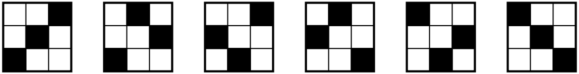
Sorting with C-machines: Enumerative and Analytic Aspects

Jay Pantone

Dartmouth College

Hanover, NH



BIRS Workshop in Analytic and Probabilistic Combinatorics

October 26, 2016

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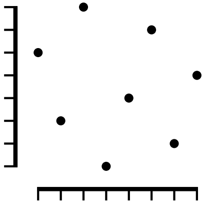

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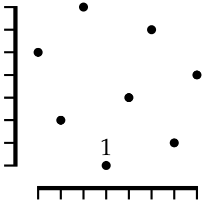

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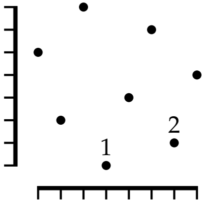

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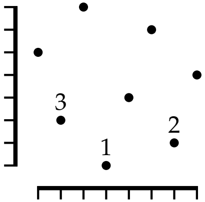

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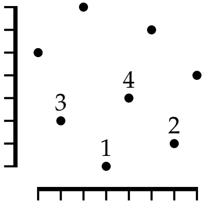

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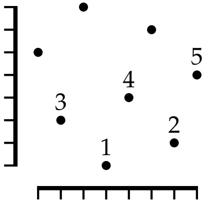

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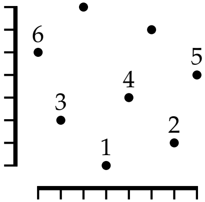
- *Example:* 63814725 $\iff$ 

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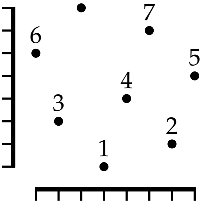

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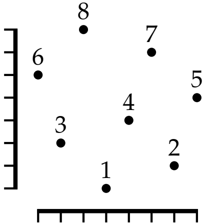

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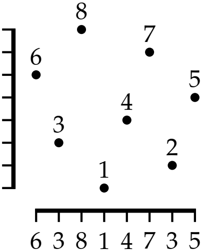

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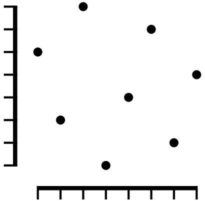

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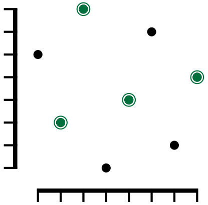

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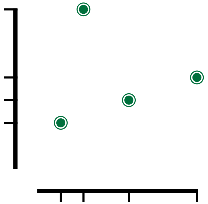

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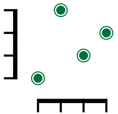

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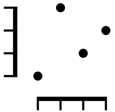

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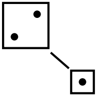

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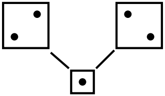

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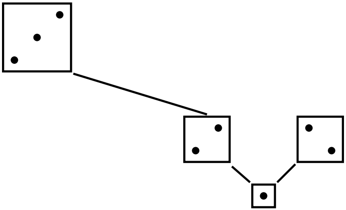

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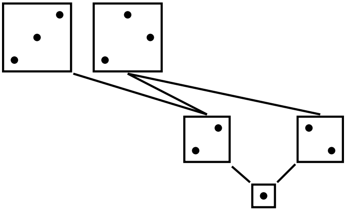

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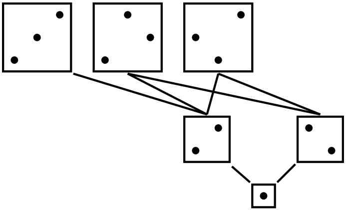

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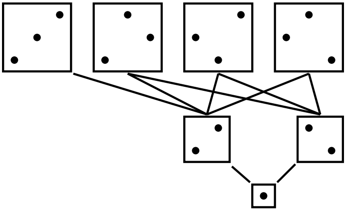

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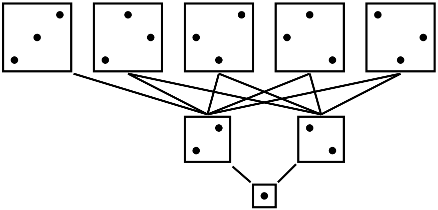

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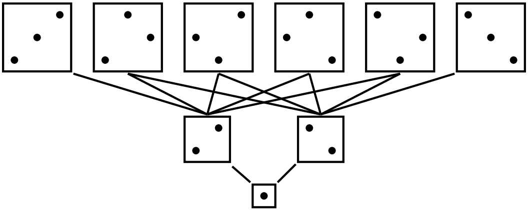

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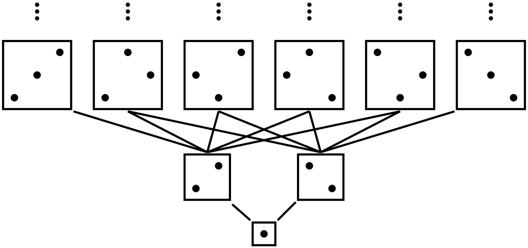

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The class with basis B is denoted $\mathrm{Av}(B)$.

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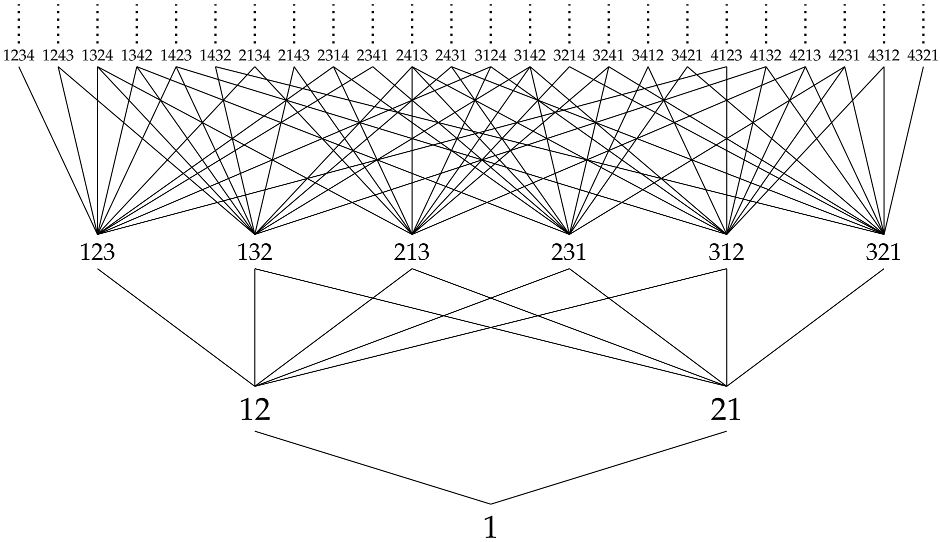

