Distribution of the Number of Factors in Monoids Generated by a Lucas Sequence

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Erdős–Kac Theorem

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Theorem (Erdős–Kac 1940)

Let N be a uniformly random integer in $[1, x]$. Then

$$\lim_{x\to\infty} \mathbb{P}\left(\frac{\omega(N) - \log\log x}{\sqrt{\log\log x}} \leq z\right) = \frac{1}{\sqrt{2\pi}} \int_0^z e^{-y^2/2}\, dy.$$

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Similar result for $\Omega(n) := \#$ prime factors of n (with multiplicities).

Primitive Divisors of Fibonacci Numbers

$F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$; $F_1 = F_2 = 1$.

1	1	2	3	5
8	13	21	34	55
89	144	233	377	610

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$$\begin{array}{ccccc} 1 = 1 & 1 = 1 & 2 = 2 & 3 = 3 & 5 = 5 \\ 8 = 2^3 & 13 = 13 & 21 = 3 \cdot 7 & 34 = 2 \cdot 17 & 55 = 5 \cdot 11 \\ 89 = 89 & 144 = 2^4 \cdot 3^2 & 233 = 233 & 377 = 13 \cdot 29 & 610 = 2 \cdot 5 \cdot 61 \end{array}$$

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Theorem (Carmichael 1913)

For $n > 12$, F_n has a primitive divisor, i.e., a prime p with

$$p \mid F_n \text{ but } p \nmid F_1 \dots F_{n-1}.$$

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$$\mathcal{M}(\mathcal{F}) = \{1, 2, 3, 4 = 2^2, 5, 6 = 2 \cdot 3, 8 = 2^3, 9 = 3^2, 10 = 2 \cdot 5, 12 = 2^2 \cdot 3, 13, 15 = 3 \cdot 5, 16 = 2^4, 18 = 2 \cdot 3^2, 20 = 2^2 \cdot 5, 21, 24 = 2^3 \cdot 3, 25 = 5^2, 26 = 2 \cdot 13, 27 = 3^3, 30 = 2 \cdot 3 \cdot 5, 32 = 2^5, 34, 36 = 2^2 \cdot 3^2, 39 = 3 \cdot 13, 40 = 2^3 \cdot 5, 42 = 2 \cdot 21, 45 = 3^2 \cdot 5, \dots\}$$

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Choose $x \in \mathcal{M}(\mathcal{F})$ minimally with two distinct factorisations

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Wlog, $n_\ell = \max\{m_1, \dots, m_k, n_1, \dots, n_\ell\}$.

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Let p primitive divisor of n_ℓ. Then $p \mid m_1 \dots m_k$. Thus n_ℓ must occur among $m_1, \dots, m_k$.
Division by n_ℓ yields smaller counterexample. Contradiction. $\square$

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Lucas Sequence

$$F_n = \frac{\phi^n - \overline{\phi}^n}{\phi - \overline{\phi}}, \qquad \phi = \frac{1+\sqrt{5}}{2}, \qquad \overline{\phi} = \frac{1-\sqrt{5}}{2}$$

All results remain valid if ϕ, $\overline{\phi}$ are replaced by any real algebraic integers such that $\phi + \overline{\phi}$ and $\phi\overline{\phi}$ are non-zero coprime rational integers with $\phi > |\overline{\phi}|$.

Number of Elements

Theorem

We have

$$|\mathcal{M}(\mathcal{F})\cap[1,x]| = k_0(\log x)^{k_1}\exp\left(\pi\sqrt{\frac{2\log x}{3\log\phi}}\right)\left(1+O\left(\frac{1}{(\log x)^{1/10}}\right)\right)$$

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for $x \to \infty$ and suitable constants k_0 and k_1. Specifically,

$$k_1 = \frac{|\mathcal{F}_0| - 13}{2} + \frac{\log(\phi - \overline{\phi})}{2 \log \phi}.$$

$\mathcal{F}_0 = \{F_n \mid n \leq 12, F_n \text{ has primitive divisor}\}$.

