Isoperimetric mass and isoperimetric surfaces in AF manifolds

Banff

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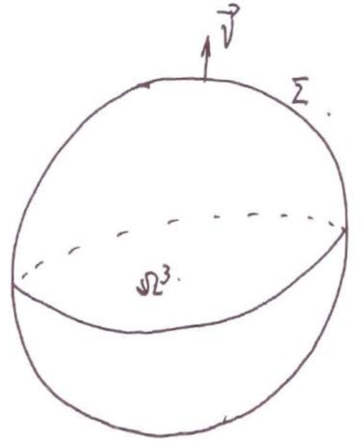
This talk is based on joint work with O.Chodosh, M.Eichmair, H. Yu

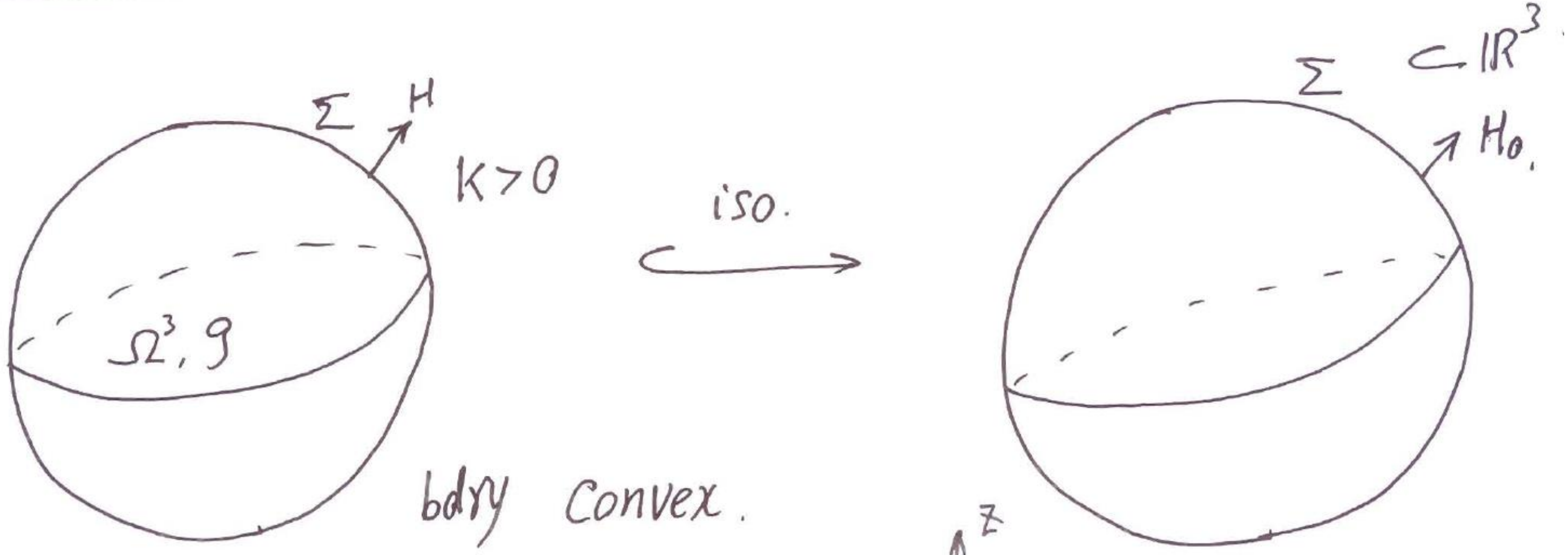
Brief descriptions of quasi local mass

- QL mass : A geometric quantity of 2-dim surface Σ that measures the mass contained in the domain that enclosed by Σ

- QL mass usually depends only on the geometry of Σ, like mean curvature area etc.

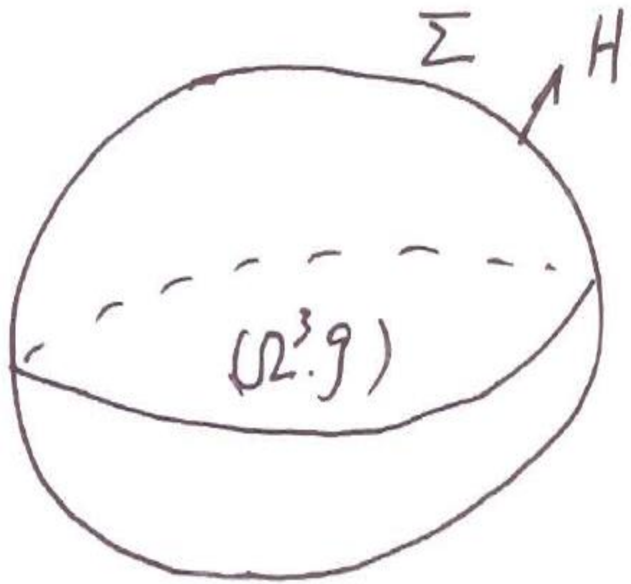


- Brown-York mass



$$M_{BY}(\Sigma) \triangleq \frac{1}{8\pi}\int_{\Sigma}(H_0 - H)\,d\sigma$$

- Hawking Mass.

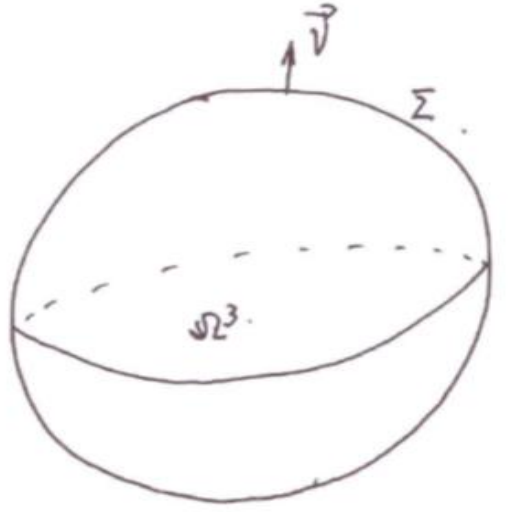


bdry may not convex.

$$m_H(\Sigma) \triangleq \frac{|\Sigma|^{1/2}}{(16\pi)^{3/2}}\left(16\pi - \int_\Sigma H^2 da\right)$$

- QL Isoperimetric mass

$$\mathcal{M}_{ISO}(\Sigma) = \frac{2}{|\Sigma|}(Vol(\Omega) - \frac{|\Sigma|^{\frac{3}{2}}}{6\sqrt{\pi}})$$



