Using Pólya urns to show normal limit laws for fringe subtrees in preferential attachment trees

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Aim of study

- To show *normal limit laws* for the number of *fringe subtrees* that are isomorphic to an arbitrary fixed tree T in preferential attachment trees.
- To show *multivariate normal limit laws* for random vectors of such numbers for different fringe subtrees in the preferential attachment trees.

The linear preferential attachment trees

- Suppose that we are given a sequence of non-negative weights $(w_k)_{k=0}^{\infty}$, with $w_0 > 0$.
- Start with a single node, the root node. Each new node is added as a child of some randomly chosen existing node.
- The probability of choosing a node v as the parent is proportional to $w_{d^+(v)}$, where $d^+(v)$ is the out-degree of v.
- When all n nodes are added we get the preferential attachment tree Λ_n.
- We will mainly consider the linear preferential attachment tree with $w_k = \chi k + \rho$.

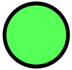
The linear preferential attachment trees

- We will mainly consider the linear preferential attachment tree with $w_k = \chi k + \rho$.
- Note that we obtain the same random tree Λ_n if we multiply w_k with some constant.
- Hence, only the quotient $\frac{\chi}{\rho}$ matters, and thus it suffices to consider $\chi \in \{-1, 0, 1\}$.
- The most interesting and most general of these cases is the case when $\chi = 1$.

Example of a random recursive tree ($w_k = 1$)

- Nodes are added one by one.
- Each new node is attached as a child of a uniformly randomly chosen node.
- Thus $w_k = 1$ and (recalling that a linear preferential attachment tree has $w_k = \chi k + \rho$) this is the only case with $\chi = 0$.

Example of a random recursive tree ($w_k = 1$)



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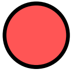

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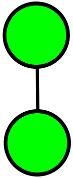

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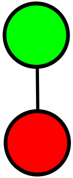

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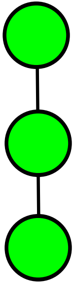

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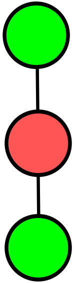

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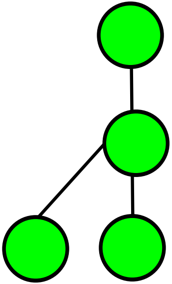

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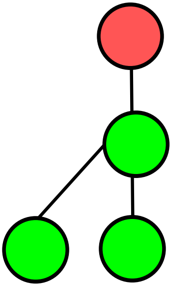

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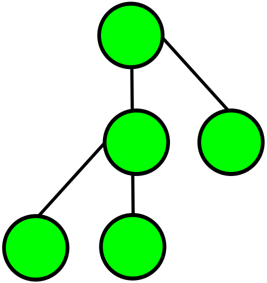

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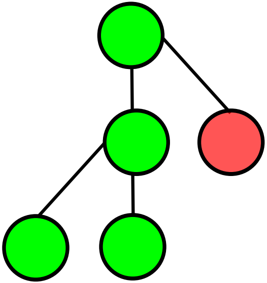

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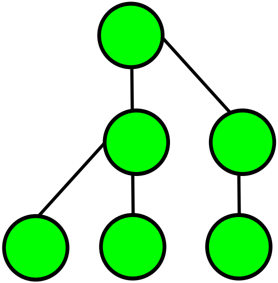

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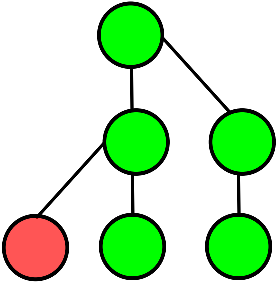

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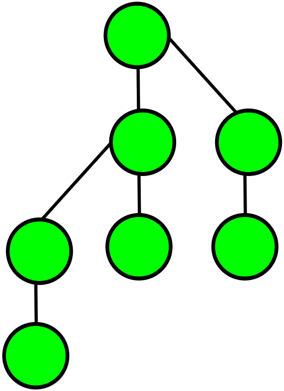

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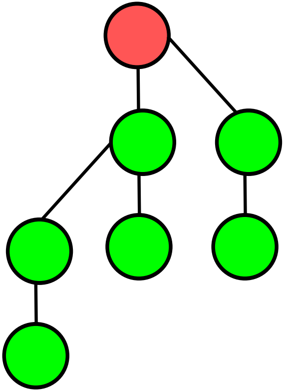

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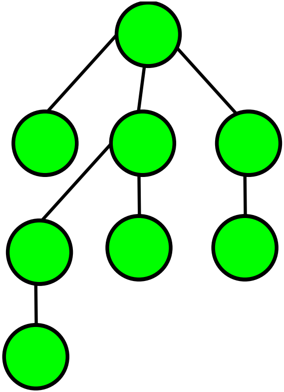


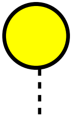
Example of a plane-oriented recursive tree ($w_k = 1$)

- This is similar to the random recursive tree, but now we consider the tree as ordered; an existing node with k children thus has $k + 1$ positions in which a new node can be added.

- All possible positions for adding the new node has the same probability.

- The probability of choosing a node v as the parent is thus proportional to $d^+(v) + 1$, so the plane oriented recursive tree is the case of a linear preferential attachment tree with $w_k = k + 1$, i.e., $\chi = \rho = 1$ (recalling that a linear preferential attachment tree has $w_k = \chi k + \rho$).

Example of a plane-oriented recursive tree ($w_k = k + 1$)



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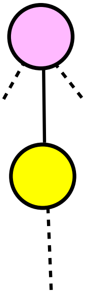

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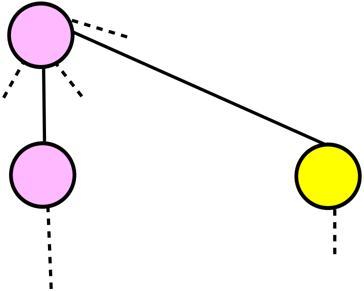

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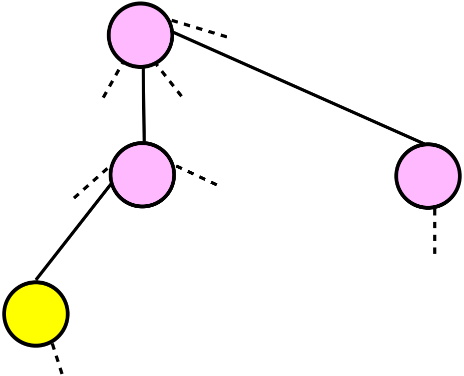

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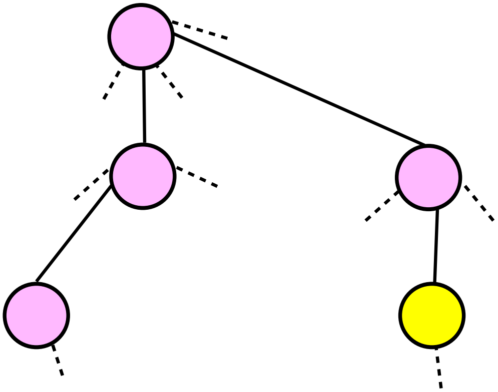