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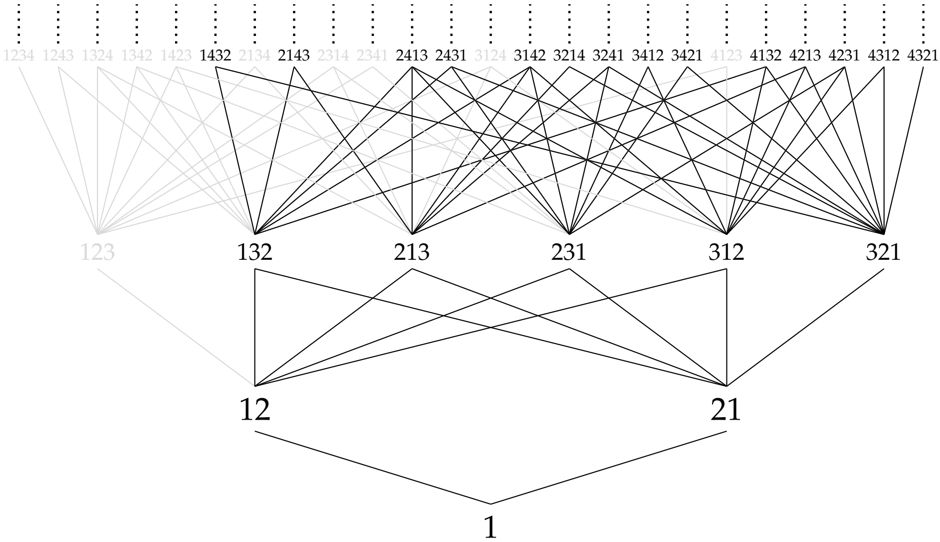

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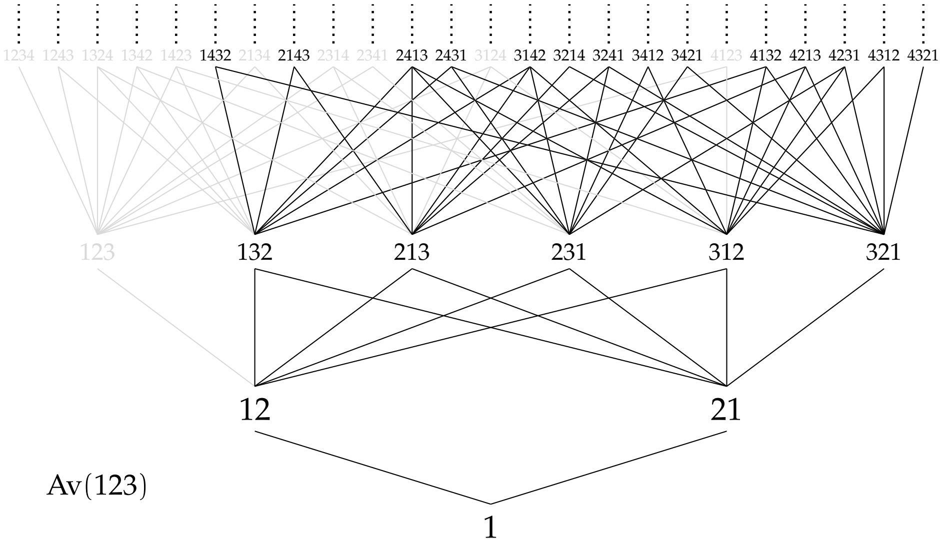
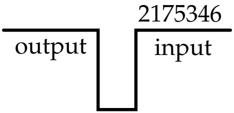


Sorting with a Stack

The permutation 2175346 can be sorted by a stack.



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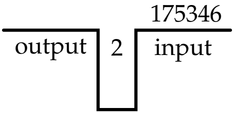

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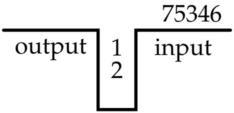

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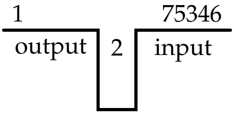

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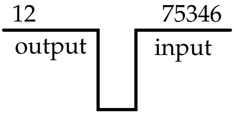

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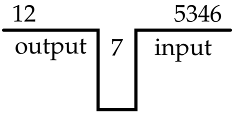

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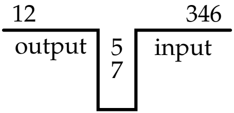

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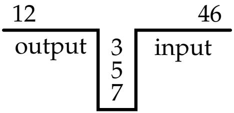

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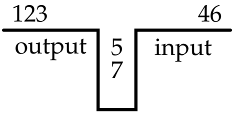

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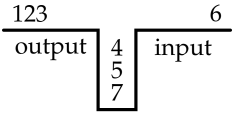

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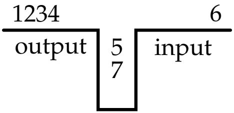

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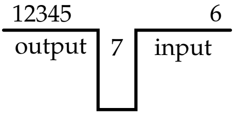

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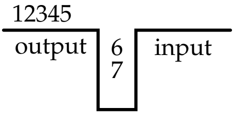

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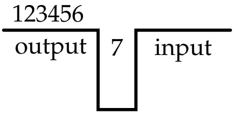

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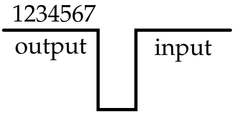

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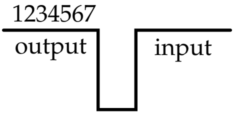

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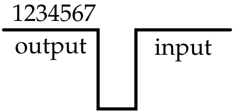

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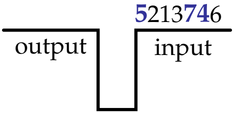

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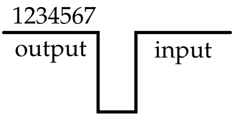

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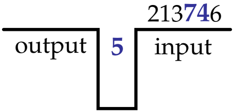

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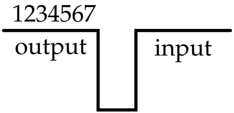