Positivity of Brown-York mass (Riemannian Case):

Thm (Shi & Tam, 2002): Suppose (Ω^3, g) is a compact mfd with $K>0$, $H>0$. If $R\geq 0$, then

$$\mathcal{M}_{BY}(\partial\Omega) \triangleq \frac{1}{8\pi}\int_{\Sigma}(H_0 - H)\,d\sigma \geq 0$$

$$\mathcal{M}_{BY}(\partial\Omega) = 0 \quad \text{iff} \quad (\Omega^3, g) \xhookrightarrow{\text{iso}} \mathbb{R}^3.$$

- (M^3, g) is AF if: $M \setminus \mathbf{K} \cong \{x \in R^3 : |x| > \frac{1}{2}\}$ with

$$g_{ij} = \delta_{ij} + \sigma_{ij}, \qquad |x|^{|\alpha|}(\partial^\alpha \sigma_{ij})(x) = O(|x|^{-\tau}), \quad \tau > \frac{1}{2}$$

- Example: Schwarzschild manifold: $(\mathbf{R}^3 \setminus \{o\}, g)$ with

$g_{ij} = (1 + \frac{m}{2r})^4 \delta_{ij}, \quad r = \frac{m}{2}$ is horizon

Thm (Christodoulou, Yau, 1988): (M^3, g) is AF with $R \geq 0$.

If Σ is "round" surface, then

$$m_H(\Sigma) \geq 0.$$

here Σ is round means if 1) Σ is topological sphere.
2) has least area in M among all surfaces in M which enclose the same volume as Σ does.

$$\mathcal{M}_H(\Sigma) \geq 0 \implies H^2 A(\Sigma) \leq 16\pi \implies A'(v)A^{\frac{1}{2}}(v) \leq 4\pi^{\frac{1}{2}}$$

$$\implies (A^{\frac{3}{2}}(v))' \leq 6\pi^{\frac{1}{2}} \implies v - (6\pi^{\frac{1}{2}})^{-1}A^{\frac{3}{2}}(v) \geq 0$$

$$\implies \mathcal{M}_{ISO}(\Sigma) = \frac{2}{|\Sigma|}(Vol(\Omega) - \frac{|\Sigma|^{\frac{3}{2}}}{6\sqrt{\pi}}) \geq 0$$

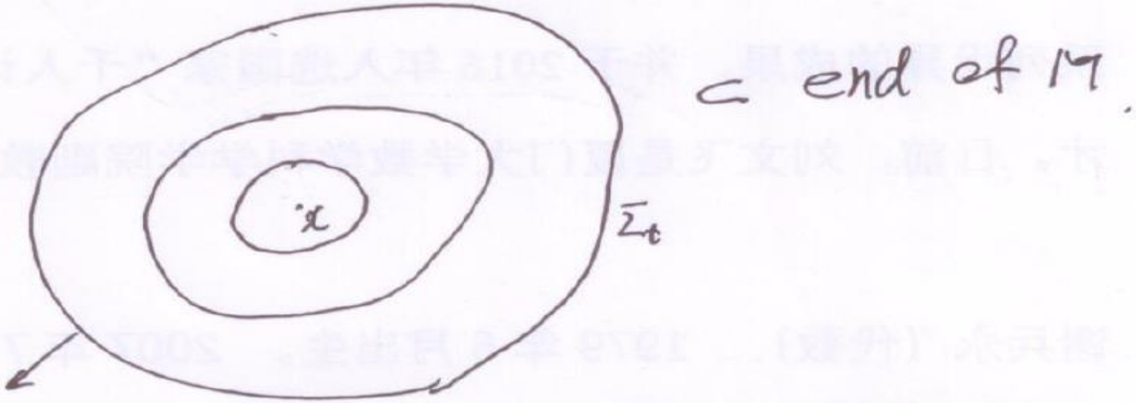
Non negativity of isoperimetric mass

- Theorem (Shi, 2016): Suppose (M^3, g) is an AF manifold with $R \geq 0$, then we have

$$A^{\frac{3}{2}}(v) \leq (6\pi^{\frac{1}{2}}) \int_0^v (1 - (16\pi)^{\frac{1}{2}} B^{-\frac{1}{2}}(t) m(t) dt \leq (6\pi^{\frac{1}{2}}) v$$

$$\Longrightarrow \quad \mathcal{M}_{ISO}(v) = \frac{2}{A(v)}(v - \frac{A(v)^{\frac{3}{2}}}{6\sqrt{\pi}}) \geq 0$$

Moreover. $\mathcal{M}_{ISO}(v) = \frac{2}{A(v)}(v - \frac{A(v)^{\frac{3}{2}}}{6\sqrt{\pi}}) = 0$ iff $(M^3, g) = R^3$



Slice of IMCF. $\Sigma_t = \partial K_t$. $Vol(K_t) = V$.

$$m(V) \triangleq m_H(\Sigma_t). \quad B(t) \triangleq Area(\Sigma_t).$$

$$A^{\frac{3}{2}}(V) = (6\pi^{\frac{1}{2}})V_0 \text{ for some } V_0 > 0 \Longrightarrow m(V) = 0 \quad \forall V \in (0, V_0).$$

$$\Longrightarrow |Ric|(x) = 0 \text{ for all } x \text{ near the infinity of } (M^3, g)$$

$$\Longrightarrow m_{ADM}(M^3, g) = 0. \Longrightarrow (M^3, g) = \mathbb{R}^3.$$

Applications

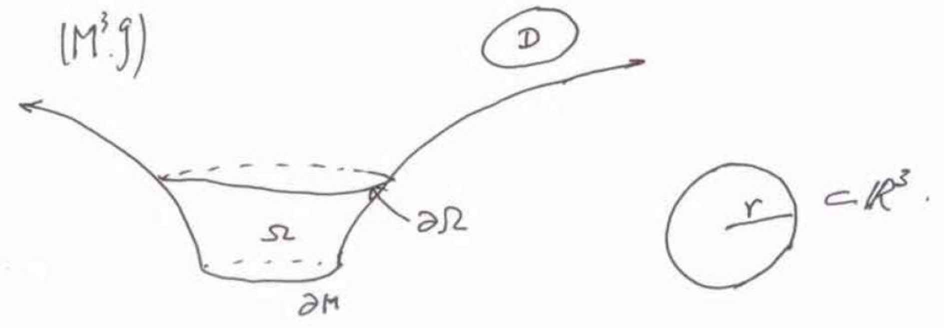
- Theorem(CCE, 2016) Suppose (M^3, g) is an AF manifold with $R \geq 0$ and positive mass and containing horizons, Then (M^3, g) admits isoperimetric region for every volume.

