Analytic combinatorics of graphs with marked subgraphs

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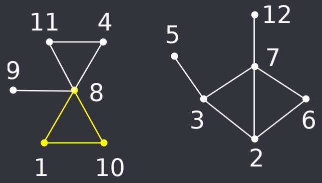
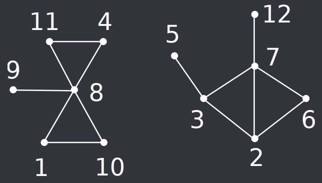
Problem



Graph with 12 vertices, 14 edges, one distinguished triangle.



Graph with 12 vertices, 14 edges, 4 triangles.

Problem

Setting

F graph

$\mathcal{X}_F$ number of copies of F contained in a random graph, with n vertices and $m \sim cn^\alpha$ edges.

Problems

- find $\alpha^\star$ such that $\begin{cases} \mathcal{X}_F = 0 & \textit{a.a.s.} \text{ if } \alpha < \alpha^\star, \\ \mathcal{X}_F \geq 1 & \textit{a.a.s.} \text{ if } \alpha > \alpha^\star. \end{cases}$
- limit law of $\mathcal{X}_F$.

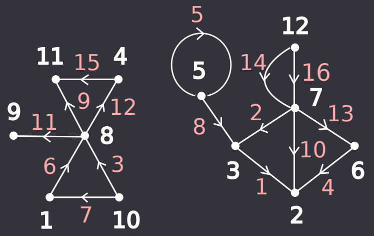
Resolution by Erdős and Rényi (1960), Bollobás (1981), Karoński and Ruciński (1983), Ruciński (1988); probabilistic approach.

Contribution: new approach based on analytic combinatorics (see the book of Flajolet and Sedgewick 2009).

Multigraphs

- Labelled vertices,
- labelled oriented edges,
- loops and multiple edges allowed,

- nb of vertices $n(G)$,
- nb of edges $m(G)$,
- nb of multigraphs n^{2m}.



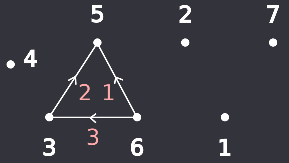
Generating function of the family $\mathcal{H}$

$$H(z,w) := \sum_{G \in \mathcal{H}} \frac{w^{m(G)}}{2^{m(G)} m(G)!} \frac{z^{n(G)}}{n(G)!} = \sum_{n,m} |\mathcal{H}_{n,m}| \frac{w^m}{2^m m!} \frac{z^n}{n!}.$$

Multigraphs with one distinguished subgraph copy of F

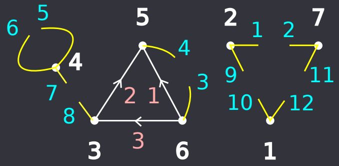
Multigraph F and a set of isolated vertices

$$F(z, w)e^z$$



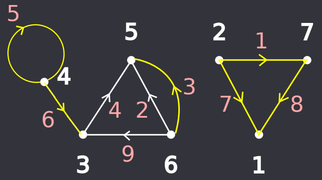
add a set of labelled half-edges to each vertex

$$F(ze^x, w)e^{z\exp(x)}$$



link the half-edges to build $2m$ edges

$$\sum_{m\geq 0}(2m)![x^{2m}]F(ze^x, w)e^{z\exp(x)}\frac{w^m}{2^m m!}$$



Asymptotics

$$|\,\mathrm{MG}^F_{n,m}\,| = n!2^m m![z^n w^m] \sum_{\ell \geq 0} (2\ell)![x^{2\ell}]F(ze^x, w)e^{z\exp(x)} \frac{w^\ell}{2^\ell \ell!}$$

Asymptotics

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Inject the relation $\frac{(2\ell)!}{2^\ell \ell!} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} t^{2\ell} e^{-t^2/2} dt$

$$n!2^m m![z^n w^m] \sum_{\ell \geq 0} [x^{2\ell}] F(ze^x, w) e^{z\exp(x)} w^\ell \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} t^{2\ell} e^{-t^2/2} dt,$$

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switch sum and integral, and apply $\sum_\ell [z^\ell]f(z)x^\ell = f(x)$

$$n!2^m m![z^n w^m] \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} F(ze^{\sqrt{w}t}, w)e^{z\exp(\sqrt{w}t)} e^{-t^2/2} dt,$$

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saddle-point method $|\,\mathrm{MG}^F_{n,m}\,| \sim n^{2m} F\left(n, m/n^2\right)$.