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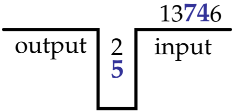

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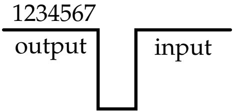

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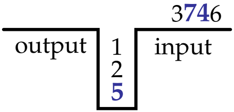

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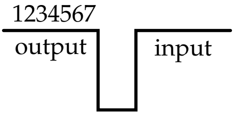

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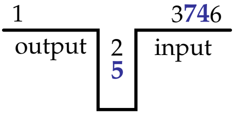

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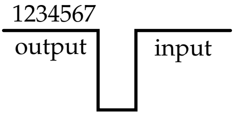

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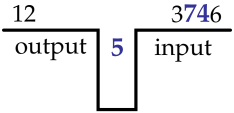

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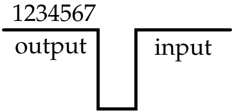
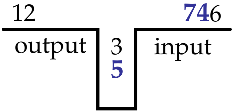


Not all permutations can be sorted by a stack.



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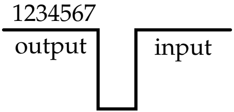

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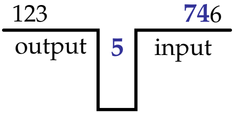

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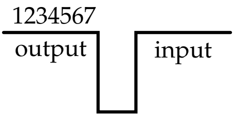

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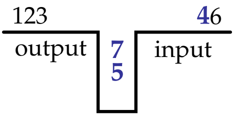

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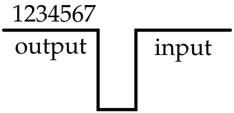

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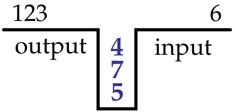

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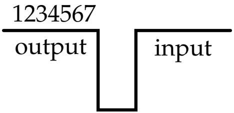

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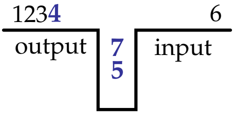

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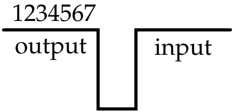

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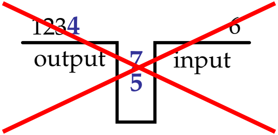

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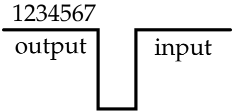

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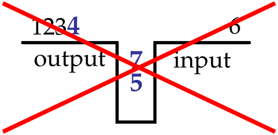
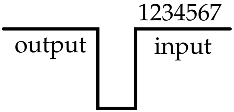


Permutations containing 231 cannot be sorted by a stack. In fact, the permutations that are stack-sortable are exactly those in Av(231).

Generating with a Stack

The permutation 2156473 can be generated by a stack.



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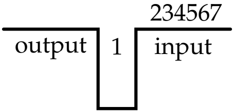

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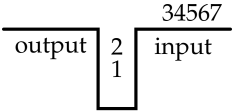

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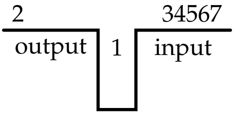

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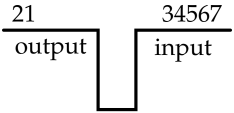

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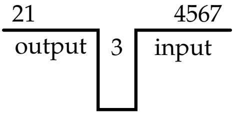

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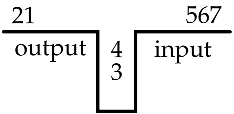

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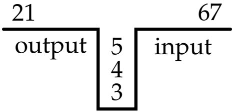

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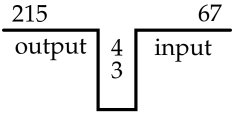

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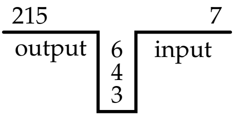

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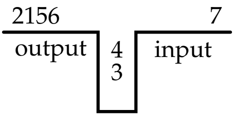

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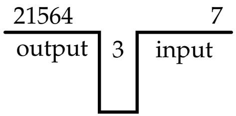

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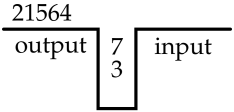

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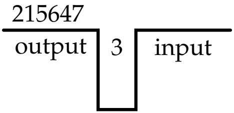

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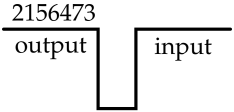

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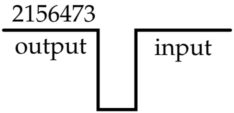

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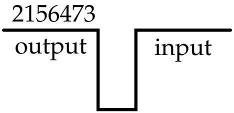

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A stack can generate the permutations in $\mathrm{Av}(231^{-1}) = \mathrm{Av}(312)$.

GENERALIZING STACKS

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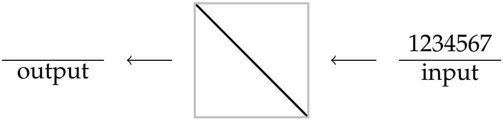

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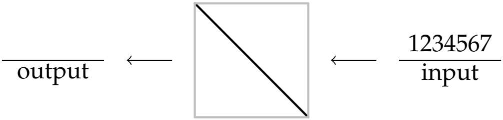

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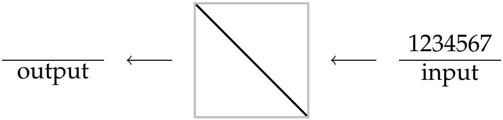

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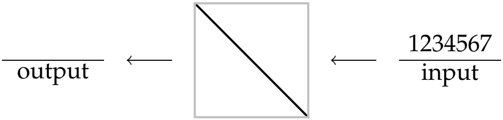

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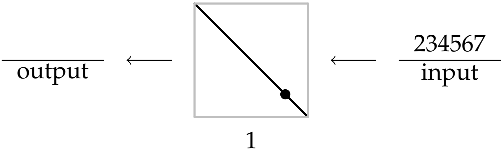

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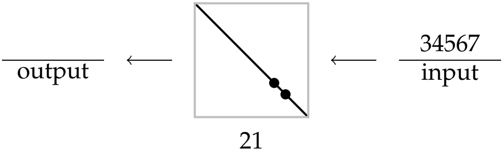

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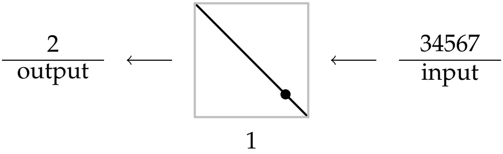