- Very brief Proof:

Proposition 4.2. *Given $V > 0$, there exists an isoperimetric region $\Omega \subset \hat{M}$ containing the horizon and a radius $r \in [0, \infty)$ such that $\mathcal{L}^3_g(\Omega) + \frac{4\pi r^3}{3} = V$ and such that $\mathcal{H}^2_g(\partial\Omega) + 4\pi r^2 = A_g(V)$. If $r > 0$ and $\mathcal{L}^3_g(\Omega) > 0$, then the mean curvature of $\partial\Omega$ equals $\frac{2}{r}$.*

$$A_g(v) = \inf\{\mathcal{H}^2(\partial\Omega) : \mathcal{L}^3(\Omega) = v,\ \Omega \subset M\}$$

$$\mathcal{M}_{ISO}(4\pi r^3/3) > 0 \Longrightarrow$$

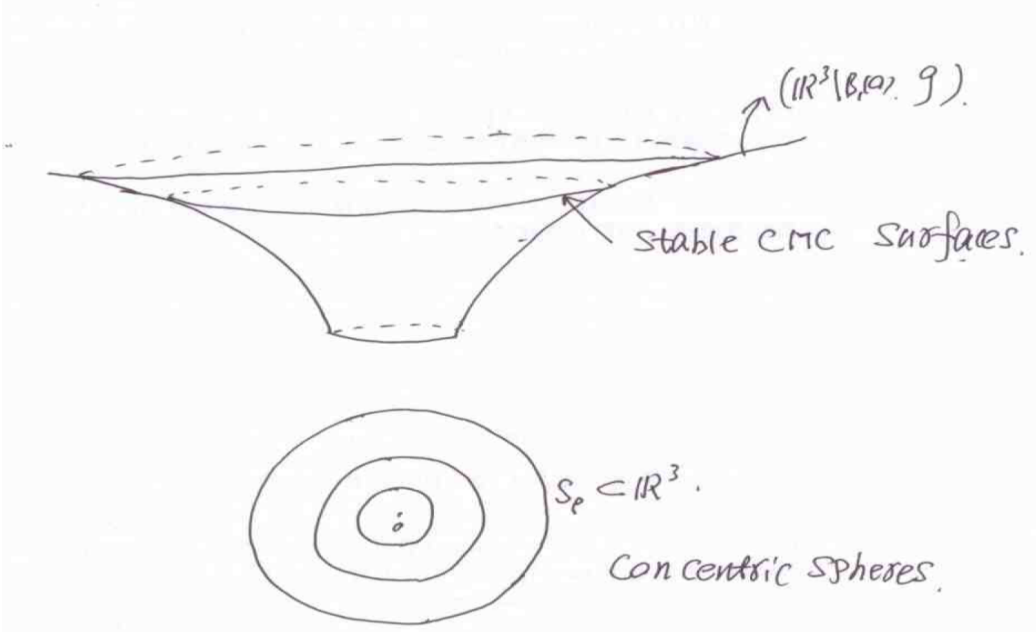


$$\left.\begin{aligned} \mathcal{L}^3_g(D) &= \tfrac{4}{3}\pi r^3 \\ A_g(\partial D) &< 4\pi r^2 \end{aligned}\right\} \Longrightarrow \mathcal{H}^2_g(\partial\Omega) + A_g(\partial D) < A_g(V)$$

$$\Longrightarrow \quad r = 0$$

- Huisken-Yau (1997): Canonical CMC foliation in AF ends

$$(\mathbf{R}^3 \setminus B_1(o), g),\ \ g_{ij} = (1 + \frac{2m}{r})\delta_{ij} + O(r^{-2})$$



- M. Eichmair and J. Metzger, On large volume preserving stable CMC surfaces in initial data sets, J.Differential Geom. 91 (2012), no. 1, 81-102.
- M. Eichmair and J. Metzger: Large isoperimetric surfaces in initial data sets, J. Differential Geom. 94 (2013), no. 1, 159-186.
- L.-H. Huang, Foliations by stable spheres with constant mean curvature for isolated systems with general asymptotics, Comm. Math. Phys. 300 (2010), no. 2, 331-373
- C. Nerz, Foliations by stable spheres with constant mean curvature for isolated systems without asymptotic symmetry, Calc. Var. Partial Differential Equations 54 (2015), no. 2, 1911-1946.

Isoperimetric mass and ADM mass

- Theorem (Huisken, 2006): Suppose (M^3, g) is an AF manifold with $R \geq 0$, let $A(v)$ be the isoperimetric profile on (M^3, g) , then

$$\lim_{v \to \infty} \mathcal{M}_{ISO}(v) = \mathcal{M}_{ADM}(M^3, g)$$

$$\mathcal{M}_{ISO}(v) = \frac{2}{A(v)}(v - \frac{A(v)^{\frac{3}{2}}}{6\sqrt{\pi}})$$

- Fan-Miao-Shi-Tam’ s result

$$\mathcal{M}_{ISO}(\Sigma_r) = \frac{2}{|\Sigma_r|}(Vol(\Omega_r) - \frac{|\Sigma_r|^{\frac{3}{2}}}{6\sqrt{\pi}}) \to \mathcal{M}_{ADM}(M, g)$$

$$\Longrightarrow \qquad \lim_{v\to\infty} \mathcal{M}_{ISO}(v) \geq \mathcal{M}_{ADM}(M, g)$$