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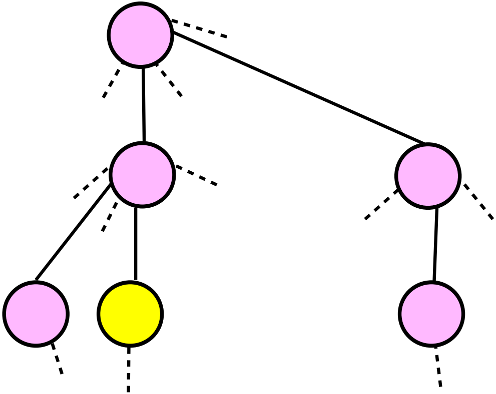

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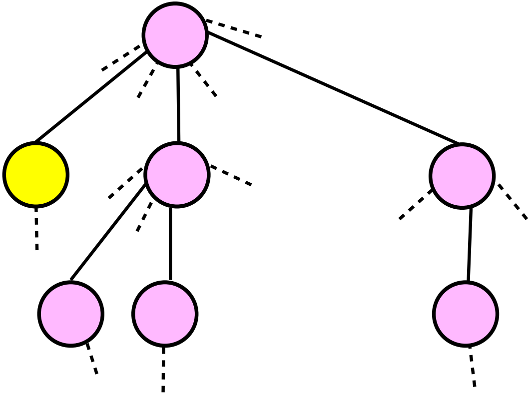

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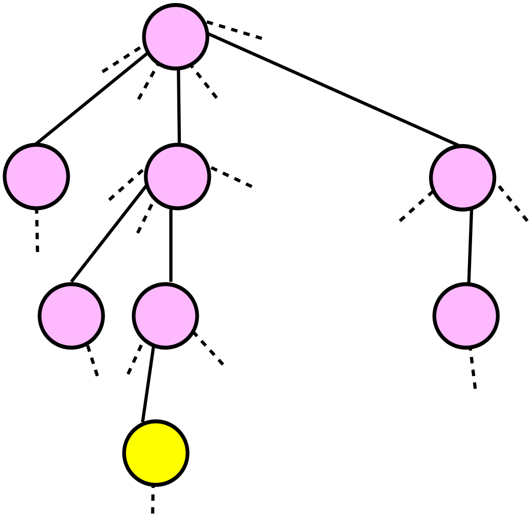

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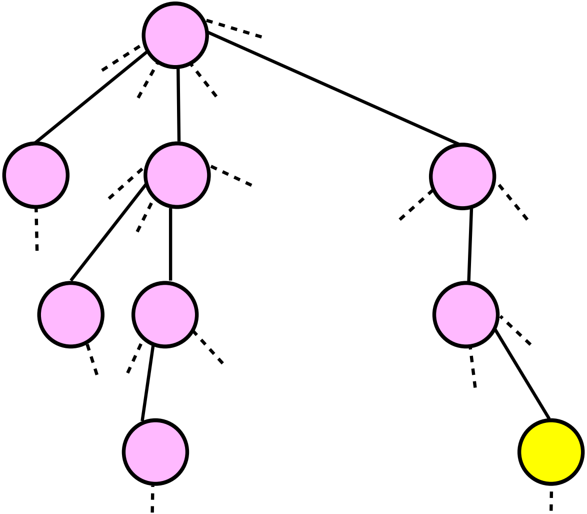

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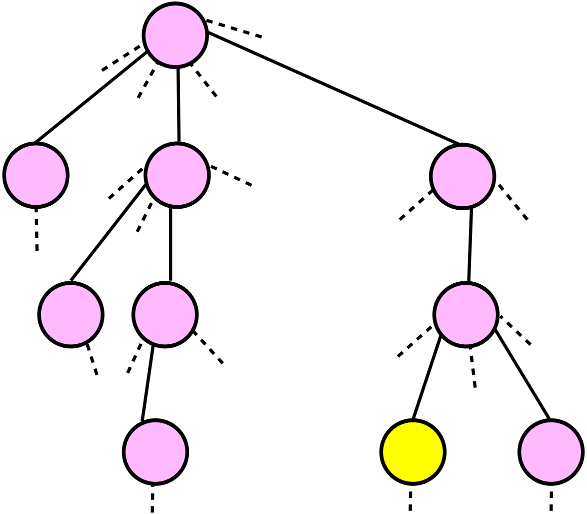

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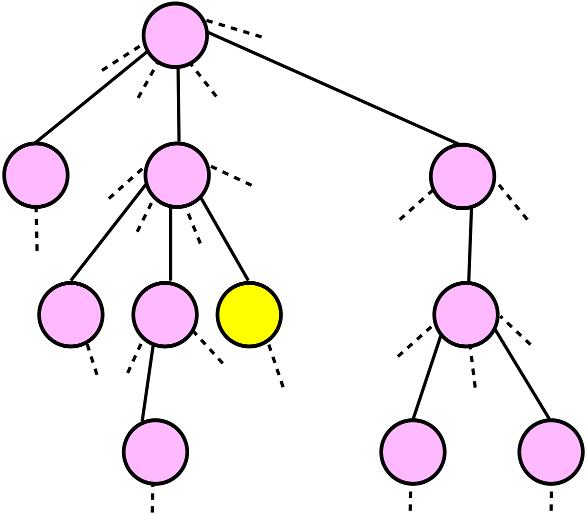

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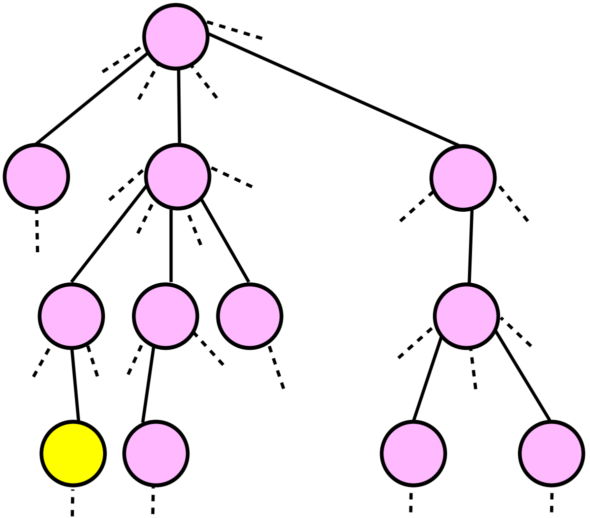

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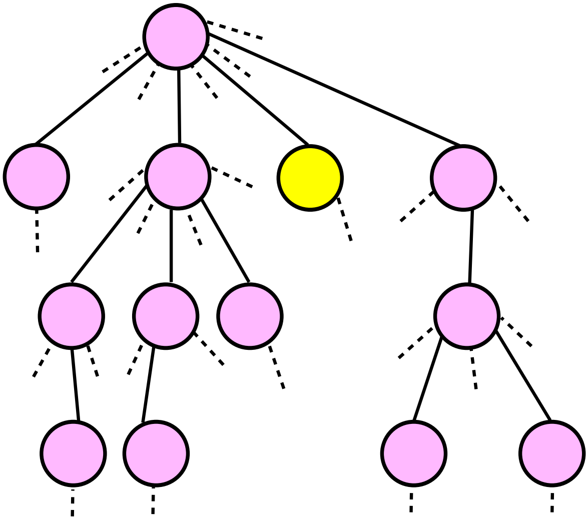