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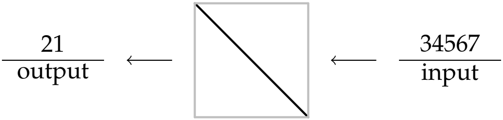

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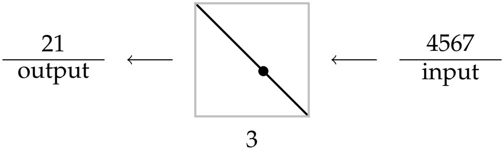

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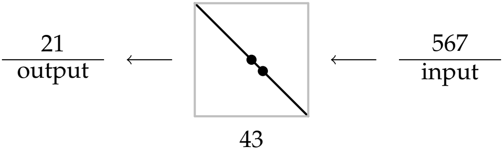

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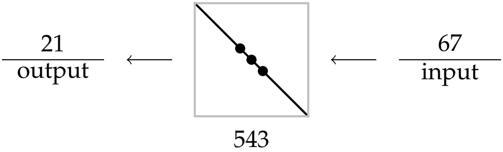

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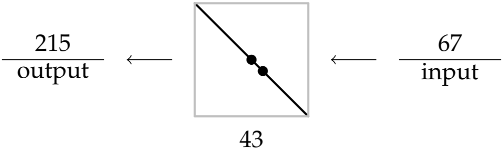

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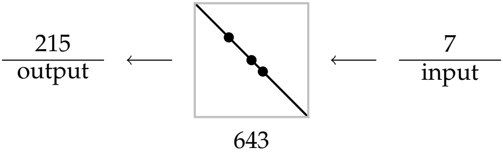

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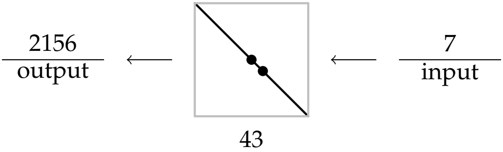

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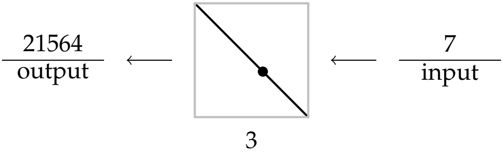

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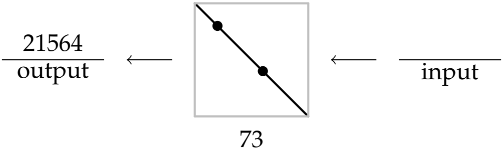

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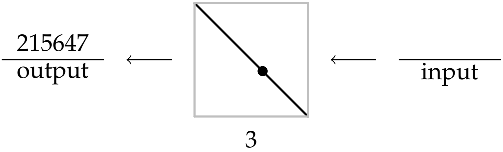

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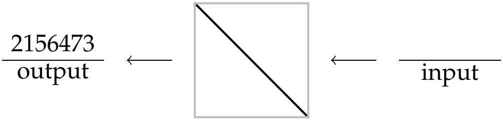

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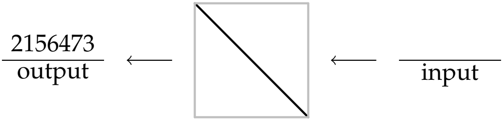

Since the container holds permutations that avoid the pattern 12, we call this the Av(12)-machine.

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We allow a third operation: an entry can bypass the container and move straight from the input to the output.

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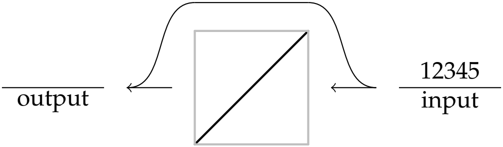

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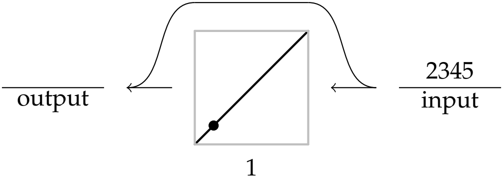

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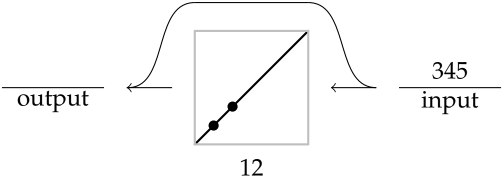

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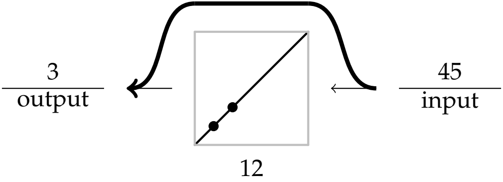

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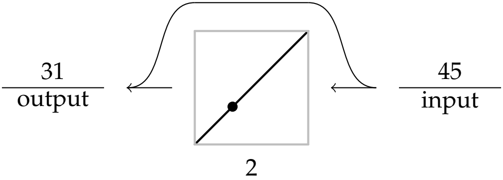

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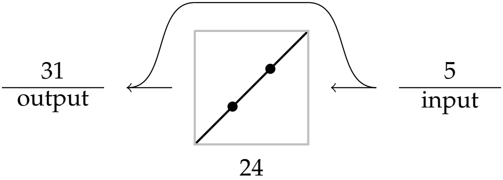

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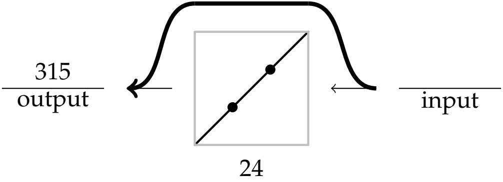

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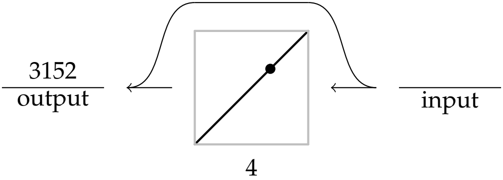

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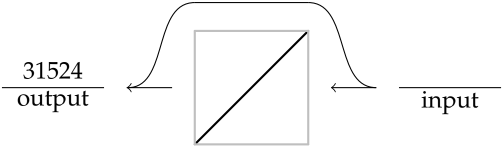

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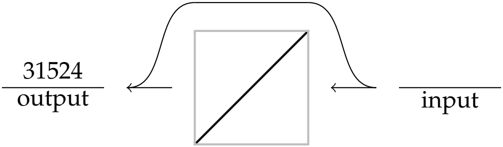
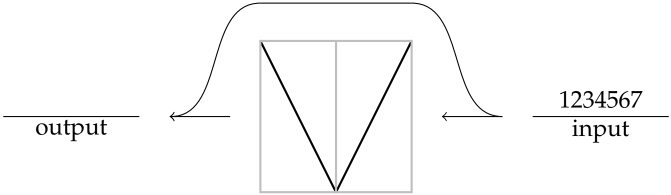


The Av(21)-machine generates the class Av(321).