- On other hand, we have

$$\mathcal{M}_H(v) = \frac{|\Sigma_v|^{\frac{1}{2}}}{16\pi^{\frac{3}{2}}}(16\pi - \int_{\Sigma_v} H^2) \leq \mathcal{M}_{ADM}(M,g)$$

$$A(v) = |\Sigma_v|$$

$$H \leq A'_-(v)$$

$\Longrightarrow$

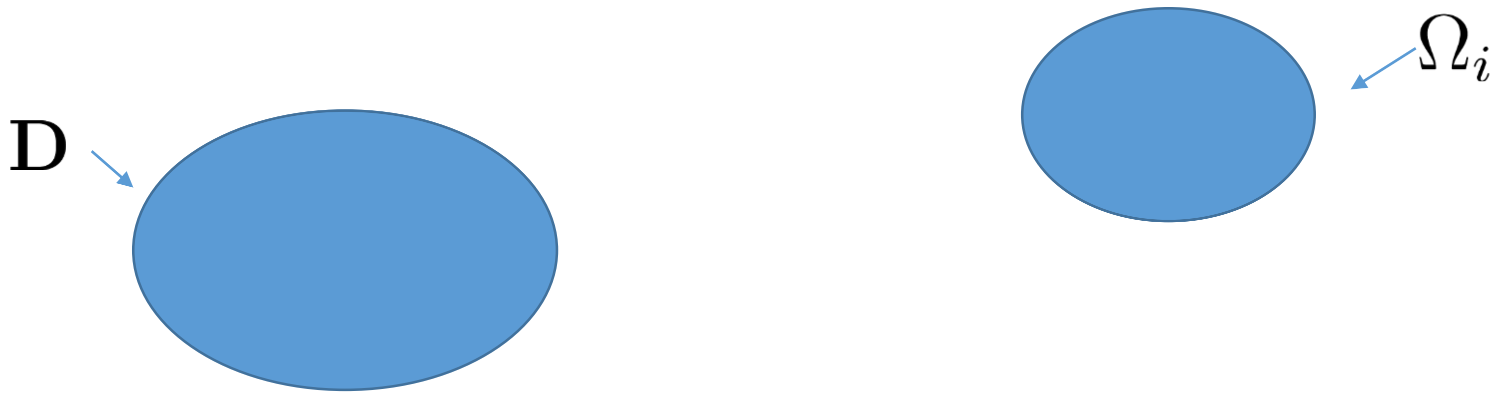
$$16\pi A^{\frac{1}{2}}(v) - (A'_-(v))^2(v)A^{\frac{3}{2}}(v) \leq 16\pi^{\frac{3}{2}}\mathcal{M}_{ADM}$$

$$\Longrightarrow \quad \mathcal{M}_{ISO}(v) \leq \mathcal{M}_{ADM}, \qquad v >> 1 \quad \Longrightarrow$$

$$\mathcal{M}_{ISO}(v) \to \mathcal{M}_{ADM}, \qquad v \to \infty$$

No drift off to infinity of isoperimetric regions

- Let $\{\Omega_i\}$ be a sequence of isoperimetric regions in M, we say $\{\Omega_i\}$ drift off to infinity, if for any compact set $\mathbf{D} \subset M$, Ω_i is disjoint with $\mathbf{D}$, for sufficiently large i.



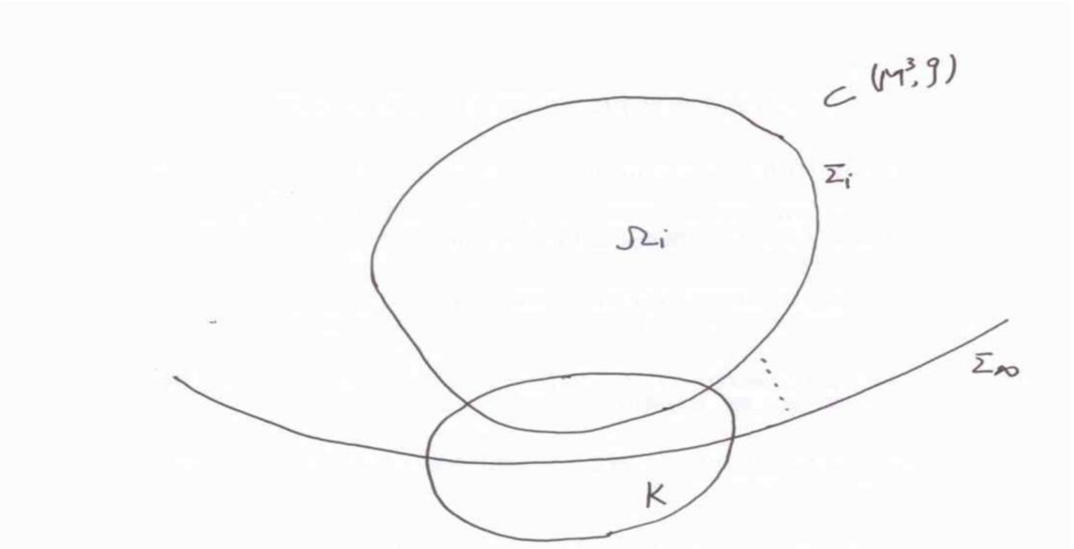
Theorem(CCE 2016). Isoperimetric regions cannot pass through a fixed compact set

- Suppose $\{\Sigma_i\}$ always pass through a fixed compact set $\mathbf{K}$
- Σ_∞ is a complete, noncompact and properly embedding area-minimizing Surface in (M^3, g)
- A result due to A.Carlotto,

O.Chodosh, M.Eichmair

$\Longrightarrow$ No such area-minimizing

Surface in (M^3, g)



- By a result due to CCE, $\{\Omega_i\}$ cannot always pass through a fixed compact set $\mathbf{D}$.

- $\{\Omega_i\}$ cannot drift off to the infinity

- $\Longrightarrow$ $\{\Omega_i\}$ is exhausting .

Proof of no drift

- Key observation: If $\{\Omega_i\}$ drift off to the infinity, then $\mathcal{M}_{ISO}(\Omega_i)$ tends to zero, contradiction to $\mathcal{M}_{ISO}(\Omega_i) \geq \mathcal{M}_{ADM}(M, g) > 0$.

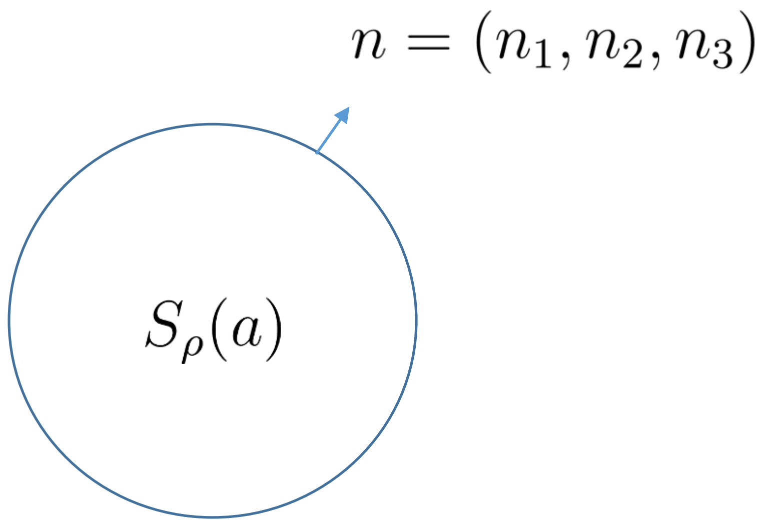
- How to estimate $\mathcal{M}_{ISO}(\Omega_i)$?

