An inverse spectral problem for the Hermite operator

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Geometric and Analytic Inequalities
Banff, July 12, 2016

Let Ω be any smooth and bounded domain of $\mathbb{R}^N$ and ν_Ω the outward normal to $\partial\Omega$ and, finally, denote with $\mu_1^{-\Delta}(\Omega)$ the first nontrivial eigenvalue of

$$\begin{cases} -\Delta u = \mu u & \text{in} \quad \Omega \\ \dfrac{\partial u}{\partial \nu_\Omega} = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

Then it holds

(SW) $$\mu_1^{-\Delta}(\Omega) \le \mu_1^{-\Delta}(\Omega^\sharp),$$

where $\Omega^\sharp$ is any ball having the same Lebesgue measure as Ω.

Remarks.

1. Equality sign holds in (SW) if and only if Ω is a ball.
2. Generalizations of (SW) can be found, for example, in Bandle 1980; Chavel 1980; Ashbaugh - Benguria 1995; Laugesen - Siudeja 2009 and 2010.

The classical Payne-Weinberger inequality

Let Ω be any bounded and convex domain of $\mathbb{R}^N$, then

(PW) $$\mu_1^{-\Delta}(\Omega) \geq \frac{\pi^2}{d(\Omega)^2},$$

where $d(\Omega)$ is the diameter of Ω .

Remarks.

1. The convexity assumption in (PW) cannot be removed.
2. (PW) is sharp, since $d(\Omega)^2\mu_1^{-\Delta}(\Omega)$ goes to π^2 for a parallelepiped all but one of whose dimensions shrink to zero, but such a value is never achieved.
3. Lower bounds for $\mu_1^{-\Delta}(\Omega)$, valid also for non convex domains, are contained, for instance, in

   Brandolini - C. - Trombetti, 2015; Gol'dshtein - Ukhlov, 2016; Brandolini - C. - Dryden - Langford, in preparation.
4. Generalizations of (PW) can be found, for example, in Ferone - Nitsch - Trombetti 2012; Valtorta 2012.

The Neumann eigenvalue problem for the Hermite operator

$$\boxed{x \in \mathbb{R}^N, \quad d\gamma_N(x) = (2\pi)^{-N/2} \exp\left(-\frac{|x|^2}{2}\right) dx}$$

$\Omega \subset \mathbb{R}^N$ smooth and possibly unbounded domain, ν_Ω outward normal to $\partial\Omega$

$$\begin{cases} -\mathrm{div}\left(\exp\left(-\frac{|x|^2}{2}\right) Du\right) = \mu \exp\left(-\frac{|x|^2}{2}\right) u & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu_\Omega} = 0 & \text{on } \partial\Omega \end{cases}$$

$$\Updownarrow$$

$$\begin{cases} -\Delta u + x \cdot \nabla u = \mu u & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu_\Omega} = 0 & \text{on } \partial\Omega \end{cases}$$

Notation and classical results

Spectral theory on compact self-adjoint operators ensures that

$$\mu_1(\Omega) = \min \left\{ \frac{\displaystyle\int_\Omega |D\psi|^2 d\gamma_N}{\displaystyle\int_\Omega \psi^2 d\gamma_N} : \psi \in H^1(\Omega, d\gamma_N) \setminus \{0\}, \int_\Omega \psi d\gamma_N = 0 \right\}$$

where

$$H^1(\Omega, d\gamma_N) = \left\{ u \in W^{1,1}_{\mathrm{loc}}(\Omega) : (u, |Du|) \in L^2(\Omega, d\gamma_N) \times L^2(\Omega, d\gamma_N) \right\}.$$

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- The case $\Omega = \mathbb{R}^N$:

$$\mu_1\left(\mathbb{R}^N\right) = 1 \iff \int_{\mathbb{R}^N} u^2 d\gamma_N \leq \int_{\mathbb{R}^N} |Du|^2 d\gamma_N$$

$$\forall u \in H^1(\mathbb{R}^N, d\gamma_N) : \int_{\mathbb{R}^N} u d\gamma_N = 0$$

Moreover, 1 is an $N-$degenerate eigenvalue with a corresponding set of independent eigenfunctions given by $u_i(x) = x_i$, with $i \in \{1, ..., N\}$.