Example: The Av$(231, 132)$-Machine



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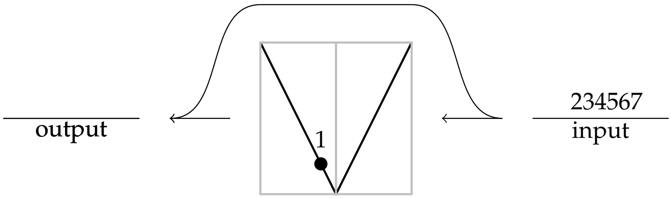

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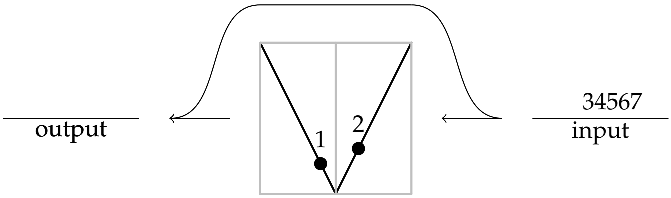

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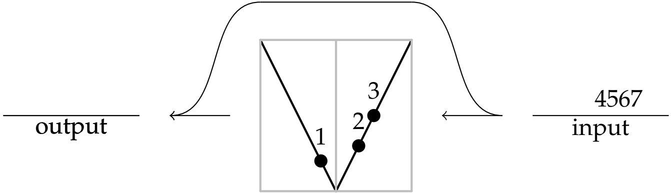

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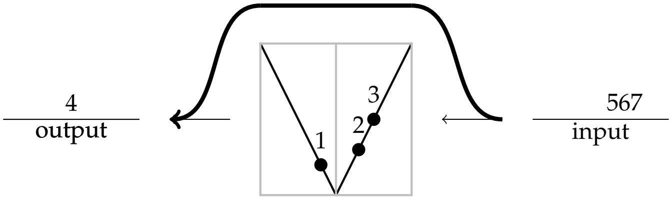

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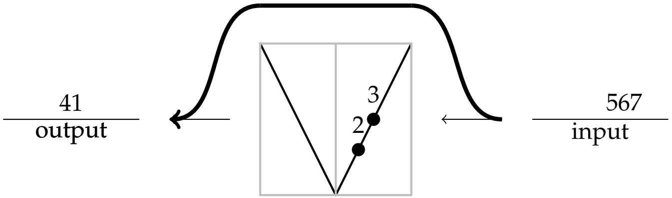

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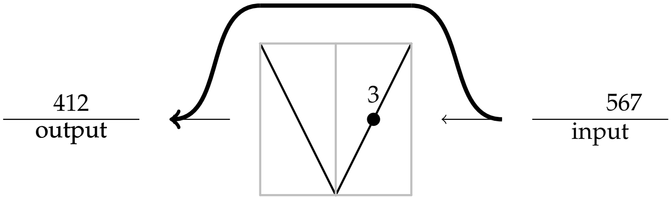

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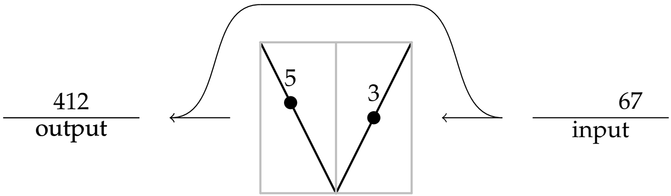

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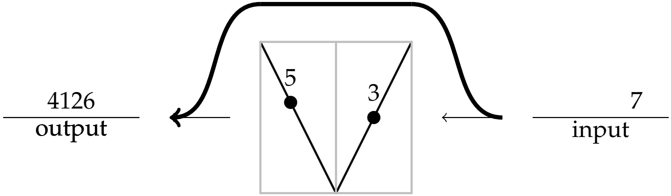

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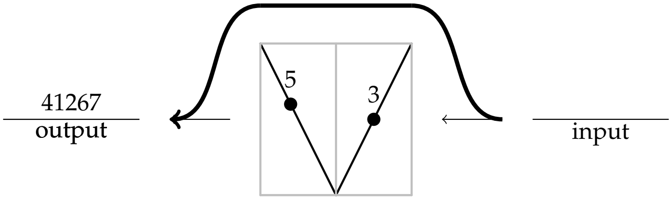

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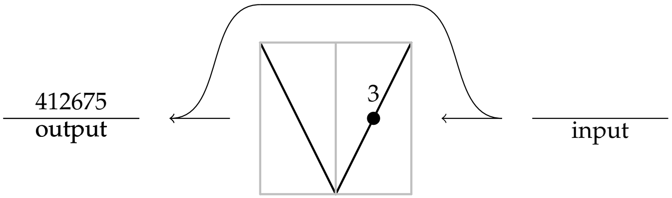

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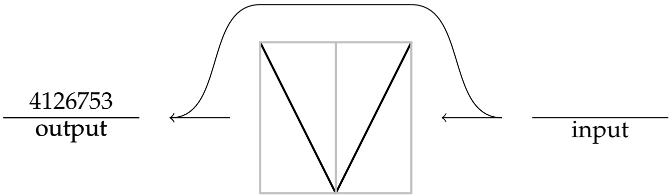

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The operation sequence of the Av(12)-machine makes it easy to find the generating function for Av(312).