- Key observation: If a Ω_i drifts off to the infinity, then the boundary looks like an Euclidean sphere

- What happen if a Euclidean sphere slides off to the infinity of an AF manifold ?

- We can get its area and volume expansion in an explicit way.

$$n^j(x) = \frac{x^j - a^j}{|x - a|}.$$



Lemma 4.1. *Let $\rho > 0$ and $a \in \mathbb{R}^3$ with $|a| - \rho > 1$. Consider a Euclidean coordinate sphere $S_\rho(a) = \{x \in \mathbb{R}^3 : |x - a| = \rho\}$. Then*

$$\sum_{i,j=1}^{3} \int_{S_\rho(a)} (\partial_i \sigma_{ij} - \partial_j \sigma_{ii})\, \partial_j |x - a| = O\left(\frac{1}{|a| - \rho}\right) \tag{13}$$

where integration is with respect to the Euclidean background metric.

$$
\begin{aligned}
\int_{B_\rho(a)} R &= \sum_{i,j=1}^{3} \int_{B_\rho(a)} (\partial_i \partial_j g_{ij} - \partial_j \partial_j g_{ii}) + O \int_{B_\rho(a)} \frac{1}{|x|^4} \\
&= \sum_{i,j=1}^{3} \int_{S_\rho(a)} (\partial_i \sigma_{ij} - \partial_j \sigma_{ii}) \partial_j |x - a| + O \int_{B_\rho(a)} \frac{1}{|x|^4}
\end{aligned}
$$

Proposition 4.2. *Let $S_\rho(a) = \{x \in \mathbb{R}^3 : |x - a| = \rho\}$ be Euclidean coordinate spheres with $\rho \to \infty$ and $|a| - \rho \to \infty$. Then*

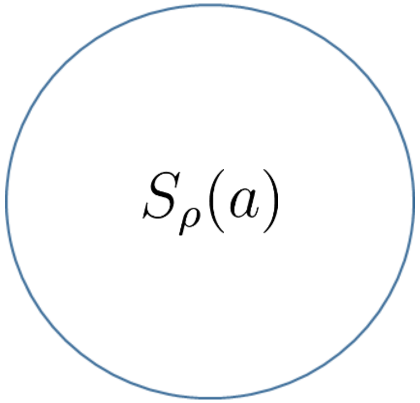
$$\operatorname{area}(S_\rho(a)) = 4\pi\rho^2 + \frac{1}{2}\int_{S_\rho(a)} (\delta^{ij} - n^i n^j)\sigma_{ij} + o(\rho) \tag{14}$$

and

$$\operatorname{vol}(B_\rho(a)) = \frac{4\pi\rho^3}{3} + \frac{\rho}{4}\int_{S_\rho(a)} (\delta^{ij} - n^i n^j)\sigma_{ij} + o(\rho^2). \tag{15}$$

$$\Longrightarrow \qquad \mathcal{M}_{ISO}(S_\rho(a)) = o(1)$$

- Proof of Lemma 4.2 :



$$d\mu = \left(1 + \frac{1}{2}(\delta^{ij} - n^i n^j)\sigma_{ij} + O(|x|^{-2})\right) d\overline{\mu}.$$

For every $t \in [1, \rho]$, we have that

$$\text{area}(S_t(a)) = 4\pi t^2 + \frac{1}{2}\int_{S_t(a)} (\delta^{ij} - n^i n^j)\sigma_{ij} + O\int_{S_t(a)} \frac{1}{|x|^2}. \tag{16}$$

Then

$$\int_{S_t(a)} \frac{1}{|x|^2} = \int_0^{2\pi}\int_0^{\pi} \frac{t^2 \sin\phi}{|a|^2 + t^2 - 2|a|t\cos\phi} d\phi d\theta = \frac{\pi t}{|a|}\log\left(\frac{|a|+\rho}{|a|-\rho}\right) = o(t) \tag{17}$$

$$
\begin{aligned}
\partial_t \operatorname{area}(S_t(a)) =& 8\pi t + \frac{1}{2}\int_{S_t(a)} n^k(\delta^{ij} - n^i n^j)\partial_k \sigma_{ij} + \frac{1}{t}\int_{S_t(a)} (\delta^{ij} - n^i n^j)\sigma_{ij} \\
&+ O\int_{S_t(a)} \frac{1}{|x|^3} + \frac{1}{t} O\int_{S_t(a)} \frac{1}{|x|^2}.
\end{aligned}
$$

By the first variation formula, we get :

$$
\begin{aligned}
\partial_t \operatorname{area}(S_t(a)) =& 8\pi t + \frac{1}{2}\int_{S_t(a)} \delta^{ik} n^j(\partial_j \sigma_{ik} - \partial_i \sigma_{kj}) + \frac{1}{t}\int_{S_t(a)} n^i n^j \sigma_{ij} \\
&+ \frac{1}{2t}\int_{S_t(a)} (\delta^{ij} - n^i n^j)\sigma_{ij} + \frac{1}{t} O\int_{S_t(a)} \frac{1}{|x|^2} + O\int_{S_t(a)} \frac{1}{|x|^3}.
\end{aligned}
$$

Note that:

$$\operatorname{area}(S_t(a)) = 4\pi t^2 + \frac{1}{2}\int_{S_t(a)} (\delta^{ij} - n^i n^j)\sigma_{ij} + O\int_{S_t(a)} \frac{1}{|x|^2}.$$

$$\begin{aligned}\partial_t \operatorname{area}(S_t(a)) =& 8\pi t + \frac{1}{2}\int_{S_t(a)} \delta^{ik} n^j (\partial_j \sigma_{ik} - \partial_i \sigma_{kj}) + \frac{1}{t}\int_{S_t(a)} n^i n^j \sigma_{ij} \\ &+ \frac{1}{2t}\int_{S_t(a)} (\delta^{ij} - n^i n^j)\sigma_{ij} + \frac{1}{t} O\int_{S_t(a)} \frac{1}{|x|^2} + O\int_{S_t(a)} \frac{1}{|x|^3}.\end{aligned}$$

$$\begin{aligned}\partial_t \operatorname{area}(S_t(a)) =& \frac{\operatorname{area}(S_t(a))}{t} + 4\pi t + \frac{1}{t}\int_{S_t(a)} n^i n^j \sigma_{ij} + O\int_{S_t(a)} \frac{1}{|x|^3} \\ &+ \frac{1}{t} O\int_{S_t(a)} \frac{1}{|x|^2} + o(1).\end{aligned}$$