Some one-dimensional eigenvalue problems

Let $a, b \in \overline{\mathbb{R}}$ with $a < b$. We denote by $\mu_1(a, b)$ the first nontrivial eigenvalue of

$$\begin{cases} -u'' + xu' = \mu u & \text{in } (a, b) \\ u'(a) = u'(b) = 0, \end{cases}$$

and by $\lambda_1(a, b)$ the first eigenvalue of the problem

$$\begin{cases} -v'' + xv' = \lambda v & \text{in } (a, b) \\ v(a) = v(b) = 0. \end{cases}$$

It is easy to verify that

$$\mu_1(a, b) = 1 + \lambda_1(a, b)$$

and

$$\begin{array}{ccl} \lambda_1(a, b) & \geq & 0 \text{ with } \lambda_1(a, b) = 0 \text{ if and only if } (a, b) = \mathbb{R} \\ & \Updownarrow & \\ \mu_1(a, b) & \geq & 1 \text{ with } \mu_1(a, b) = 1 \text{ if and only if } (a, b) = \mathbb{R} \end{array}$$

The Szegö-Weinberger inequality in Gauss space

Theorem [C. - di Blasio, 2012]

Let Ω be any smooth domain of $\mathbb{R}^N$ symmetric about the origin. Let $B_{R_\Omega}(0)$ be the ball centered at the origin such that $\gamma_N(\Omega) = \gamma_N(B_{R_\Omega}(0))$. Then

$$\mu_1(\Omega) \leq \mu_1(B_{R_\Omega}(0))$$

and equality holds if and only of $\Omega = B_{R_\Omega}(0)$.

Remark 1.

We also show that, even removing the assumption on the symmetry, the half-spaces (i.e. the isoperimetric sets) cannot be optimal in the "Gaussian Szegö-Weinberger" inequality. To this aim we study the behavior of $\mu_1(a, b)$ when the interval (a, b) slides along the x-axis, keeping $\gamma_1(a, b)$ fixed.

Remark 2.

Szegö-Weinberger type inequalities for log-convex weight are contained in Brock - C. - di Blasio, 2016.

The Payne-Weinberger inequality in Gauss space

Theorem [Bakry-Qian, 2000]

If $\Omega \subset \mathbb{R}^N$ is a **bounded**, convex domain, then

$$\mu_1(\Omega) \geq \mu_1(-d(\Omega)/2, d(\Omega)/2).$$

Remark 1.

The assumption on the convexity cannot be removed.

Remark 2.

By the results on the one-dimensional case we have

$$\mu_1(\Omega) \geq \mu_1(-d(\Omega)/2, d(\Omega)/2) = 1 + \lambda_1(-d(\Omega)/2, d(\Omega)/2).$$

Hence any convex domain $\Omega \subset \mathbb{R}^N$ such that $\mu_1(\Omega) = 1$ must be unbounded.

Theorem [Brandolini - C. - Henrot -Trombetti, 2013]

Let $\Omega \subset \mathbb{R}^N$ be any convex domain then

$$\mu_1(\Omega) \geq 1.$$

Equality sign is achieved for if Ω is any $N-$dimensional strip.

The proof is divided into the following steps.

(1) We provide an extension Theorem in $H^1(\Omega, d\gamma_N)$.

(2) We find a sequence of convex, bounded domains $\{\Omega_k\}_{k\in\mathbb{N}}$ invading Ω such that $\lim_{k\to\infty} \mu_1(\Omega_k) = \mu_1(\Omega)$.

(2) We conclude by using the Bakry-Qian estimate

$$\mu_1(\Omega) = \lim_{k\to\infty} \mu_1(\Omega_k) \geq \lim_{k\to\infty} \mu_1\left(-d(\Omega_k), d(\Omega_k)\right) \geq 1.$$

Remark 1

Let $S_{a,b} = \{(x_1, x_2, ..., x_N) \in \mathbb{R}^N : \ a < x_N < b\}$, with $-\infty < a < b < +\infty$. Any eigenfunction corresponding to $\mu_1(S_{a,b})$ must depend on one variable only. Since, as we said before,

$$\mu_1(a, b) = \lambda_1(a, b) + 1 > 1 = \mu_1(\mathbb{R}),$$

we get $\mu_1(S_{a,b}) = 1$ for every $a, b \in \mathbb{R}$ and a corresponding eigenfunction is, for instance, $H_1(x_1) = x_1$.