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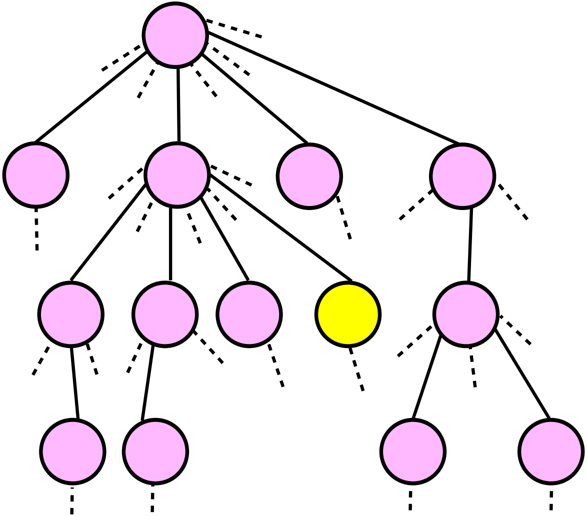

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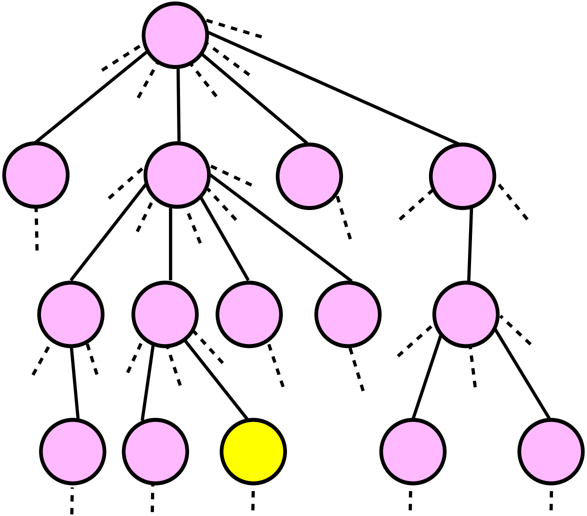

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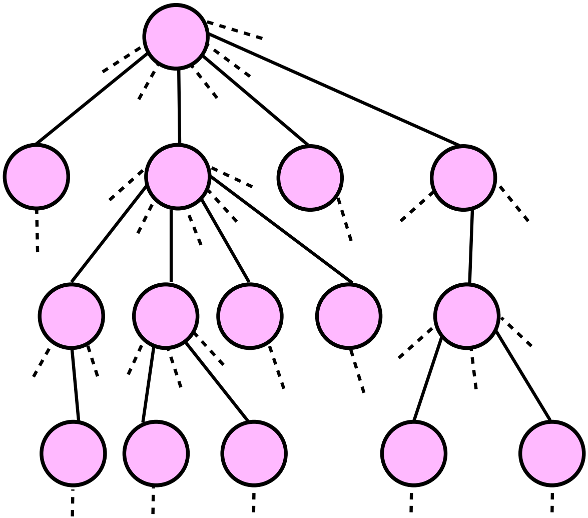
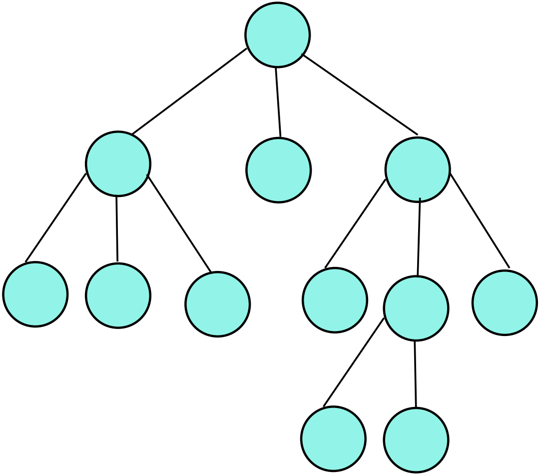


The linear preferential attachment trees

- It is often natural to consider preferential attachment trees as unordered.
- It is also possible to consider them as ordered, either by assigning random orders as for the plane oriented recursive tree or by ordering the children of each node in the order that they are added to the tree.

What is a fringe subtree?



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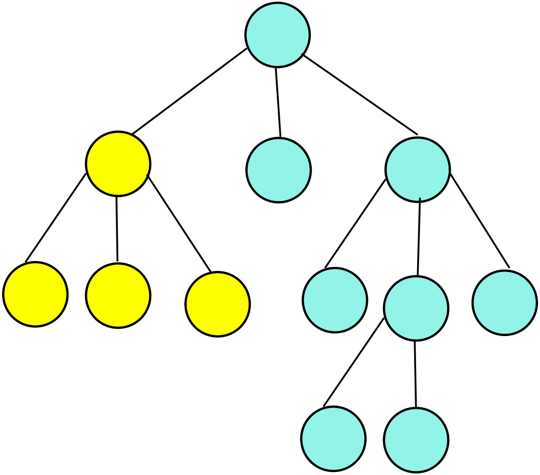

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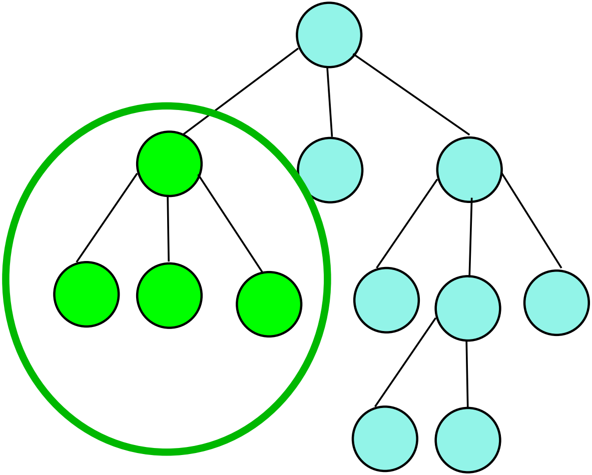

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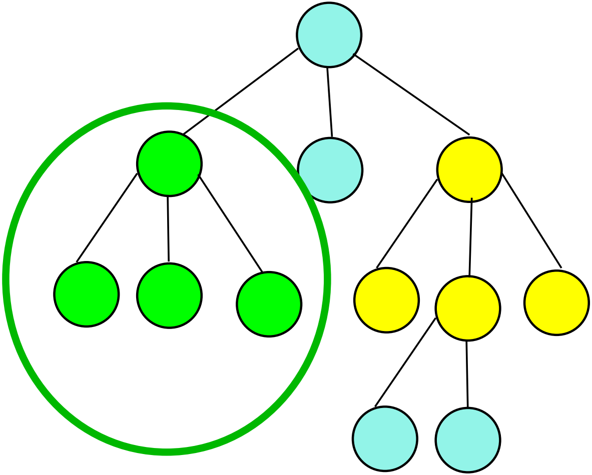