- By the co-area formula, we have:

$$\partial_t \operatorname{vol}(B_t(a)) = \int_{S_t(a)} \frac{t\, d\mu}{\sqrt{g_{ij}(x^i - a^i)(x^j - a^j)}}$$
$$= \operatorname{area}(S_t(a)) + \frac{1}{2}\int_{S_t(a)} n^i n^j \sigma_{ij} + O\int_{S_t(a)} \frac{1}{|x|^2}$$

- We get

$$\partial_t(t\,\operatorname{area}(S_t(a))) = 4\pi t^2 + 2\partial_t \operatorname{vol}(B_t(a)) + t\,O\int_{S_t(a)} \frac{1}{|x|^3} + O\int_{S_t(a)} \frac{1}{|x|^2} + o(t).$$

$$\int_1^\rho \int_{S_t(a)} \frac{t}{|x|^3} dt = o(\rho^2). \qquad \int_1^\rho \int_{S_t(a)} \frac{1}{|x|^2} dt = o(\rho^2).$$

$\Longrightarrow$

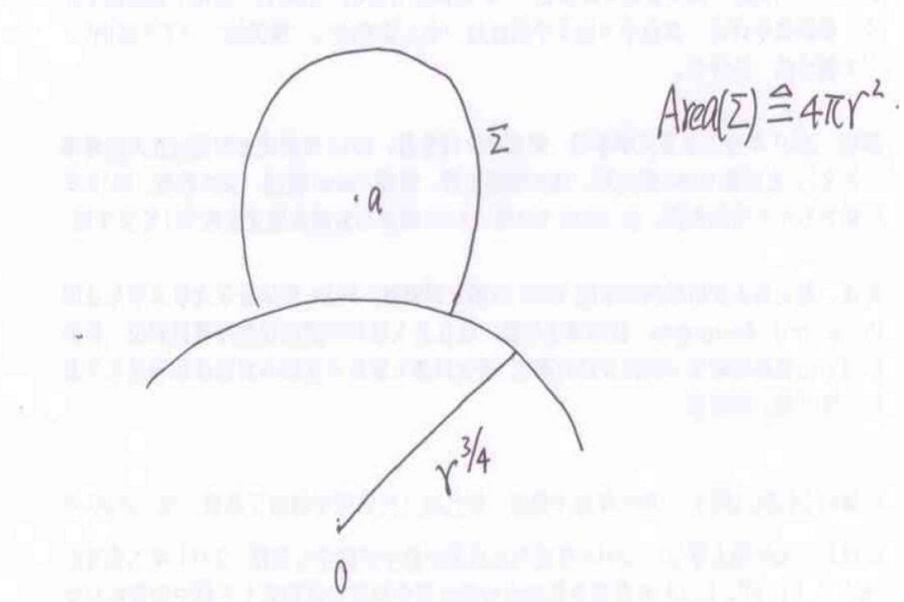
- $$\mathrm{vol}(B_\rho(a)) = \frac{4\pi\rho^3}{3} + \frac{\rho}{4} \int_{S_\rho(a)} (\delta^{ij} - n^i n^j)\sigma_{ij} + o(\rho^2).$$

- $$\mathrm{area}(S_\rho(a)) = 4\pi\rho^2 + \frac{1}{2} \int_{S_\rho(a)} (\delta^{ij} - n^i n^j)\sigma_{ij} + o(\rho)$$

Estimate for isoperimetric surfaces

- Observation: Isoperimetric surfaces in AF manifolds look like Euclidean spheres outside a large compact set.

- $$0 \leq \mathcal{M}_H(\Sigma) = \frac{|\Sigma|^{\frac{1}{2}}}{(16\pi)^{\frac{3}{2}}}\left(16\pi - \int_\Sigma H^2\right) \leq m$$



$$\Longrightarrow \quad \frac{2}{r} - \frac{8m}{r^2} + O(r^{-3}) \leq H \leq \frac{2}{r}$$

$$2|\mathring{h}|^3 + \Delta|\mathring{h}| \geq -c(H|\mathring{h}|^2 + H|Rm| + |\mathring{h}||Rm| + |\nabla Rm|)$$

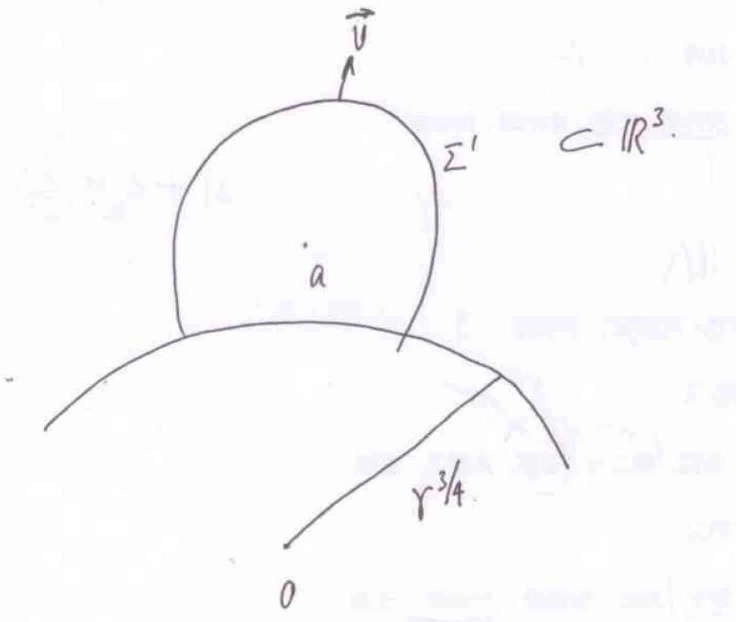
$$\Longrightarrow \quad |\mathring{h}(x)| \leq c\, r^{-\frac{5}{4}} \qquad \textit{for all } x \in \Sigma \textit{ such that } |x| \geq r^{3/4}.$$

•

Lemma 4.3. *There is $a \in \mathbb{R}^3$ with $|a| > r + r^{3/4}$ such that*

$$\left|\overline{\nu}(x) - \frac{x-a}{r}\right| = O(r^{-1/4}) \tag{27}$$

for all $x \in \Sigma'$.