Remark 2

One can prove the estimare $\mu_1(\Omega) \geq 1$ by using some results contained in Brascamp-Lieb 1976 or in Caffarelli 2000, concerning Poincaré - Wirtinger type inequalities for measures obtained as log-concave perturbations of the Gaussian.

An inverse spectral problem: the equality case in $\mu_1(\Omega) \geq 1$

Theorem [Brandolini - C. - Krejčiřík - Trombetti, to appear]

Let Ω be a convex subset of $S_{y_1,y_2} = \left\{(x, y) \in \mathbb{R}^2 : y_1 < y < y_2\right\}$ for some y_1, $y_2 \in \mathbb{R}$.
If $\mu_1(\Omega) = 1$ then Ω is a strip.

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Remark 1.

As we noticed before, if $\mu_1(\Omega) = 1$, then Ω must be unbounded.

Remark 2.

By employing a separation of variables, we also deduce that Ω is not a semi-strip.

An inverse spectral problem: the equality case in $\mu_1(\Omega) \geq 1$

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Let Ω be a convex subset of $S_{y_1,y_2} = \{(x,y) \in \mathbb{R}^2 : y_1 < y < y_2\}$ for some y_1, $y_2 \in \mathbb{R}$.
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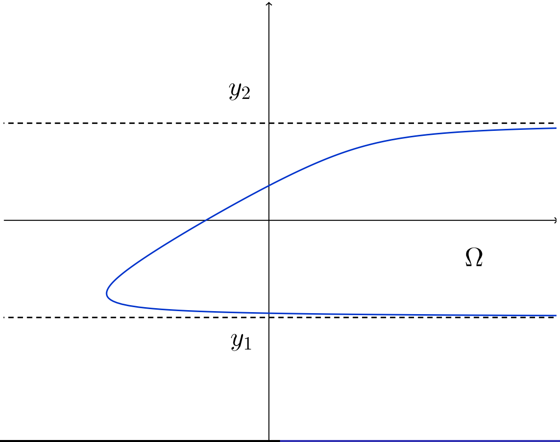

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If $\mu_1(\Omega) = 1$ then Ω is a strip.

The proof is divided in two steps.

(1) Slicing.
We find a sequence $\{\Omega_\varepsilon\}$ of thinner and thinner horizontal slices of Ω such that $\mu_1(\Omega_\varepsilon) = 1 \ \forall \varepsilon > 0$.

(2) Asymptotics.
We show that $\exists (x_0, y_0) \in \partial\Omega : \lim_{\varepsilon \to 0} \mu_1(\Omega_\varepsilon) = \mu_1(x_0, +\infty)$.

Proposition

Let Ω be a convex subset of S_{y_1,y_2} such that $\mu_1(\Omega) = 1$ and let $\bar{y} \in (y_1, y_2)$ be such that the straight-line $y = \bar{y}$ bisects the area $\gamma_2(\Omega)$. Then

$$\mu_1(\Omega \cap \{y < \bar{y}\}) = \mu_1(\Omega \cap \{y > \bar{y}\}) = 1.$$

Proof Let u be an eigenfunction corresponding to $\mu_1(\Omega)$; then

$$\frac{\int_\Omega |Du|^2 d\gamma_2}{\int_\Omega u^2 d\gamma_2} = 1 \qquad \text{and} \qquad \int_\Omega u \, d\gamma_2 = 0.$$

For every $\alpha \in [0, 2\pi]$ there is a unique straight line r_α orthogonal to $(\cos\alpha, \sin\alpha)$ dividing Ω into two convex sets Ω'_α, Ω''_α with equal Gaussian area. Let $\mathcal{I}(\alpha) = \int_{\Omega'_\alpha} u d\gamma_2$. Since $\mathcal{I}(\alpha) = -\mathcal{I}(\alpha + \pi)$, by continuity there is $\bar{\alpha}$ such that

$$\gamma_2\left(\Omega'_{\bar{\alpha}}\right) = \gamma_2\left(\Omega''_{\bar{\alpha}}\right) = \frac{\gamma_2(\Omega)}{2} \text{ and } \int_{\Omega'_{\bar{\alpha}}} u d\gamma_2 = \int_{\Omega''_{\bar{\alpha}}} u d\gamma_2 = 0.$$