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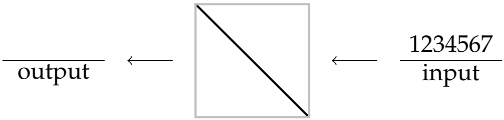

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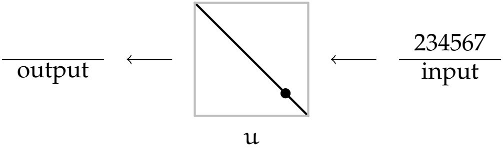

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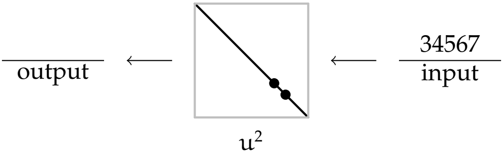

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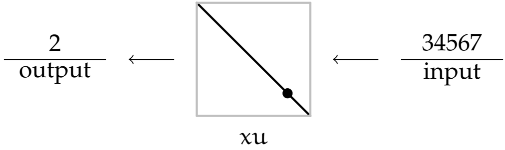

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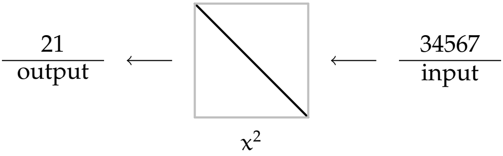

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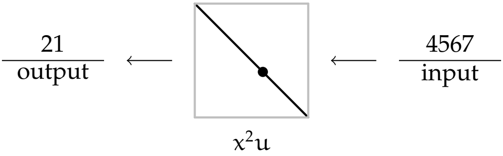

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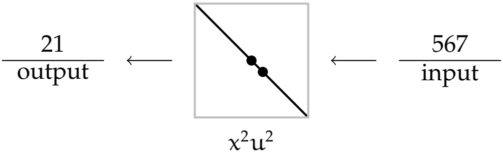

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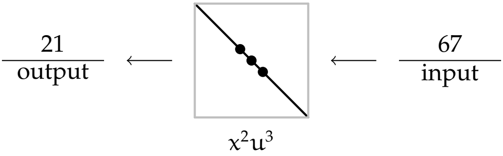

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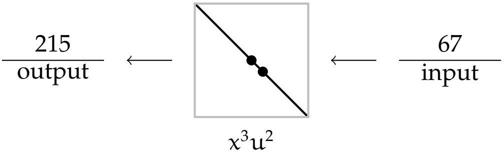

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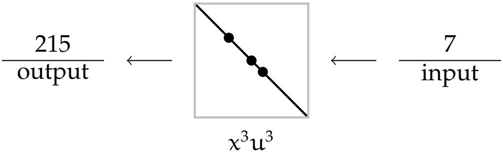

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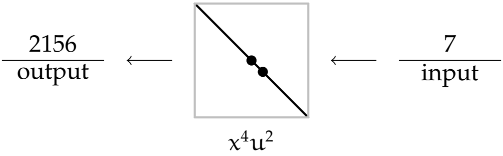

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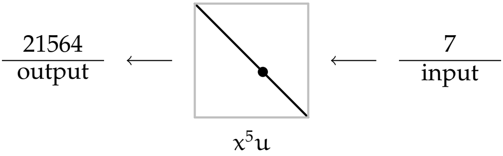

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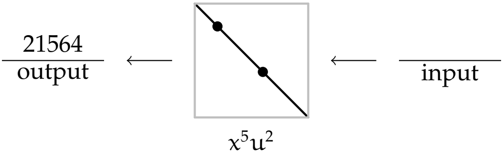

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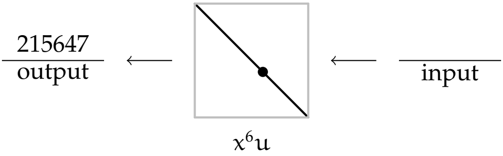

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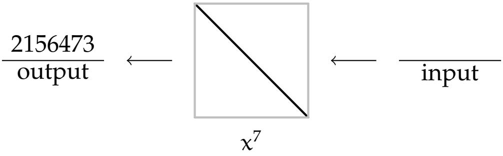

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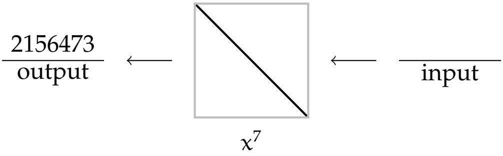

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Hence the generating function for these states is $f(x, 0)$.

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$$f(x,u) = 1 + uf(x,u) + \frac{x}{u}\left(f(x,u) - f(x,0)\right)$$

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(There are some uniqueness issues I'm sweeping under the rug.)

2×4 Classes

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These three can all be modeled with $\mathcal{C}$-machines.

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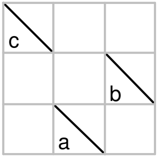

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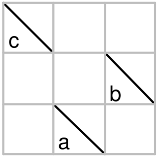
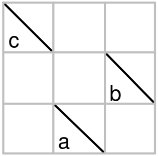


The states of the machine can be represented by 4-tuples (a, b, c, P) where a, b, and c are the number of a, b, c entries, and P is a boolean representing whether we can or can't pop.

Av(4123, 4231)



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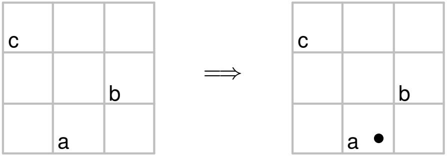
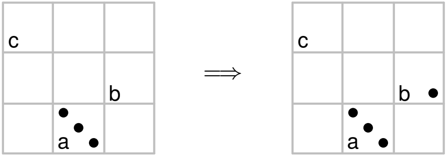


We can describe the state transitions:

- $(0, 0, 0, \mathsf{T}) \longrightarrow \{(1, 0, 0, \mathsf{F}), (0, 0, 0, \mathsf{T})\}$

$\mathrm{Av}(4123, 4231)$



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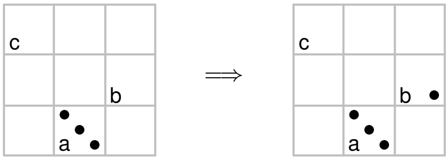

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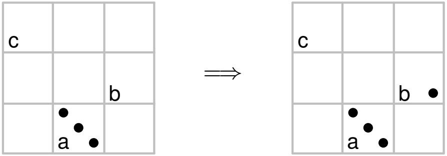

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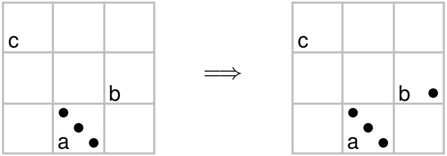

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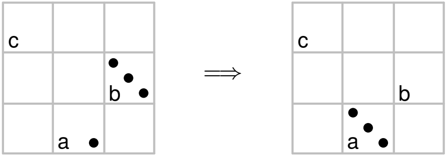

