarrow \quad$ EC, Lahbabi, Lewin, J. Pure App. Math. ' 13


Very large or infinite inhomogenous system

- focused models: QM/PCM, QM/MM, QM/MM/PCM, QCM, ...

QM/PCM (quantum mechanics/polarizable continuum models)


Born ('20), Kirkwood ('34), Onsager ('36)
Rivail et al. ('76), Tomasi et al. ('81), Klamt et al. ('93), Cramer and Truhlar ('94), ... see also the talk by S. Gusarov on reference interaction site models (RISM)

Very large or infinite inhomogenous system

- focused models: QM/PCM, QM/MM, QM/MM/PCM, QCM, ...

QM/MM (quantum mechanics/molecular mechanics)


Warshel and Levitt ('76), Singh and Kollman ('86), Field, Bash and Karplus ('90)

Morokuma et al. (ONIOM method, '96), ...

Note that the ONIOM method also allows one to carry out QM/QM (high-accuracy/low-accuracy) quantum mechanical calculations, e.g. MP4/HF.

Very large or infinite inhomogenous system

- focused models: QM/PCM, QM/MM, QM/MM/PCM, QCM, ...


## QM/MM/PCM



QM/MM/PCM simulation (Mennucci et al.)
Dynamical simulation of very large systems made possible by recent advances in numerical methods (EC, Lagardère, Lipparini, Maday, Mennucci, Piquemal, Scalmani, Stamm, ... '13-'16)

Very large or infinite inhomogenous system

- focused models: QM/PCM, QM/MM, QM/MM/PCM, QCM, ...

QCM (quasi-continuum model, Ortiz, Phillips and Tadmor '96)


Simulation of an AFM tip (Whalen, Shi, Tadmor)
Further developments and applications: Bernstein, Bhattacharya, Carter, Csanyi, Gavini, Kaxiras, Kermode, Knapp, Miller, Molinari, Perez, Rodney, Voter, ... Mathematical analysis by Blanc, Dobson, E, Le Bris, Legoll, Lu, Luskin, Ortner, Plechac, Shapeev, Suli, ...

Very large or infinite inhomogenous system

- focused models: QM/PCM, QM/MM, QM/MM/PCM, QCM, ...
- (Schur complement-like) self-energy / Green's function methods

Schur complement method: if the square matrix $D$ is invertible,

$$
\underbrace{\left(\begin{array}{cc}
A & B \\
C & D
\end{array}\right)}_{M}\binom{u}{v}=\binom{f}{g} \quad \Leftrightarrow \quad\left\{\begin{array}{l}
\left(A-B D^{-1} C\right) u=f-B D^{-1} g \\
v=D^{-1}(g-C u)
\end{array}\right.
$$

Terminology coined by Haynsworth in 1968.
Decomposition used by Schur in his determinant formula ('17)

$$
\operatorname{det}(M)=\operatorname{det}(D) \operatorname{det}\left(A-B D^{-1} C\right)
$$

and already used by Laplace (1812) and Sylvester (1851) in earlier works.

Very large or infinite inhomogenous system

- focused models: QM/PCM, QM/MM, QM/MM/PCM, QCM, ...
- (Schur complement-like) self-energy / Green's function methods

Self-energy formalism: for all $E \notin \sigma\left(H_{\mathrm{e}}\right)$,

$$
\left(\begin{array}{cc}
H_{\mathrm{s}} & H_{\mathrm{es}}^{\dagger} \\
H_{\mathrm{es}} & H_{\mathrm{e}}
\end{array}\right)\binom{\psi_{\mathrm{s}}}{\psi_{\mathrm{e}}}=E\binom{\psi_{\mathrm{s}}}{\psi_{\mathrm{e}}} \quad \Leftrightarrow \quad\left\{\begin{array}{l}
\left(H_{\mathrm{s}}+\Sigma(E)\right) \psi_{\mathrm{s}}=E \psi_{\mathrm{s}} \\
\psi_{\mathrm{e}}=\left(E-H_{\mathrm{e}}\right)^{-1} H_{\mathrm{es}} \psi_{\mathrm{s}}
\end{array}\right.
$$

where $\Sigma(E)=H_{\mathrm{es}}^{\dagger}\left(E-H_{\mathrm{e}}\right)^{-1} H_{\mathrm{es}}$ is called the self-energy operator.

Somehow related to Feshbach-Schur perturbation method.

Very large or infinite inhomogenous system

- focused models: QM/PCM, QM/MM, QM/MM/PCM, QCM, ...
- (Schur complement-like) self-energy / Green's function methods

Non-equilibrium Green's function (NEGF) for coherent electron transport



$$
\left(\begin{array}{ccc}
z-H_{\mathrm{l}} & -H_{\mathrm{ls}} & 0 \\
-H_{\mathrm{ls}}^{\dagger} & z-H_{\mathrm{s}} & -H_{\mathrm{rs}}^{\dagger} \\
0 & -H_{\mathrm{rs}} & z-H_{r}
\end{array}\right)\left(\begin{array}{ccc}
G_{\mathrm{l}}(z) & G_{\mathrm{ls}}(z) & G_{\mathrm{lr}}(z) \\
G_{\mathrm{sl}}(z) & G_{\mathrm{s}}(z) & G_{\mathrm{sr}}(z) \\
G_{\mathrm{rl}}(z) & G_{\mathrm{rs}}(z) & G_{\mathrm{r}}(z)
\end{array}\right)=\left(\begin{array}{ccc}
I & 0 & 0 \\
0 & I & 0 \\
0 & 0 & I
\end{array}\right)
$$

Very large or infinite inhomogenous system

- focused models: QM/PCM, QM/MM, QM/MM/PCM, QCM, ...
- (Schur complement-like) self-energy / Green's function methods

Non-equilibrium Green's function (NEGF) for coherent electron transport



$$
G_{\mathrm{s}}(z)=\left(z-\left(H_{\mathrm{s}}+\Sigma_{\mathrm{l}}(z)+\Sigma_{\mathrm{r}}(z)\right)\right)^{-1}
$$

with

$$
\Sigma_{\mathrm{l}}(z)=H_{l s}^{\dagger}\left(z-H_{\mathrm{l}}\right)^{-1} H_{\mathrm{ls}} \quad \text { and } \quad \Sigma_{\mathrm{r}}(z)=H_{r s}^{\dagger}\left(z-H_{\mathrm{r}}\right)^{-1} H_{\mathrm{rs}}
$$

Very large or infinite inhomogenous system

- focused models: QM/PCM, QM/MM, QM/MM/PCM, QCM, ...
- (Schur complement-like) self-energy / Green's function methods
- superimposed multiscale models:

- systematic upscaling (Brandt '90s)
- concurrent coupling (Kaxiras et al. '99)
- equation-free method (Kevrekidis et al. '03)
- HMM: heterogeneous multiscale method (E and Engquist '03)
- coarse-grained models (talk by N. Zabaras)



## Conclusion

How can mathematicians help?

- analysis of the models
- rule out or fix models with bad properties,
- establish rigorous connections between models (e.g. at different scales),
- construct suitable approximations/discretizations with controlled accuracy,
- contribution to the design a new generation of codes
- generating numerical results supplemented with error bars,
- optimizing the available computational resources,
- adapted to massively parallel and hybrid architectures.
- knowledge transfer
- between pure mathematics and applied fields,
- between different applied fields.