Slicing

Observe that

$$\frac{\int_{\Omega'_{\bar\alpha}} |Du|^2 d\gamma_2}{\int_{\Omega'_{\bar\alpha}} u^2 d\gamma_2} \geq \mu_1(\Omega'_{\bar\alpha}) \geq 1, \qquad \frac{\int_{\Omega''_{\bar\alpha}} |Du|^2 d\gamma_2}{\int_{\Omega''_{\bar\alpha}} u^2 d\gamma_2} \geq \mu_1(\Omega''_{\bar\alpha}) \geq 1$$

and

$$\begin{aligned} 1 \quad &= \quad \mu_1(\Omega) = \frac{\int_{\Omega'_{\bar\alpha}} |Du|^2 d\gamma_2 + \int_{\Omega''_{\bar\alpha}} |Du|^2 d\gamma_2}{\int_{\Omega'_{\bar\alpha}} u^2 d\gamma_2 + \int_{\Omega''_{\bar\alpha}} u^2 d\gamma_2} \\ &\geq \quad \min\left\{ \frac{\int_{\Omega'_{\bar\alpha}} |Du|^2 d\gamma_2}{\int_{\Omega'_{\bar\alpha}} u^2 d\gamma_2}, \frac{\int_{\Omega''_{\bar\alpha}} |Du|^2 d\gamma_2}{\int_{\Omega''_{\bar\alpha}} u^2 d\gamma_2} \right\} \geq 1 \end{aligned}$$

$$\Downarrow$$

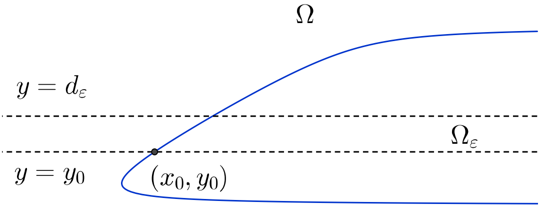
$$1 = \mu_1(\Omega) = \mu_1(\Omega'_{\bar\alpha}) = \mu_1(\Omega''_{\bar\alpha}).$$

By repeating the procedure described in the above Proposition, since at any step we are bisecting the Gaussian area, we can obtain a sequence of unbounded convex domains Ω_ε such that

$$\Omega_\varepsilon = \left\{(x,y) \in \mathbb{R}^2 : \ x > x_0, \ y_0 < y < \min\{f(x), d_\varepsilon\}\right\}$$

$$\mu_1(\Omega_\varepsilon) = 1, \qquad \varepsilon = d_\varepsilon - y_0 \longrightarrow 0$$

(here f is a concave and nondecreasing function such that $f'(x_0) < +\infty$).



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It remains to prove that

$$\boxed{1 = \lim_{\varepsilon \to 0} \mu_1(\Omega_\varepsilon) = \mu_1(x_0, +\infty) = 1 + \lambda_1(x_0, +\infty)}$$

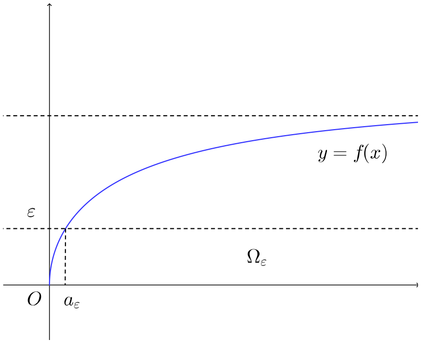
$$\Downarrow$$

$$\lambda_1(x_0, +\infty) = 0 \quad \Rrightarrow \quad x_0 = -\infty.$$

This means that Ω contains a straight-line and then it is a strip.

f concave, nondecreasing, continuous, $f(0) = 0$, $f'(0) < +\infty$

$$\Omega_\varepsilon = \{x > 0,\ 0 < y < f_\varepsilon = \min\{f(x), \varepsilon\}\}$$



$(x_0, y_0) \longrightarrow (0,0)$:

$$\begin{cases} -\mathrm{div}\left(\exp\left(-\frac{(x+x_0)^2+(y+y_0)^2}{2}\right) Du\right) = \mu \exp\left(-\frac{(x+x_0)^2+(y+y_0)^2}{2}\right) u & \text{in } \Omega_\varepsilon \\ \frac{\partial u}{\partial \nu_{\Omega_\varepsilon}} = 0 & \text{on } \partial\Omega_\varepsilon \end{cases}$$

We consider the one-dimensional problem

$$\begin{cases} -\left(\exp\left(-\frac{(x+x_0)^2+y_0^2}{2}\right) v'\right)' = \nu \exp\left(-\frac{(x+x_0)^2+y_0^2}{2}\right) v & \text{in } (0,+\infty) \\ v'(0) = 0 \end{cases}$$