Lower bound on $\alpha^\star$

Multigraphs with n vertices and $m \sim cn^\alpha$ edges.

Nb of multigraphs that contain at least one copy of F
$\leq$ nb of multigraphs with one distinguished copy

$$\mathbb{P}(\mathcal{X}_F \geq 1) \leq \frac{|\,\mathrm{MG}_{n,m}^F\,|}{n^{2m}} \sim F(n, 2m/n^2).$$

By definition, we have

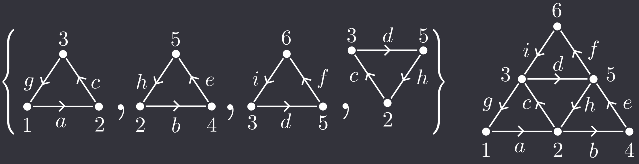
$$F(n, 2m/n^2) = \frac{(2m/n^2)^{m(F)}}{2^{m(F)} m(F)!} \frac{n^{n(F)}}{n(F)!} = \frac{c^{m(F)}}{m(F)!n(F)!} n^{n(F)-(2-\alpha)m(F)}$$

which tends to 0 if $\alpha < 2 - \frac{n(F)}{m(F)}$. Thus $\alpha^\star \geq 2 - \frac{n(F)}{m(F)}$.

Multigraphs with all subgraphs F marked

MG(z, w, u): gf of multigraphs where each subgraph F is marked by u.

Patchwork: set of copies of F that might share vertices and edges. Generating function $P(z, w, u)$.



Inclusion-exclusion: consider MG$(z, w, u+1)$.
Now each subgraph is either marked or left unmarked.
By definition, the marked subgraphs form a patchwork

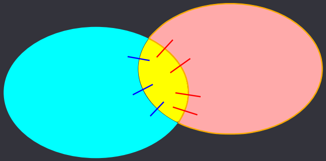
$$\mathrm{MG}(z, w, u+1) = \sum_{m \geq 0} (2m)! [x^{2m}] P(ze^{x}, w, u) e^{z \exp(x)} \frac{w^m}{2^m m!}.$$

Application: strictly balanced multigraphs

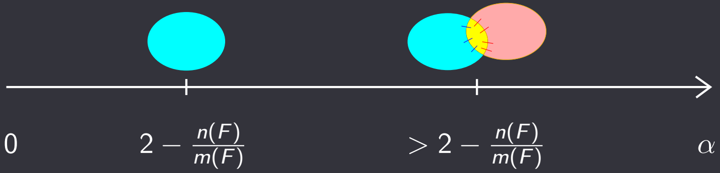
F is strictly balanced if all its strict subgraphs are less dense

$$\frac{m(F)}{n(F)} > \max_{H \subsetneq F} \frac{m(H)}{n(H)}.$$

In that case, any pair of non-disjoint copies has a higher density



so they typically do not appear for $m = \Theta\left(n^{2-\frac{n(F)}{m(F)}}\right)$



Application: strictly balanced multigraphs

Thus for $m \sim cn^{\alpha^\star}$, we need only consider disjoint patchworks

$$P(z, w, u) \approx e^{uF(z,w)}.$$

Nb of multigraphs with n vertices, $m \sim cn^{\alpha^\star}$ edges, that contain exactly t copies of F

$$|\,\mathrm{MG}_{n,m,t}\,| = n!2^m m![z^n w^m u^t] \sum_{\ell \geq 0} (2\ell)![x^{2\ell}] P(ze^x, w, u-1) e^{z \exp(x)} \frac{w^m}{2^\ell \ell!}$$

$$\sim n^{2m}[u^t] e^{(u-1)F(n, 2m/n^2)} = n^{2m}[u^t] e^{(u-1)\frac{c^{m(F)}}{m(F)!n(F)!}}$$

Thus the limit law of $\mathcal{X}_F$ is Poisson $\left(\frac{c^{m(F)}}{m(F)!n(F)!}\right)$.

Conclusion

Results presented

- exact expression for the nb of multigraphs with a given number of vertices, edges, and subgraphs copies of F,
- new proof of the limit law of $\mathcal{X}_F$ in the critical window when F is strictly balanced.

Other results

- from multigraphs to simple graphs,
- induced subgraphs,
- marked subgraphs and degree constraints.

In progress

- limit law of $\mathcal{X}_F$ outside the critical window,
- and when F is not strictly balanced,
- phase transition of 2-SAT.