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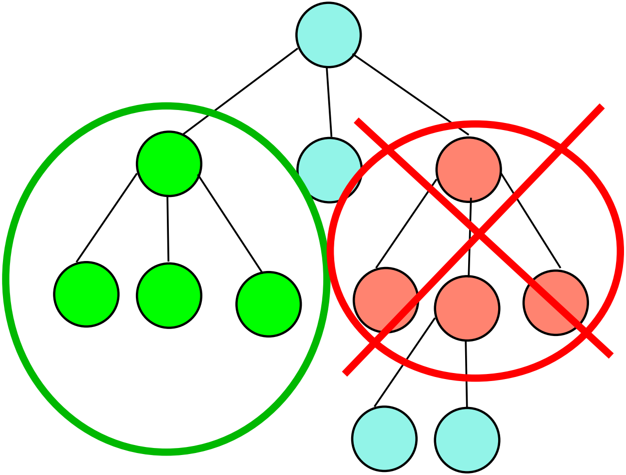
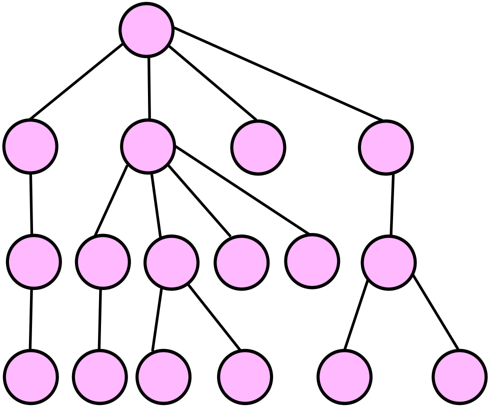


Counting fringe subtrees



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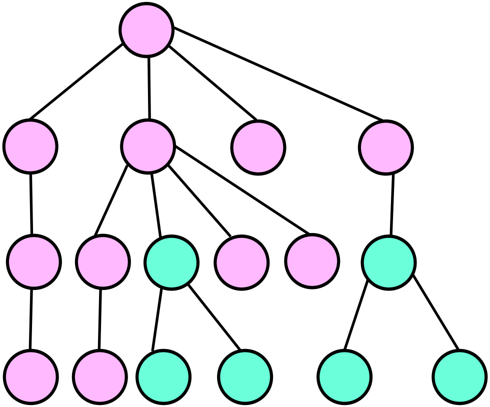

Main Result

- Let $\Lambda^1, \ldots, \Lambda^d$ be a fixed sequence of non-isomorphic unordered (or ordered) trees.
- Let $\mathbf{Z}_n = (X_n^{\Lambda^1}, X_n^{\Lambda^2}, \ldots, X_n^{\Lambda^d})$, where $X_n^{\Lambda^i}$ is the number of fringe subtrees that are isomorphic to Λ^i in the linear preferential attachment tree Λ_n.
- Let k_i be the number of nodes in Λ^i.
- Let

$$\boldsymbol{\mu}_n := \mathbb{E}\,\mathbf{Z}_n = \left(\mathbb{E}(X_n^{\Lambda^1}), \mathbb{E}(X_n^{\Lambda^2}), \ldots, \mathbb{E}(X_n^{\Lambda^d})\right).$$

Main Result

Recall that $\mathbf{Z}_n = (X_n^{\Lambda^1}, X_n^{\Lambda^2}, \dots, X_n^{\Lambda^d})$, and that $\boldsymbol{\mu}_n = \mathbb{E}\,\mathbf{Z}_n$.

Theorem

Then, as $n \to \infty$,

$$n^{-1/2}(\mathbf{Z}_n - \boldsymbol{\mu}_n) \xrightarrow{\mathrm{d}} \mathcal{N}(0, \Sigma), \tag{1}$$

where the vector $\boldsymbol{\mu}_n$ can be replaced with the vector $\hat{\boldsymbol{\mu}}_n := n\hat{\boldsymbol{\mu}}$ with

$$\hat{\boldsymbol{\mu}} := \Big(\frac{\mathbb{P}(\Lambda_{k_1} = \Lambda^1)\cdot\kappa}{(k_1+\kappa-1)(k_1+\kappa)}, \dots, \frac{\mathbb{P}(\Lambda_{k_d} = \Lambda^d)\cdot\kappa}{(k_d+\kappa-1)(k_d+\kappa)}\Big), \tag{2}$$

with

$$\kappa := \frac{\rho}{\chi+\rho} = \frac{w_0}{w_1}, \tag{3}$$

and $\Sigma = (\sigma_{ij})_{i,j=1}^d$ is some non-singular covariance matrix .

Corollary of Main Result

Let k be an arbitrary fixed integer. Let $Y_{n,k}$ be the number of subtrees with k nodes in the linear preferential attachment tree.

Corollary 1

As $n \to \infty$,

$$n^{-1/2}\left(Y_{n,k} - \mathbb{E}\, Y_{n,k}\right) \xrightarrow{\mathrm{d}} \mathcal{N}(0, \sigma_k^2), \tag{4}$$

where σ_k^2 is some constant with $\sigma_k^2 > 0$. Furthermore, we also have

$$n^{-1/2}\left(Y_{n,k} - \frac{\kappa}{(k+\kappa-1)(k+\kappa)}\, n\right) \xrightarrow{\mathrm{d}} \mathcal{N}(0, \sigma_k^2), \tag{5}$$

again with

$$\kappa = \frac{\rho}{\chi + \rho} = \frac{w_0}{w_1}.$$

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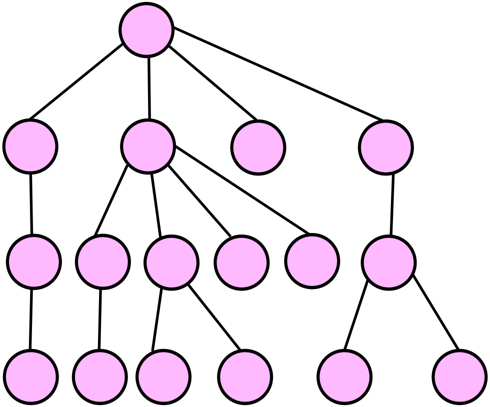
Counting fringe subtrees of size 3



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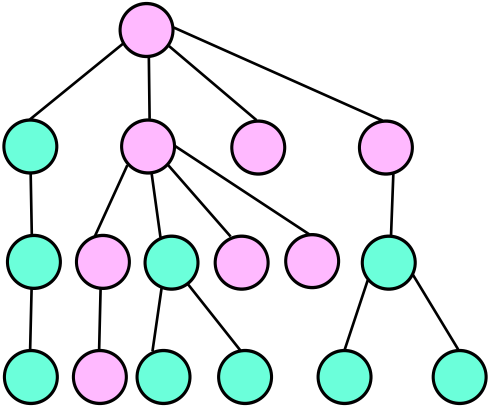


Generalised Pólya urns

- There are balls of q types (or colours) $1, \dots, q$, and for each n a random vector $\mathcal{X}_n = (X_{n,1}, \dots, X_{n,q})$, where $X_{n,i}$ is the number of balls of type i in the urn at time n.