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Let N be a uniformly random positive integer in $\mathcal{M}(\mathcal{F}) \cap [1, x]$ and let

$$a_1 = \frac{1}{\pi}\sqrt{\frac{6}{\log \phi}}, \qquad a_2 = \frac{\pi^2 - 6}{2\pi^3}\sqrt{\frac{6}{\log \phi}}.$$

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The random variable $\omega_{\mathcal{F}}(N)$ is asymptotically normal: we have

$$\lim_{x\to\infty} \mathbb{P}\left(\frac{\omega_{\mathcal{F}}(N) - a_1 \log^{1/2} x}{\sqrt{a_2}\log^{1/4} x} \leq z\right) = \frac{1}{\sqrt{2\pi}}\int_0^z e^{-y^2/2}\, dy.$$

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Let N be a uniformly random positive integer in $\mathcal{M}(\mathcal{F}) \cap [1, x]$, a_1, b_1, b_2 suitable constants.
The random variable $\Omega_{\mathcal{F}}(N)$, suitably normalised, converges weakly to a sum of shifted exponentially distributed random variables:

$$\frac{\Omega_{\mathcal{F}}(N) - \frac{a_1}{2} \log^{1/2} x \log\log x - b_1 \log^{1/2} x}{b_2 \log^{1/2} x} \xrightarrow{(d)} \sum_{m \in \mathcal{F}} \left(X_m - \frac{1}{\log m} \right),$$

where $X_m \sim \mathrm{Exp}(\log m)$.

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Number of Factors With Multiplicities—Constants

$$a_1 = \frac{1}{\pi}\sqrt{\frac{6}{\log\phi}}$$

$$b_1 = \frac{\sqrt{6\log\phi}}{\pi}\left(\frac{2\gamma - \log(\pi^2\log\phi/6)}{2\log\phi} + \sum_{m\in\mathcal{F}_0}\frac{1}{\log m} + \frac{1}{\log v_{13}(\phi,\overline{\phi})} + \sum_{k\geq 1}\left(\frac{1}{\log v_{k+13}(\phi,\overline{\phi})} - \frac{1}{k\log\phi}\right)\right),$$

$$b_2 = \frac{\sqrt{6\log\phi}}{\pi}.$$

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Sketch of Proof: without Multiplicities

Let u be real, $u \approx 1$.

$$d(z, u) := \sum_{n \in \mathcal{M}(\mathcal{F})} \frac{u^{\omega_{\mathcal{F}}(n)}}{n^z}$$

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Mellin–Perron summation formula:

$$I_{\omega_{\mathcal{F}}}(x,u) := \sum_{\substack{n \in \mathcal{M}(\mathcal{F}) \\ n \le x}} u^{\omega_{\mathcal{F}}(n)} \left(1 - \frac{n}{x}\right) = \frac{1}{2\pi i} \int_{r-i\infty}^{r+i\infty} \frac{d(z,u)}{z(z+1)} x^z \, dz.$$

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Use saddle point approach for computing the asymptotic behaviour of the integral.

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Saddle Point: Central Approximation

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Asymptotic behaviour for $z \to 0$ via Mellin transform.

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for

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Harmonic sum.

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Harmonic sum. Mellin transform:

$$g^\star(s,u) = (\zeta(s+1) - \mathrm{Li}_{s+1}(1-u))\Gamma(s) \sum_{m\in\mathcal{F}} \frac{1}{(\log m)^s}.$$

Li denotes the polylogarithm.

Saddle Point: Central Approximation (2)

Lemma

Let $r > 0$, $z = r + it$ *with* $|t| \le r^{7/5}$, *and* $|1 - u| < 1$.
Then

$$d(z,u) = d(r,u)\exp\left(-\frac{ia(u)t}{r^2} - \frac{a(u)t^2}{r^3} + O(r^{1/5})\right),$$

$$d(r,u) = \exp\left(\frac{a(u)}{r} + b\log r + c(u) + O(r)\right)$$

for $r \to 0^+$ *and*

$$a(u) = \frac{\pi^2/6 - \mathrm{Li}_2(1-u)}{\log\phi}.$$

b: constant; $c(u)$ *analytic around* 1.

Mellin–Perron Summation

$$I_{\omega_{\mathcal{F}}}(x,u) = \sum_{\substack{n\in\mathcal{M}(\mathcal{F})\\ n\le x}} u^{\omega_{\mathcal{F}}(n)}\left(1-\frac{n}{x}\right) = \frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty} \frac{d(z,u)}{z(z+1)} x^z\, dz.$$

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Putting Everything Together

Lemma

We have

$$\sum_{\substack{n\in\mathcal{M}(\mathcal{F})\\ n\leq x}} u^{\omega_{\mathcal{F}}(n)} =$$

$$\frac{1}{2\sqrt{\pi}}\exp\Big(2\sqrt{a(u)}\sqrt{\log x}-\frac{2b+1}{4}\log\log x+\frac{2b-1}{4}\log a(u)+c(u)\Big)$$

$$\times\Big(1+O\Big(\frac{1}{(\log x)^{1/10}}\Big)\Big)$$

for $x\to\infty$ *and* $1/2<u<3/2$.

Use Curtiss' theorem to obtain the central limit theorem.