Σ' is a Connected Component of $\Sigma \setminus B_{r^{3/4}}$

$$\mathrm{area}(\Sigma) - \overline{\mathrm{area}}(\Sigma) = \mathrm{area}(S_r(a)) - 4\pi r^2 + o(r)$$

$$\mathrm{vol}(\Omega) - \overline{\mathrm{vol}}(\Omega) = \mathrm{vol}(B_r(a)) - \frac{4\pi r^3}{3} + o(r^2).$$

$$\mathrm{area}(S_\rho(a)) = 4\pi\rho^2 + \frac{1}{2}\int_{S_\rho(a)} (\delta^{ij} - n^i n^j)\sigma_{ij} + o(\rho)$$

$$\mathrm{vol}(B_\rho(a)) = \frac{4\pi\rho^3}{3} + \frac{\rho}{4}\int_{S_\rho(a)} (\delta^{ij} - n^i n^j)\sigma_{ij} + o(\rho^2).$$

$$\mathrm{vol}(\Omega) - \frac{\mathrm{area}(\Sigma)^{3/2}}{6\sqrt{\pi}} = \overline{\mathrm{vol}}(\Omega) - \frac{\overline{\mathrm{area}}(\Sigma)^{3/2}}{6\sqrt{\pi}} + o(r^2) \leq o(r^2).$$

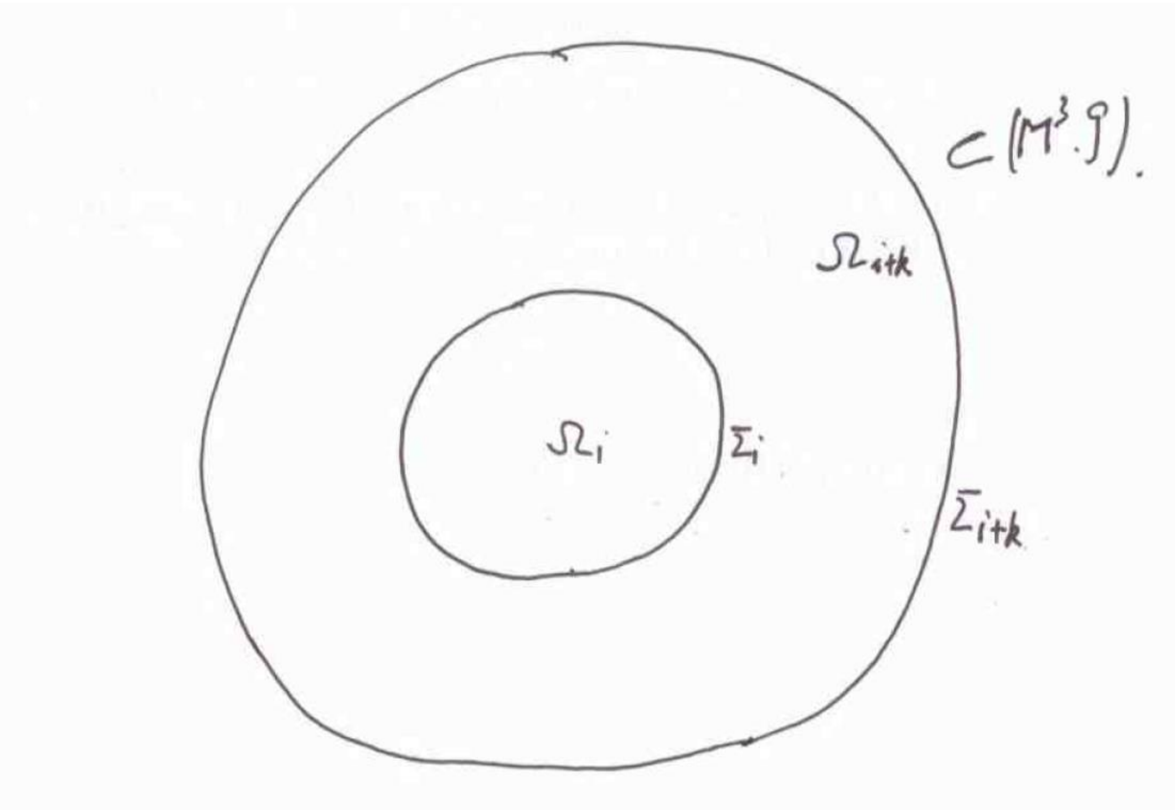
- $\Longrightarrow$ If $\sum$ is an isoperimetric surface with the topology of sphere and drift off to the infinity, then $\mathcal{M}_{ISO}(\Sigma) = o(1)$

- Conclusion: Suppose (M^3, g) is an AF manifold with $R \geq 0$ $\{\Sigma_i\}$ is a sequence of isoperimetric surfaces with the topology of spheres, and areas approach to infinity, then $\{\Sigma_i\}$ cannot drift off to the infinity.

- Σ_i separates a compact set in (M^3, g) from the infinity for i large enough.

⇨

- Conclusion: Σ_i is a leaf of canonical foliation in (M^3, g)



Canonical foliations in AF manifolds and isoperimetric surfaces with large enclosed volume

- Conjecture(Bray, 1998): The volume-preserving stable CMC surfaces in the end of AF manifold (M^3, g) are isoperimetric surfaces.

- In 2013, M.Eichmair & J. Metzger proved above conjecture in the case that (M^3, g) is asymptotically Schwarzchild of positive mass.