- The urn starts with a given vector $\mathcal{X}_0$. Each type i is given an activity $a_i \geq 0$ and a random vector $\xi_i = (\xi_{i1}, \dots, \xi_{iq})$, which describes the change of the composition of balls in the urn when a ball of type i is drawn. (In fact it often happens that ξ_i is deterministic, and thus the randomness in the urn process only comes from drawing of the balls.) We will assume that $\xi_{ii} \geq -1$ and $\xi_{ij} \geq 0, i \neq j$.

- The urn evolves according to a discrete time Markov process. At each time $n \geq 1$, one ball is drawn at random, with the probability of any ball proportional to its activity.

Generalised Pólya urns

- If the drawn ball has type i, it is replaced by $\Delta X_{n,j}^{(i)}$ balls of type j, $j = 1, \dots, q$, where the random vector $\Delta X_n^{(i)} = (\Delta X_{n,1}^{(i)}, \dots, \Delta X_{n,q}^{(i)})$ has the same distribution as $\xi_i = (\xi_{i1}, \dots, \xi_{iq})$. (We allow $\Delta X_{n,i}^{(i)} = -1$, which means that the drawn ball is *not* replaced.)

- We let A denote the $q \times q$ matrix

$$A = (a_j \, \mathbb{E}\, \xi_{ji})_{i,j=1}^{q}.$$

The intensity matrix A with its eigenvalues and eigenvectors is central for proving limit theorems for $\mathcal{X}_n = (X_{n,1}, \dots, X_{n,q})$.

Generalised Pólya urns and normal limit theorem

The following theorem holds under some conditions of the random process (these are often easy to verify using the Perron-Frobenius theory). Recall that $\mathcal{X}_n = (X_{n,1}, \ldots, X_{n,q})$, where $X_{n,i}$ is the number of balls of type i in the urn at time n. Let λ_1 denote the largest real eigenvalue of A and a certain right eigenvector v_1 corresponding to λ_1, i.e., $Av_1 = \lambda_1 v_1$.

Theorem (Janson (2004) Theorem 3.22)

Assume that $\mathrm{Re}\lambda < \lambda_1/2$ *for each eigenvalue* $\lambda \neq \lambda_1$. *Then, as* $n \to \infty$,

$$n^{-1/2}(\mathcal{X}_n - n\mu) \xrightarrow{d} \mathcal{N}(0, \Sigma),$$

with $\mu = \lambda_1 v_1$ *and some covariance matrix* Σ.

The covariance matrix Σ is expressed in Janson (2004) and can be evaluated using the intensity matrix A.

Applying Pólya urns to count fringe subtrees

- Recall that we want to count the total number of fringe subtrees that are isomorphic to some fixed trees $\Lambda^1, \ldots, \Lambda^d$ in the linear preferential attachment tree Λ_n.

- We will model our process as a Pólya urn, by subdividing the linear preferential attachment tree into subtrees that represent the types $\{1, 2, \ldots, q\}$, where some of the types represent the fringe subtrees isomorphic to $\Lambda^1, \ldots, \Lambda^d$ in the tree.

- We consider adding a node to the tree in terms of the Pólya urn process of drawing a ball. In particular this will show a multivariate normal limit law for the vector $\mathbf{Z}_n = (X_n^{\Lambda^1}, X_n^{\Lambda^2}, \ldots, X_n^{\Lambda^d})$, i.e., the number of fringe subtrees that are isomorphic to $\Lambda^1, \ldots, \Lambda^d$ in Λ_n.

Types in the Pólya urn counting fringe subtrees

- Let Λ_n be a given linear preferential attachment tree with n nodes. Let $\Lambda_n(v)$ be the fringe subtree of Λ_n rooted at node v.

- We may consider either ordered or unordered trees.

- There is a natural partial order on the set of ordered or unordered trees, such that $T \preceq T'$ if T' can be obtained from T by adding nodes (including the case $T' = T$).

Types in the Pólya urn counting fringe subtrees

- Recall that we want to count the number of fringe subtrees $\Lambda^1, \ldots, \Lambda^d$ in the preferential attachment tree Λ_n.

- We say that a node v is *living* if $\Lambda_n(v) \preceq \Lambda^i$ for some $i \in \{1, \ldots, d\}$, i.e., if $\Lambda_n(v)$ is isomorphic to some Λ^i or can be grown to one of them by adding more nodes. We let all descendants of a living node be living (all nodes of $\Lambda_n(v)$ are living if v is living). All other nodes of Λ_n are *dead*.

- Now erase all edges from dead nodes to their children. This yields a forest of small trees, where each tree either consists of a single dead node or is living (all nodes are living) and can be grown to one of the Λ^i.

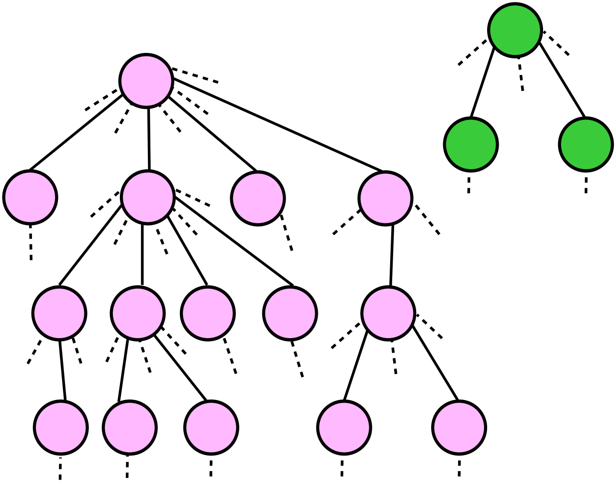
- We regard these small trees as the balls in the Pólya urn.

Types in the Pólya urn counting fringe subtrees

- However, we can not ignore the dead nodes, since they may get new children; furthermore, the probability of this depends on their degree. Hence we label each dead node by the number of children it has in Λ_n.

- Hence, the types in this urn are all trees Λ such that $\Lambda \preceq \Lambda^i$ for some $i \in \{1, \dots, d\}$ (these are called *normal types*), plus one type $*_k$ for each positive integer k, consisting of a single dead node labelled by k (these are called *special types*).

Types in the Pólya urn counting fringe subtrees



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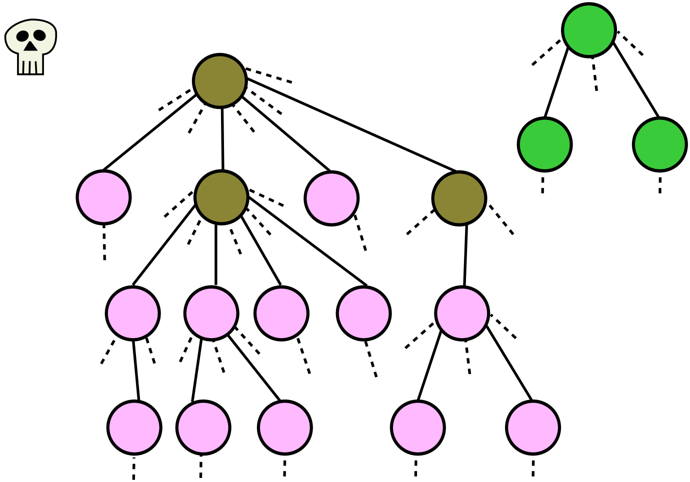