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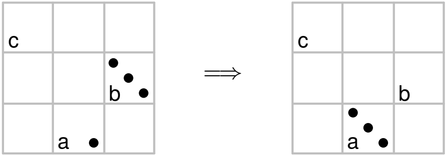

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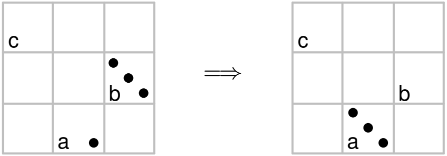

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Av(4123, 4231)

With a little work, we can translate the state transitions into a set of functional equations:

$$
\begin{aligned}
A(a,x) &= 1 + \frac{x}{a}(A(a,x) - A(0,x)) + aA(a,x) + xB(0,a,x),\\
B(a,b,x) &= \frac{1}{a}(A(a,x) - A(0,x))\frac{bC}{1-b} + B(a,b,x)\frac{bC}{1-b}\\
&\qquad + \frac{x}{a}(1+C)(B(a,b,x) - B(0,b,x)).
\end{aligned}
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(C is the generating function for the Catalan numbers.)

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(C is the generating function for the Catalan numbers.)

The generating function for the class is $A(0,x)$, but we have no idea how to solve these equations.

Av(4123, 4312)

$$
\begin{aligned}
A(a,x) &= 1 + \frac{x}{a}(A(a,x) - A(0,x)) + aA(a,x) + xB(a,0,a,x) \\
B(a,b,c,x) &= \frac{cx}{(1-c)(1-x)}\left(\frac{A(a,x)-A(b,x)}{a-b}\right) + \frac{cx}{(1-c)(1-x)}B(a,b,c,x) \\
&\quad + \frac{x}{b(1-x)}(B(a,b,c,x) - B(a,0,c,x))
\end{aligned}
$$

Av(4231, 4321)

$$
\begin{aligned}
A_p &= 1 + x(A_p(a,x) + A_n(a,x)) + \frac{x}{a}(A_p(a,x) - A_p(0,x)) + \frac{x}{a}B_p(a,a,0,x) \\
A_n &= a(A_p(a,x) + A_n(a,x)) \\
B_p &= x(B_p(a,b,c,x) + B_n(a,b,c,x)) + \frac{x}{c}(B_p(a,b,c,x) - B_p(a,b,0,x)) \\
B_n &= a(B_p(a,b,c,x) + B_n(a,b,c,x)) + \frac{b^2}{c-b}((A_p(c,x) - A_p(b,x)) \\
&\qquad + (A_n(c,x) - A_n(b,x)))\frac{b^2}{c-b}((B_p(c,c,c,x) - B_p(b,c,c,x)) \\
&\qquad + +(B_n(c,c,c,x) - B_n(b,c,c,x)))
\end{aligned}
$$

Initial terms

Although we don't know how to solve these functional equations, they can be used to obtain many initial terms of each sequence. (Between 600 and 5000 terms. Probably more are possible.)

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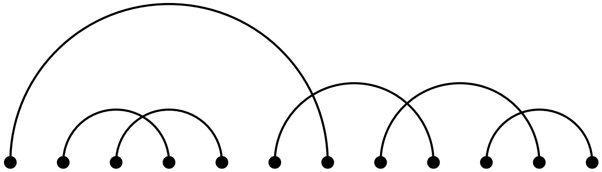
Although we don't know how to solve these functional equations, they can be used to obtain many initial terms of each sequence. (Between 600 and 5000 terms. Probably more are possible.)

We can use two tools to analyze these terms:

- Automated Guessing
- The Method of Differential Approximants

Quick Detour: 2-Colorable Matchings

To illustrate the power of these two methods, consider the class of 2-*colorable matchings*.



I've computed 1500 terms of this sequence, but the generating function is still unknown.

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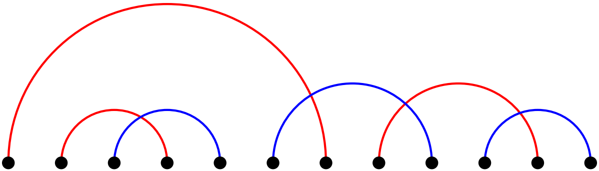

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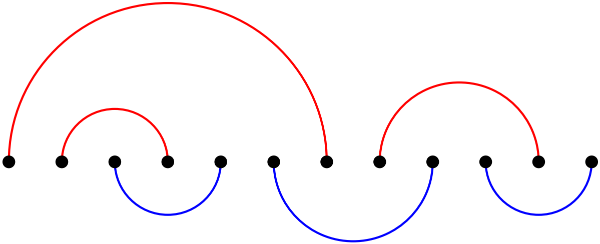

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Quick Detour: 2-Colorable Matchings

Using a few hundred terms, automated guessing says the generating function $F(z)$ satisfies the *algebraic differential equation*

$$\begin{aligned}0 = {} & (-8+12z)\,F(z)^7 + (-12z+20)\,F(z)^6 + (3z-18)\,F(z)^5 - F(z)^3 - 8z^3F'(z)^3 + 7F(z)^4 \\ & - 12z^2F(z)\,F'(z)^2 + 6z\,(3z-10)\,F(z)^4F'(z) + z^2F(z)^3F''(z)/2 + 6z^3\,(4z-3)\,F(z)^2F'(z)^3 \\ & - 8z^2\,(8z-3)\,F(z)^4F'(z)^2 + 32zF(z)^3F'(z) + 4z^2\,(4z-1)\,F(z)^6F''(z) \\ & - 12z\,(5z-4)\,F(z)^5F'(z) - 2z^2\,(4z-3)\,F(z)^5F''(z) + 16z\,(3z-1)\,F(z)^6F'(z) \\ & - 6zF(z)^2F'(z) + 42z^2F(z)^2F'(z)^2 + 12z^2\,(3z-4)\,F(z)^3F'(z)^2 - 8z^3\,(4z-1)\,F(z)^3F'(z)^3 \\ & + 15z^3F(z)\,F'(z)^3 - 3z^2F(z)^4F''(z)\end{aligned}$$

Quick Detour: 2-Colorable Matchings

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1. $z = 0.07490791222594518 \pm 10^{-17}$

   Growth rate $\approx 13.349724618992102$

Quick Detour: 2-Colorable Matchings

The method of differential approximants predicts that $\mathsf{F}(z)$ has singularities at:

1. $z = 0.0749079122259451\,8 \pm 10^{-17}$

   $$\text{Growth rate} \approx 13.349724618992102$$

2. $z = -2.88202477916 \pm 10^{-11}$

QUICK DETOUR: 2-COLORABLE MATCHINGS

The method of differential approximants predicts that $\mathsf{F}(z)$ has singularities at:

1. $z = 0.074907912225945 18 \pm 10^{-17}$

   Growth rate $\approx 13.349724618992102$

2. $z = -2.88202477916 \pm 10^{-11}$

$$\left(= -\frac{256}{9\pi^2} = -2.8820247791598\ldots\right).$$

Back to the Three Unknown Permutation Classes

Applying the method of automated guessing to our three sequences:

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- Av$(4123, 4231)$, 1000 terms, no results

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- $\mathrm{Av}(4123, 4231)$, 1000 terms, no results
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- $\mathrm{Av}(4231, 4321)$, 600 terms, no results

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Conjecture. The generating functions for these three permutation classes are non-differentially-algebraic.