- Some references for uniqueness of stable CMC surfaces in the end of AF manifolds

- R. Ye, Foliation by constant mean curvature spheres on asymptotically flat manifolds Geometric analysis and the calculus of variations, Int. Press, Cambridge,MA, 1996, pp. 369 - 383, MR 1449417
- G. Huisken and S.-T. Yau, Definition of center of mass for isolated physical systems and unique foliations by stable spheres with constant mean curvature, Invent. Math. 124 (1996), no. 1-3, 281-311. MR 1369419
- J. Qing and G. Tian, On the uniqueness of the foliation of spheres of constant mean curvature in asymptotically at 3-manifolds, J. Amer. Math. Soc. 20 (2007), no. 4, 1091-1110. MR 2328717
- L-H Huang, Foliations by stable spheres with constant mean curvature for isolated systems with general asymptotics, Comm. Math. Phys. 300 (2010), no. 2, 331{373. MR 2728728 (2012a:53045)
- C. Nerz, Foliations by stable spheres with constant mean curvature for isolated systems without asymptotic symmetry, Calc. Var. Partial Differential Equations 54 (2015), no. 2, 1911-1946. MR 3396437
- S. Ma, On the radius pinching estimate and uniqueness of the CMC foliation in asymptotically at 3-manifolds, Adv. Math. 288 (2016), 942-984. MR MR3436403

- Theorem(CESY, 2016): Suppose (M^3, g) is an AF manifold with $R \geq 0$, and positive mass. There is $V_0 > 0$ with the following property. Let $V \geq V_0$ there is a unique region $\Omega(V) \in R_V$ such that $area(\partial\Omega_V) \leq area(\partial\Omega)$, for all $\Omega \in R_V$. The boundary of Ω_V consist of ∂M and a leaf of the canonical foliation of the end of (M^3, g) .

$$R_V = \{\Omega : \Omega \subset M \text{ is a compact region with } \partial M \subset \partial\Omega,\ Vol(\Omega) = V\}$$

- Proof of Theorem: only consider the case $\tau = 1$
- Step 1: All leaves of canonical foliation are isoperimetric surfaces

Consider the canonical foliation: $\{\Sigma^H = (\partial\Omega^H) \setminus \partial M : H \in (0, H_0)\}$

$$J = \{\mathrm{vol}(\Omega^H) : H \in (0, H_0) \text{ and } \Omega^H \text{ is an isoperimetric region}\}$$

sed

- Claim: J is unbounded.

- Recall: Suppose (M^3, g) is an AF manifold with $R \geq 0$
  $\{\Sigma_i\}$ is a sequence of isoperimetric surfaces with the topology of spheres, and areas approach to infinity, then Σ_i is a leaf of canonical foliation.

J is bounded $\Longrightarrow$ there is $V_0 > 0$, for all isoperimetric surfaces Σ_V with enclosed volume $V \geq V_0$, $g(\Sigma_V) \geq 1$

- $\Longrightarrow \quad \int_{\Sigma_V} H^2 \geq 16\pi + \delta_0$

$\Longrightarrow \quad 16\pi - A'(V)^2 A(V) \leq 0, \quad \text{for all } V \geq V_0$

$\Longrightarrow \quad \limsup_{V \to \infty} \mathcal{M}_{ISO}(\Omega(V)) \leq 0$

contradicts Fan-Miao- Shi-Tam' s estimate .

- $J \cap [V_0, \infty)$ is connected when V_0 is large enough.

Suppose not $\Longrightarrow$ $(L_i, R_i) \subset R \setminus J \quad L_i \in J, \quad R_i \in J$

$\Longrightarrow$ $V_i \in (L_i, R_i), \quad A'(V_i)$ exist, $\quad V_i \searrow L_i$

Let Ω_i be the isoperimetric region with volume V_i , and $\Sigma_i = \partial\Omega_i$

$\Longrightarrow$ $g(\Sigma_i) \geq 1$

$$A'(V_i) = H_i.$$

$$16\pi - A'(V_i)^2 A(V_i) \leq 16\pi - H_i^2 \text{area}(\Sigma_i) \leq 0.$$

$$\lim_{V \searrow L_i} A'^+(V) \leq A'^+(L_i).$$

$$\Longrightarrow \quad \sqrt{\frac{A(L_i)}{16\pi}} \left(1 - \frac{A'^+(L_i)^2 A(L_i)}{16\pi}\right) \leq \lim_{V \searrow L_i} \sqrt{\frac{A(V)}{16\pi}} \left(1 - \frac{A'^+(V)^2 A(V)}{16\pi}\right) \leq 0$$

Let $\Sigma^{H(L_i)}$ be the leaf of the canonical foliation corresponding to volume L_i. Recall that $\Sigma^{H(L_i)}$ has least area for the volume it encloses since $L_i \in J$. In particular,

$$\text{area}(\Sigma^{H(L_i)}) = A(L_i) \qquad \text{and} \qquad A'^{+}(L_i) \leq H(L_i) \leq A'^{-}(L_i)$$

so that

$$\sqrt{\frac{\text{area}(\Sigma^{H(L_i)})}{16\pi}}\left(1 - \frac{H(L_i)^2\,\text{area}(\Sigma^{H(L_i)})}{16\pi}\right) \leq \sqrt{\frac{A(L_i)}{16\pi}}\left(1 - \frac{A'^{+}(L_i)^2 A(L_i)}{16\pi}\right).$$

$$0 < m_{ADM} = \lim_{i\to\infty} \sqrt{\frac{\text{area}(\Sigma^{H(L_i)})}{16\pi}} \left(1 - \frac{H(L_i)^2 \ \text{area}(\Sigma^{H(L_i)})}{16\pi}\right)$$

$$\Longrightarrow \quad \le \liminf_{i\to\infty} \sqrt{\frac{A(L_i)}{16\pi}} \left(1 - \frac{A'^{+}(L_i)^2 A(L_i)}{16\pi}\right)$$

$$\le 0.$$

$\Longrightarrow$ $g(\Sigma_i) = 0$ $\Longrightarrow$ $V_i \in J$

$\Longrightarrow$ $J \cap [V_0, \infty)$ is connected

- $A(V)$ is smooth for $V \geq V_0$

$J \cap [V_0, \infty)$ is connected $\Longrightarrow$ $J = (V_0, \infty)$

$\Longrightarrow$ $A(V)$ is smooth for $V \geq V_0$

$\Longrightarrow$ Hawking mass for each isoperimetric surface with large enclosed volume is positive

$\Longrightarrow$ all isoperimetric surfaces with large enclosed volumes are topological spheres, hence are unique and are leaves of canonical foliation.

- Example(A. Carlotto and R.Schoen) There is an AF Riemannian metric $g = g_{ij}dx_i dx_j$ on $\mathbf{R}^3$ that has non-negative scalar curvature and positive mass such that $g_{ij} = \delta_{ij}$ on $\mathbf{R}^2 \times (0, \infty)$

- There is no uniqueness for stable CMC surfaces in an AF manifold with asymptotical order $\tau < 1$