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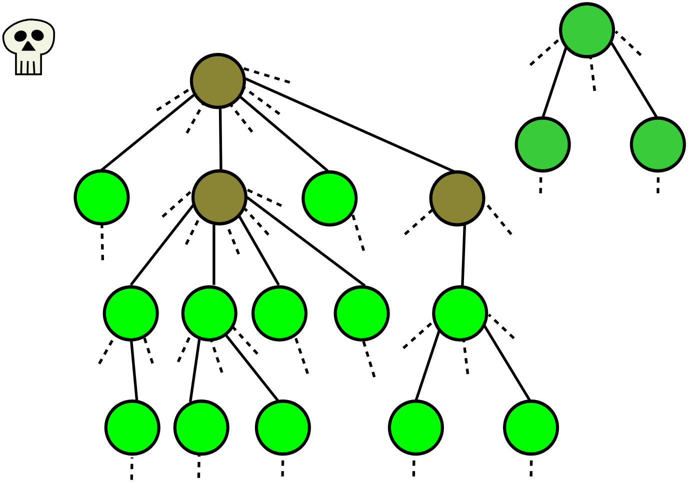
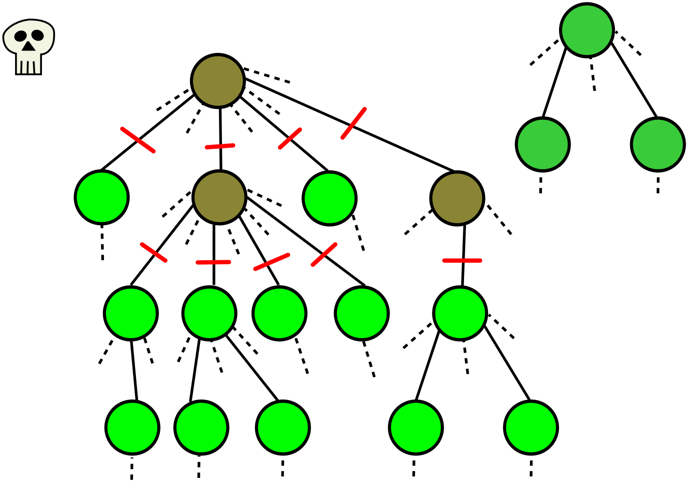


Types in the Pólya urn counting fringe subtrees



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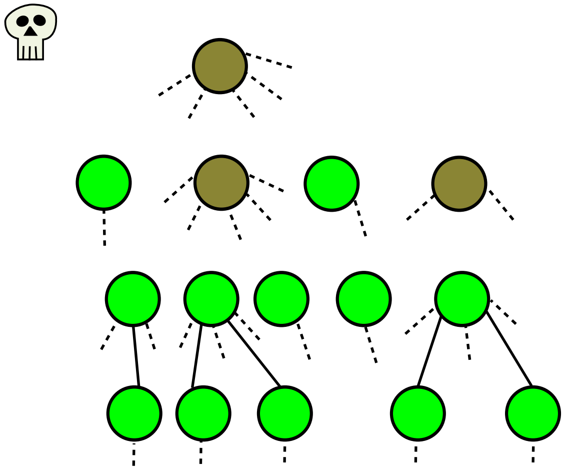

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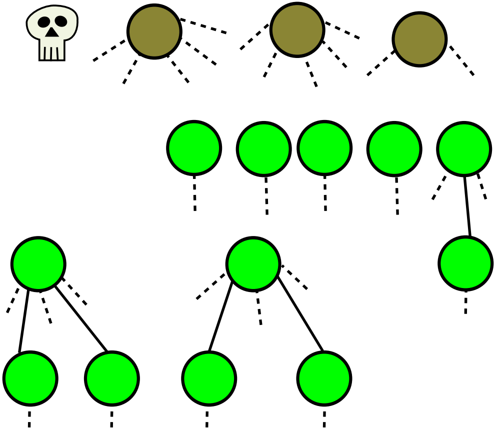
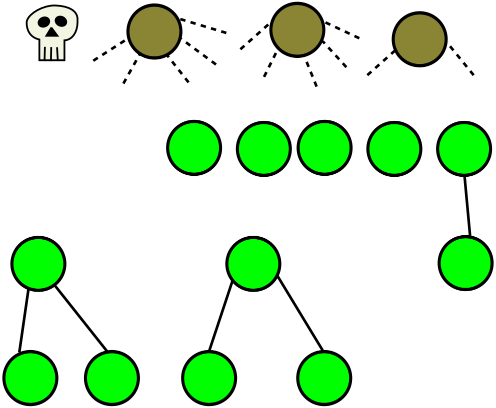


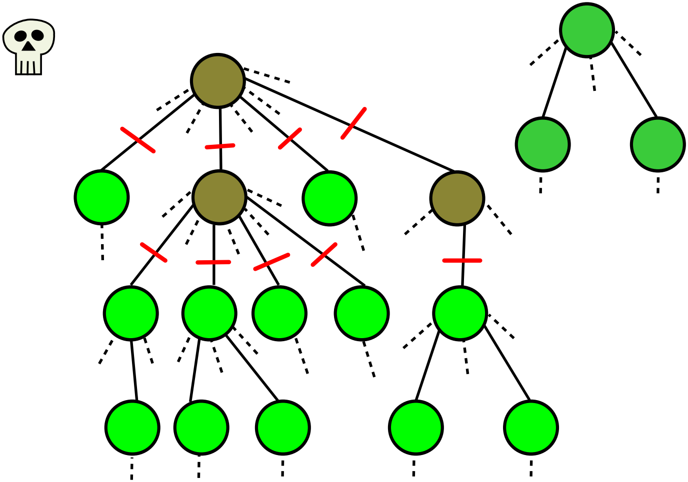
Types in the Pólya urn counting fringe subtrees



The replacement rule

- If a ball of a special type $*_k$ is drawn, it is replaced by one ball of type $*_{k+1}$ and one living single node.
- If a normal type is drawn, we add a new child to one of its nodes, with probabilities determined by the weights w_k.
- If the root of that subtree still is living after the addition, then that subtree becomes a living subtree of a different type; if the root becomes dead, then the subtree is further decomposed into one or several dead nodes and one or several living subtrees.
- The random evolution of the forest obtained by decomposing Λ_n is thus described by a Pólya urn, where each type has activity equal to the sum of all weights for the nodes in that type.