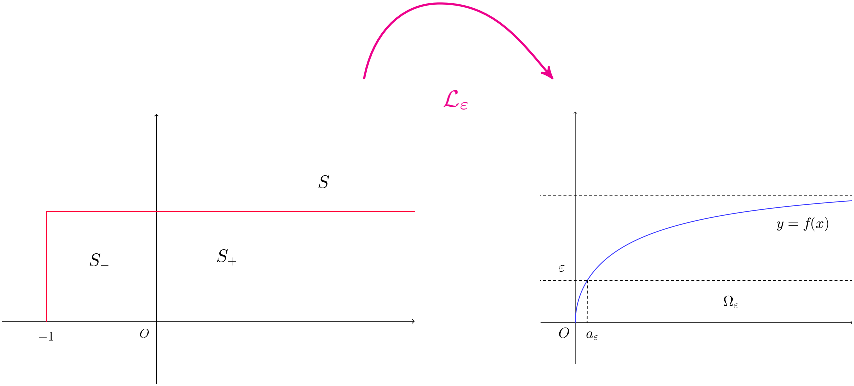
and we prove

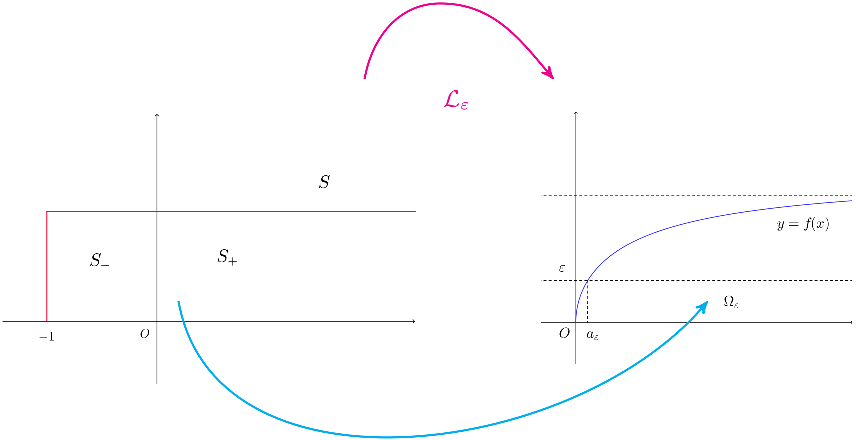
Theorem

For every $k \in \mathbb{N}$

$$\lim_{\varepsilon\to 0} \mu_k(\Omega_\varepsilon) = \nu_k = \mu_k(x_0, +\infty).$$

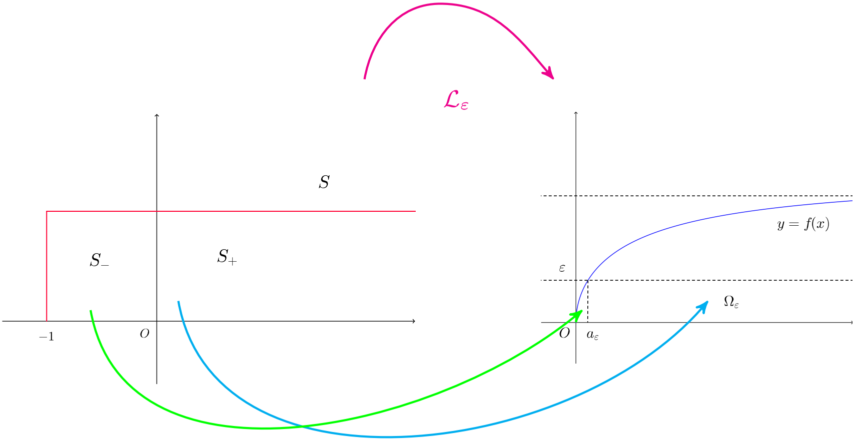
Main strategy



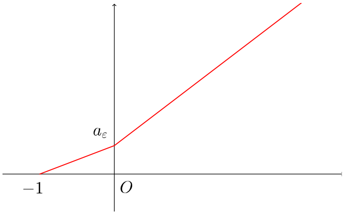


$\mathcal{L}_\varepsilon$
S
S_-
S_+
-1
O
ε
$y = f(x)$
Ω_ε
O
a_ε

Main strategy



Let $\color{red}{g_\varepsilon(s)} = \begin{cases} a_\varepsilon s + a_\varepsilon & \text{if} \quad s \in [-1, 0) \\ s + a_\varepsilon & \text{if} \quad s \in [0, +\infty) \end{cases}$



and

$$\mathcal{L}_\varepsilon : (s, t) \in S \longrightarrow \big(g_\varepsilon(s), f_\varepsilon(g_\varepsilon(s))\, t\big) \in \Omega_\varepsilon$$