Av(4123,4231)

Dominant singularity at

$$x = 0.20092861430290850630066749465511761874842136315274588937508575345652139 \pm 10^{-71}$$

Av(4123,4231)

Dominant singularity at

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Other singularities:

- $x = 0.2061466445892927115969855884 0 \pm 10^{-29}$
- $x = 0.20724531832263 \pm 10^{-14}$
- $x = 0.24870945696 \pm 0.00390217832i \pm (1 + i)10^{-11}$

Av(4123,4312)

Dominant singularity at

$$x = 0.1715728752538099023966225515806038428606562492461038536466405240185\ldots$$
$$\ldots 3504307578592229922493134471685452997230753817540595015032 \pm 10^{-125} = 3-2\sqrt{2}$$

Av(4123,4312)

Dominant singularity at

$$x = 0.17157287525380990239662255158060384286065624924610385364664052401 85\ldots$$
$$\ldots 3504307578592229922493134471685452997230753817540595015032 \pm 10^{-125} = 3-2\sqrt{2}$$

Other singularities:

- $x = 0.1597726221747308831 \pm 10^{-19}$
- $x = 0.2439516153417787 \pm 0.1200274949895230i \pm (1+i)10^{-16}$
- $x = 0.643104131 \pm 10^{-9}$

Av(4231,4321)

Dominant singularity at

$$x = 0.1697075539292771109936127228338072222506560152943900573 3\ldots$$
$$\ldots 31923504338040028420722677 40380 \pm 10^{-93}$$

Av(4231,4321)

Dominant singularity at

$$x = 0.16970755392927711099361272283380722225065601529439005733\ldots$$
$$\ldots 319235043380400284207226774038 0 \pm 10^{-93}$$

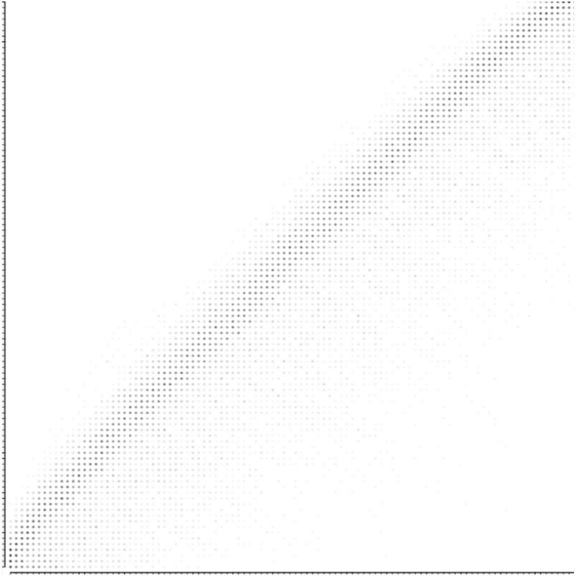
Other singularities:

- $x = .1715728752538099023966225515806038 42\ldots \pm 10^{-70}$
- $x = 0.167216154\ldots \pm 0.004184699\ldots i \pm (1+i)10^{-37}$
- $x = 0.16666666666666666666666666666666666666 \pm 10^{-37}$
- $x = 0.14589803375031545 \pm 10^{-17}$
- $x = 0.241270585132 \pm 0.024119881572i \pm (1+i)10^{-12}$
- $\ldots$

One More Epsilon of Information

Random Sampling!

$\mathrm{Av}(4123, 4231)$



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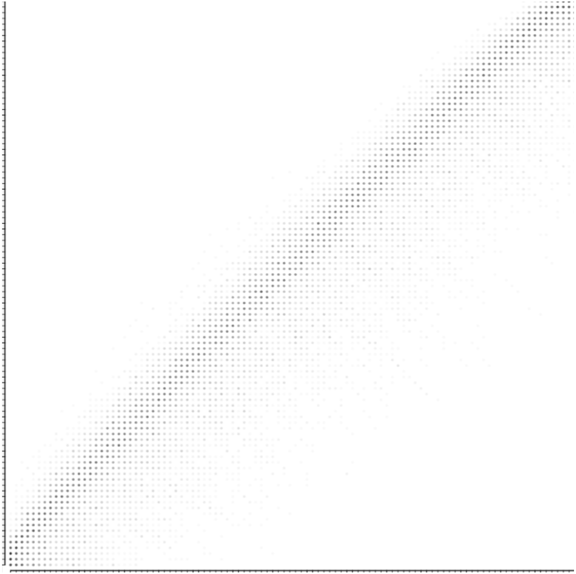

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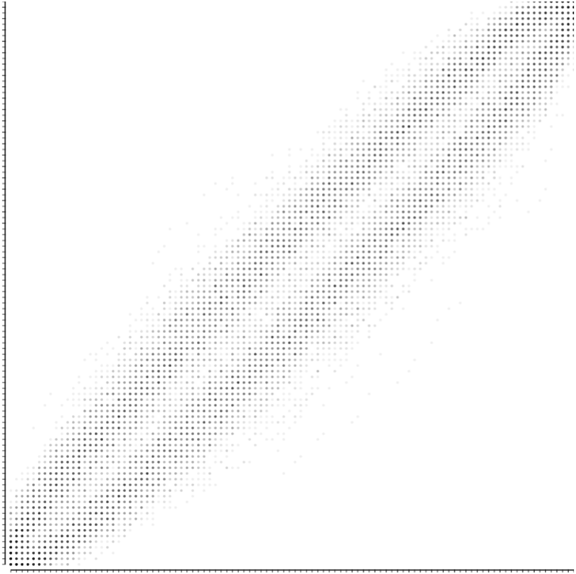

Thanks!