The replacement rule of a living type



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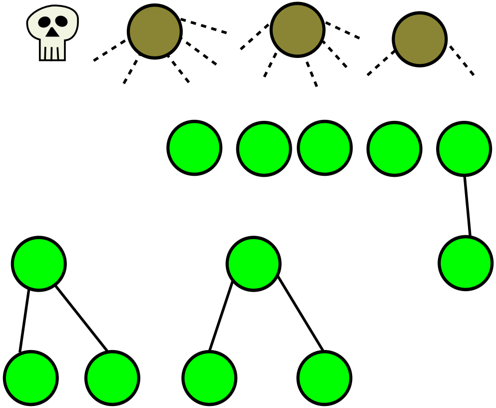
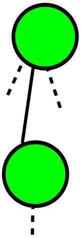


The replacement rule of a living type



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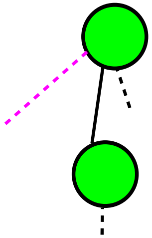

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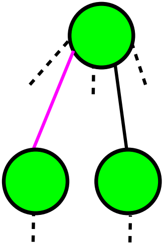

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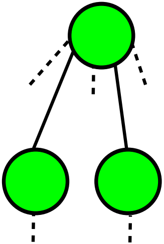

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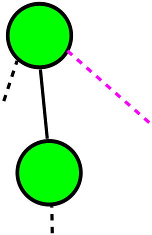

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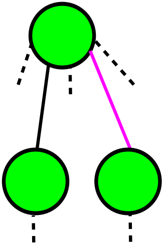

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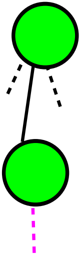

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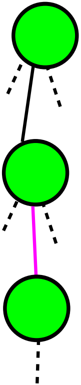

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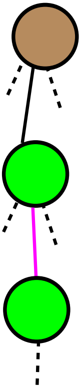

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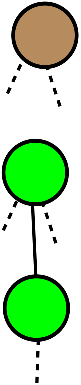
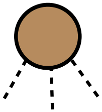


The replacement rule of a dead type



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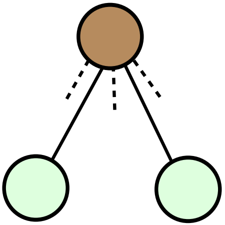

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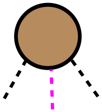

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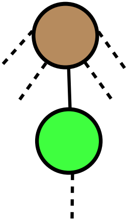

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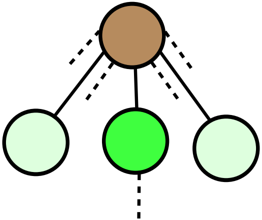
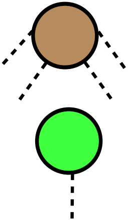
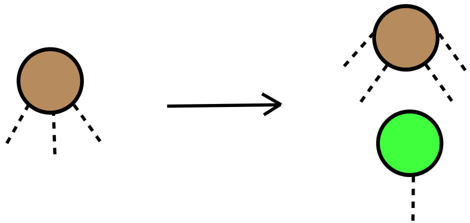


The replacement rule of a dead type



The replacement rule of a dead type



An infinite Pólya urn

- Unfortunately, the number of dead (special) types is infinite since they depend on the number of children of the single node that corresponds to the the dead node.

- Hence, this is a Pólya urn with infinitely many types. The normal limit theorem for generalised Pólya urns stated as

Theorem (Janson (2004) Theorem 3.22)

Assume that $\mathrm{Re}\lambda < \lambda_1/2$ for each eigenvalue $\lambda \neq \lambda_1$. Then, as $n \to \infty$, $n^{-1/2}(\mathcal{X}_n - n\mu) \xrightarrow{d} \mathcal{N}(0, \Sigma)$, with $\mu = \lambda_1 v_1$ and some covariance matrix Σ.

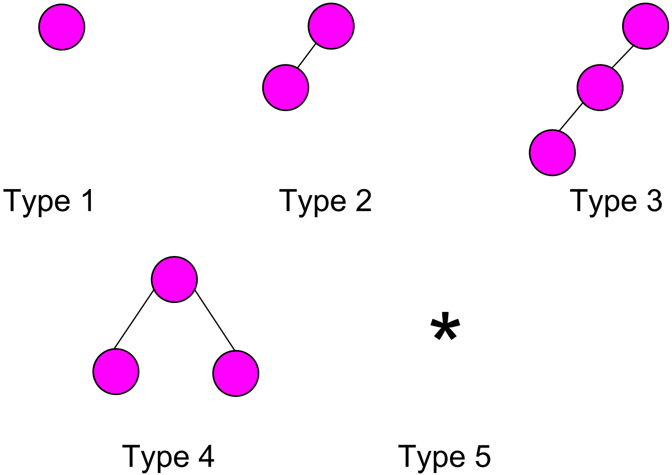
does not apply to such urns.

- However, for the linear case i.e., $w_k = \chi k + \rho$, we can reduce the urn to a finite-type one.

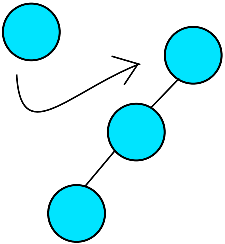
Trick to reduce the infinite urn to a finite urn

- Recall that in the preferential attachment tree $w_k = \chi k + \rho$ and we can assume that $\chi \in \{-1, 0, 1\}$.
- For simplicity consider the case when ρ is an integer and thus w_k is an integer (when ρ is real-valued the same trick works by interpreting the number of balls as a real number).
- Change each dead (special) ball of type $*_k$ to w_k balls of a new type $*$. Let $*$ have activity 1; then the activities are preserved by the change.
- Recall that if a ball of type $*_k$ is drawn, it is replaced by one ball of type $*_{k+1}$ and one single living node; after the change, this means that the number of balls $*$ is increased by $w_{k+1} - w_k = \chi$. (This is where the linearity is essential.)

Number of unordered fringe subtrees with 3 nodes



Finding the intensity matrix A



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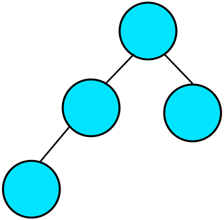

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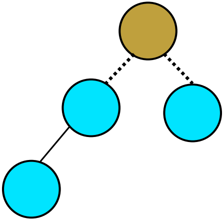
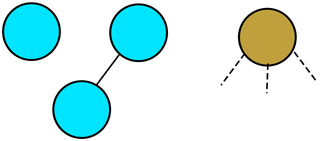


Finding the intensity matrix A



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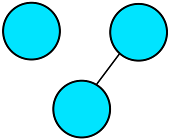
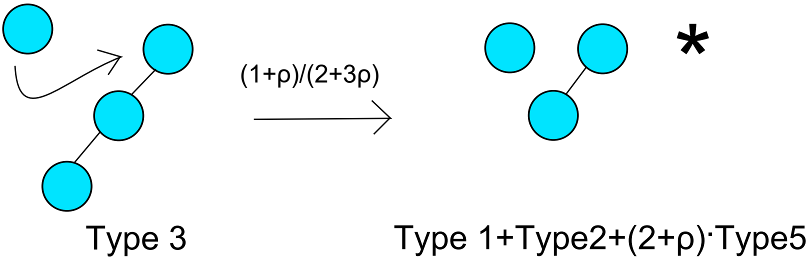
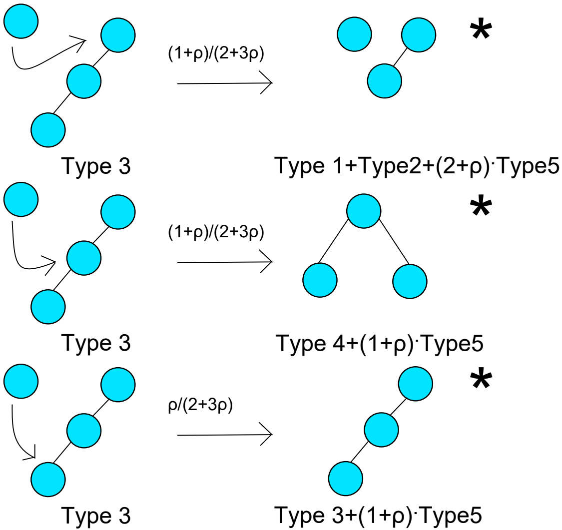