$\Rightarrow$ $\mathcal{L}_\varepsilon$ is a $C^{0,1}$ diffeomorphism between S and Ω_ε, whose jacobian

$$j_\varepsilon(s, t) = g'_\varepsilon(s) f_\varepsilon(g_\varepsilon(s))$$

is independent of t and singular at $s = -1$

With the notation

$$\gamma(x,y) = \exp\left(-\frac{(x_0+x)^2+(y_0+y)^2}{2}\right)$$

$$\gamma_\varepsilon(s,t) = (\gamma\circ\mathcal{L}_\varepsilon)(s,t) = \exp\left(-\frac{[x_0+g_\varepsilon(s)]^2+[y_0+f_\varepsilon(g_\varepsilon(s))t]^2}{2}\right)$$

we get that, for any $v\in H^1(\Omega_\varepsilon, d\gamma)$,

$$\int_{\Omega_\varepsilon}|Dv|^2 d\gamma = \int_S\left[\left(\frac{\partial_s v}{g_\varepsilon'} - \frac{f_\varepsilon'\circ g_\varepsilon}{f_\varepsilon\circ g_\varepsilon}\,t\,\partial_t v\right)^2 + \frac{(\partial_t v)^2}{(f_\varepsilon\circ g_\varepsilon)^2}\right]\gamma_\varepsilon\, g_\varepsilon'\,\frac{f_\varepsilon\circ g_\varepsilon}{\varepsilon}\,ds\,dt$$

$$\int_{\Omega_\varepsilon} v^2 d\gamma = \int_S v^2\,\gamma_\varepsilon\, g_\varepsilon'\,\frac{f_\varepsilon\circ g_\varepsilon}{\varepsilon}\,ds\,dt$$

Passing to the limit

Denoted by ψ_ε an eigenfunction corresponding to $\mu_k(\Omega_\varepsilon)$, the weak formulation of the eigenvalue equation reads as

$$\int_{\Omega_\varepsilon} D\psi_\varepsilon \, D\varphi \, d\gamma = \mu_k(\Omega_\varepsilon) \int_{\Omega_\varepsilon} \psi_\varepsilon \, \varphi \, d\gamma \qquad \forall \varphi \in H^1(\Omega_\varepsilon, d\gamma)$$

$$\Updownarrow$$

$$\int_S \left[\left(\frac{\partial_s \psi_\varepsilon}{g_\varepsilon'} - \frac{f_\varepsilon' \circ g_\varepsilon}{f_\varepsilon \circ g_\varepsilon} t \partial_t \psi_\varepsilon \right) \left(\frac{\partial_s \phi}{g_\varepsilon'} - \frac{f_\varepsilon' \circ g_\varepsilon}{f_\varepsilon \circ g_\varepsilon} t \partial_t \phi \right) + \frac{(\partial_t \psi_\varepsilon)}{(f_\varepsilon \circ g_\varepsilon)} \frac{(\partial_t \phi)}{(f_\varepsilon \circ g_\varepsilon)} \right] \gamma_\varepsilon \, g_\varepsilon' \, \frac{f_\varepsilon \circ g_\varepsilon}{\varepsilon} \, dsdt$$

$$= \mu_k(\Omega_\varepsilon) \int_S \psi_\varepsilon \, \phi \, \gamma_\varepsilon \, g_\varepsilon' \, \frac{f_\varepsilon \circ g_\varepsilon}{\varepsilon} \, dsdt \qquad \forall \phi \in H^1 \left(S, \gamma_\varepsilon g_\varepsilon' \, f_\varepsilon \circ g_\varepsilon \, dsdt \right)$$

We want to pass to the limit as $\varepsilon \to 0$.

Proposition.

For any $k \in \mathbb{N}$ there exists $C_k > 0$ such that, for all $\varepsilon \leq 1$, $\mu_k(\Omega_\varepsilon) \leq C_k$.