$(2+\rho)$ *

Finding the intensity matrix A



Finding the intensity matrix A



Finding the intensity matrix A

Thus, we get the intensity matrix A as

$$A = \begin{pmatrix} -\rho & 0 & \rho+1 & 5\rho+6 & 1 \\ \rho & -2\rho-1 & \rho+1 & 2\rho & 0 \\ 0 & \rho & -2\rho-2 & 0 & 0 \\ 0 & \rho+1 & \rho+1 & -3\rho-2 & 0 \\ 0 & 0 & 3(\rho+1)^2 & 3(\rho+1)(\rho+2) & 1 \end{pmatrix}.$$

The eigenvalues are

$$\rho+1, -\rho, -2\rho-1, -3\rho-2, -3\rho-2.$$

Theorem (Janson (2004) Theorem 3.22)

Assume that $\mathrm{Re}\lambda < \lambda_1/2$ *for each eigenvalue* $\lambda \neq \lambda_1$.
Then, as $n \to \infty$, $n^{-1/2}(\mathcal{X}_n - n\mu) \xrightarrow{d} \mathcal{N}(0, \Sigma)$, *with* $\mu = \lambda_1 v_1$ *and some covariance matrix* Σ.

The eigenvalues for the general intensity matrices

- To show asymptotic normality in general Polya urns one needs to check $\mathrm{Re}\lambda < \lambda_1/2$ for each eigenvalue $\lambda \neq \lambda_1$ (e.g., Janson 2004).

- We will find the eigenvalues of A by using induction on the size of the set of different types.

- Let q be the number of types and $*$ the dead type. Choose a numbering $T_1, \dots, T_{q-1}$ of these $q-1$ types that is compatible with the partial order $\preceq$. For $k \leq q$, let

$$\mathcal{S}_k := \{T_1, \dots, T_{k-1}, *\} .$$

The eigenvalues for the general intensity matrices

- The activities of the types in $\mathcal{S}_k := \{T_1, \ldots, T_{k-1}, *\}$ are $(a_1, \ldots, a_{k-1}, 1)$, where

$$a_i = w_{T_i} = |T_i|(\chi + \rho) - \chi,$$

  which is the sum of all weights w_k in T_i. (Recall that $*$ always has weight 1.)

- We may thus consider the Pólya urn with the k types in $\mathcal{S}_k$ constructed by chopping the whole tree $\mathcal{T}_n$ into a forest of small trees in $\mathcal{S}_k$. Let A_k be the intensity matrix of this Pólya urn. Thus $A = A_q$.

The eigenvalues for the general intensity matrices

Proposition

Let $2 \leq k \leq q$.

1. *For every normal (living) type T_j, $j = 1, \ldots, k-1$, except the type that is a path (rooted at an endpoint) of maximal length,*
$$(A_k)_{jj} = -a_j.$$
2. *For the normal type T_i that is a path of maximal length among all paths in $\mathcal{S}_k$, we have $(A_k)_{ii} = \rho - a_i$.*
3. *For the special type $*$, we have $(A_k)_{kk} = \chi$.*

Consequently,

$$\mathrm{tr}(A_k) = \chi + \rho - \sum_{j=1}^{k-1} a_j.$$

Idea of the proof

Recall that

$$A = (a_j \, \mathbb{E}\, \xi_{ji})_{i,j=1}^{q}. \tag{6}$$

- The diagonal entry $(A)_{jj}$ is the expected change of the number of balls of type j when such a ball is drawn multiplied by the activity of type j.

- The idea is that if a living ball of type j is drawn it is "usually" not replaced by balls of the same type, thus $\xi_{jj} = -1$ and, by (6), $(A_k)_{jj} = -a_j$.

- The only way such a drawn ball can be replaced by itself is if it corresponds to a path of maximal length. Then it is replaced by itself in case the new node is attached to the end-node of this path.

The eigenvalues for the general intensity matrices

Proposition

Let $2 \leq k \leq q$.

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$$\mathrm{tr}(A_k) = \chi + \rho - \sum_{j=1}^{k-1} a_j.$$

The eigenvalues for the general intensity matrices

Recall that we deduced from the proposition that for $2 \le k \le q$,

$$\mathrm{tr}(A_k) = \chi + \rho - \sum_{j=1}^{k-1} a_j.$$

Applying this result we show by induction on k that the eigenvalues correspond to the activities of the types in the urn.

Theorem

For the linear preferential attachment trees, the eigenvalues of the intensity matrix A are $\chi + \rho$ and $-a_i$ for $i \in \{1, \ldots, q-1\}$, where a_i is given by $a_i = w_{T_i} = |T_i|(\chi + \rho) - \chi$.

The eigenvalues for the general intensity matrices

Theorem

For the linear preferential attachment trees, the eigenvalues of the intensity matrix A are $\chi + \rho$ and $-a_i$ for $i \in \{1, \dots, q-1\}$, where a_i is given by $a_i = w_{T_i} = |T_i|(\chi + \rho) - \chi$.

- We prove by induction on $k \geq 2$ that the theorem holds for A_k. For A_2 the eigenvalues are $\chi + \rho$ and $-a_1 = -\rho$.
- We show that the eigenvalues of A_{k+1} are inherited from A_k, i.e., that the eigenvalues of A_{k+1} can be listed (with multiplicities) as $\lambda_1, \dots, \lambda_{k+1}$, where $\lambda_1, \dots, \lambda_k$ are the eigenvalues of A_k.

The eigenvalues for the general intensity matrices

- The trace of a matrix is equal to the sum of the eigenvalues; hence,

$$\operatorname{tr} A_{k+1} = \lambda_1 + \cdots + \lambda_{k+1} = \operatorname{tr} A_k + \lambda_{k+1}.$$

Recall that we deduced from the proposition that for $2 \leq k \leq q$,

$$\operatorname{tr}(A_k) = \chi + \rho - \sum_{j=1}^{k-1} a_j.$$

- Hence, this proposition implies

$$\lambda_{k+1} = \operatorname{tr}(A_{k+1}) - \operatorname{tr}(A_k) = -a_{k+1}.$$

- Thus, by induction the theorem holds for every A_k, with $2 \leq k \leq q$, and in particular for $k = q$.

Summary

- We studied fringe subtrees in preferential attachment trees, by putting them in context of generalised Pólya urns.
- We showed that the number of fringe subtrees that are isomorphic to $\Lambda^1, \ldots, \Lambda^d$ in the linear preferential attachment tree Λ_n has a multivariate normal distribution.
- As a corollary we showed that the number of fringe subtrees with k nodes in such a tree has a normal distribution.
- By using induction and the traces of the intensity matrices we saw that the eigenvalues for any Pólya urn corresponding to a set of finite fringe subtrees in the linear preferential attachment tree correspond to the activities of the types in the urns. Thus, it followed that $\mathrm{Re}\lambda < \lambda_1/2$.