Proof. Recall that $\mu_k(\Omega_\varepsilon)$ can be characterized by the following Rayleigh-Ritz variational formula

$$\mu_k(\Omega_\varepsilon) = \inf_{\dim V = k+1} \sup_{v \in V} \frac{\displaystyle\int_{\Omega_\varepsilon} |Dv|^2 d\gamma}{\displaystyle\int_{\Omega_\varepsilon} v^2 d\gamma}$$

$$= \inf_{\dim V = k+1} \sup_{v \in V} \frac{\displaystyle\int_S \left[\left(\frac{\partial_s v}{g_\varepsilon'} - \frac{f_\varepsilon' \circ g_\varepsilon}{f_\varepsilon \circ g_\varepsilon} \, t \, \partial_t v \right)^2 + \frac{(\partial_t v)^2}{(f_\varepsilon \circ g_\varepsilon)^2} \right] \gamma_\varepsilon \, g_\varepsilon' \, \frac{f_\varepsilon \circ g_\varepsilon}{\varepsilon} \, ds \, dt}{\displaystyle\int_S v^2 \, \gamma_\varepsilon \, g_\varepsilon' \, \frac{f_\varepsilon \circ g_\varepsilon}{\varepsilon} \, ds \, dt}.$$

In $S_+ = (0, +\infty) \times (0, 1)$ it holds that

$$\gamma_-(s) \leq \gamma_\varepsilon(s,t) \leq \gamma_+(s), \quad g_\varepsilon' = 1, \quad \frac{f_\varepsilon \circ g_\varepsilon}{\varepsilon} = 1.$$

A uniform upper bound for $\mu_k(\Omega_\varepsilon)$

Hence choosing a test function independent of t, $v \in C_0^\infty(0,+\infty)$, we get

$$\frac{\displaystyle\int_{\Omega_\varepsilon} |Dv|^2 d\gamma}{\displaystyle\int_{\Omega_\varepsilon} v^2 d\gamma} \le \frac{\displaystyle\int_0^{+\infty} v'(s)^2 \gamma_+(s) ds}{\displaystyle\int_0^{+\infty} v(s)^2 \gamma_-(s) ds}.$$

$$\Downarrow$$

$$\mu_k(\Omega_\varepsilon) \le \inf_{\substack{\dim V = k+1 \\ V \subset C_0^\infty(0,+\infty)}} \sup_{v \in V} \frac{\displaystyle\int_0^{+\infty} v'(s)^2 \gamma_+(s) ds}{\displaystyle\int_0^{+\infty} v(s)^2 \gamma_-(s) ds}$$

$\uparrow$

eigenvalues of the one-dimensional operator

$$-\gamma_-^{-1} \partial_s \gamma_+ \partial_s,$$

subject to Dirichlet boundary conditions

What happens in S_-?

Take a normalized eigenfunction ψ_ε corresponding to $\mu_k(\Omega_\varepsilon)$:

$$\begin{aligned}
\int_{\Omega_\varepsilon} \psi_\varepsilon^2 d\gamma &= \int_S \psi_\varepsilon^2 \, \gamma_\varepsilon \, g_\varepsilon' \, \frac{f_\varepsilon \circ g_\varepsilon}{\varepsilon} ds\, dt = 1 \\
\int_{\Omega_\varepsilon} |D\psi_\varepsilon|^2 d\gamma &= \int_S \left[\left(\frac{\partial_s \psi_\varepsilon}{g_\varepsilon'} - \frac{f_\varepsilon' \circ g_\varepsilon}{f_\varepsilon \circ g_\varepsilon} \, t \, \partial_t \psi_\varepsilon \right)^2 + \frac{(\partial_t \psi_\varepsilon)^2}{(f_\varepsilon \circ g_\varepsilon)^2} \right] \gamma_\varepsilon \, g_\varepsilon' \, \frac{f_\varepsilon \circ g_\varepsilon}{\varepsilon} \, ds\, dt \\
&= \mu_k(\Omega_\varepsilon)
\end{aligned}$$

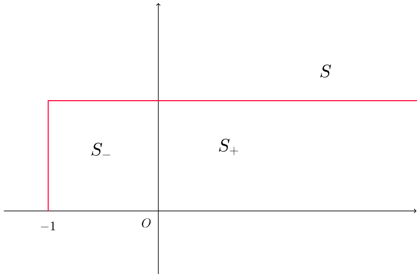
Take a normalized eigenfunction ψ_ε corresponding to $\mu_k(\Omega_\varepsilon)$:

$$\int_{\Omega_\varepsilon} \psi_\varepsilon^2 d\gamma = \int_S \psi_\varepsilon^2 \gamma_\varepsilon\, g_\varepsilon' \frac{f_\varepsilon \circ g_\varepsilon}{\varepsilon} ds\, dt = 1$$

$$\int_{\Omega_\varepsilon} |D\psi_\varepsilon|^2 d\gamma = \int_S \left[\left(\frac{\partial_s \psi_\varepsilon}{g_\varepsilon'} - \frac{f_\varepsilon' \circ g_\varepsilon}{f_\varepsilon \circ g_\varepsilon}\, t\, \partial_t \psi_\varepsilon\right)^2 + \frac{(\partial_t \psi_\varepsilon)^2}{(f_\varepsilon \circ g_\varepsilon)^2}\right] \gamma_\varepsilon\, g_\varepsilon' \frac{f_\varepsilon \circ g_\varepsilon}{\varepsilon}\, ds\, dt$$

$$= \mu_k(\Omega_\varepsilon)$$

- $\gamma_\varepsilon(s,t) \geq c_0 > 0$,
- $g_\varepsilon' = a_\varepsilon$,
- $1 \geq \dfrac{f_\varepsilon \circ g_\varepsilon}{\varepsilon} \geq s + 1.$



What happens in S_-?

$$\boxed{\int_{S_-} (\partial_s \psi_\varepsilon)^2 (s+1)\, ds\, dt + \left(\frac{a_\varepsilon}{\varepsilon}\right)^2 \int_{S_-} (\partial_t \psi_\varepsilon)^2 (s+1)\, ds\, dt \le C a_\varepsilon}$$

$$f'(0) < +\infty \quad \oplus \quad f(s) \le f'(0)s$$

$$\Downarrow$$

$$\epsilon = f(a_\epsilon) \le f'(0) a_\epsilon \quad \text{and} \quad \frac{1}{f'(0)^2} \le \left(\frac{a_\varepsilon}{\varepsilon}\right)^2$$

$$\Downarrow$$

$$\int_{S_-} |\nabla \psi_\epsilon|^2 (s+1) ds dt \le C a_\epsilon \to 0 \quad \text{as} \quad \epsilon \to 0.$$

What happens in S_-?

Writing $\psi_\varepsilon(s,t) = \varphi_\varepsilon + \eta_\varepsilon(s,t)$ with φ_ε *constant*, and

$$\int_{S_-} \eta_\varepsilon(s,t)(s+1)ds\,dt = 0,$$

we get

$$\pi^2 \int_{S_-} \eta_\varepsilon^2(s+1)ds\,dt \leq \int_{S_-} |D\eta_\varepsilon|^2(s+1)ds\,dt \leq Ca_\varepsilon.$$

We then prove, with some more effort, that

$$\varphi_\varepsilon^2 \leq C \qquad \text{on } S_-.$$

Hence

$$\int_{S_-} \psi_\varepsilon^2 \gamma_\varepsilon\, g_\varepsilon' \frac{f_\varepsilon \circ g_\varepsilon}{\varepsilon} dsdt \leq Ca_\varepsilon \int_{S_-} (\varphi_\varepsilon + \eta_\varepsilon)^2 (s+1)dsdt \to 0 \quad \text{as} \quad \epsilon \to 0.$$

This means that what happens in S_- becomes more and more negligible as ϵ goes to zero. In the limit only S_+ does matter.

What happens in S_+?

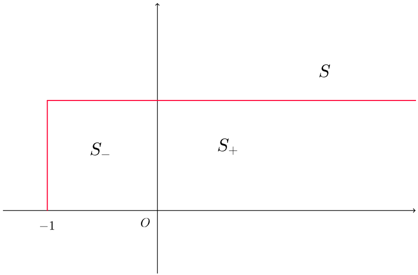
- $\gamma_\varepsilon(s,t) \geq \rho_\varepsilon(s)\gamma_0(s)$ with

$$\lim_{\varepsilon \to 0} \rho_\varepsilon = 1$$

and

$$\gamma_0(s) = \exp\left(-\frac{(x_0+s)^2 + y_0^2}{2}\right),$$

- $g'_\varepsilon = 1, \qquad \dfrac{f_\varepsilon \circ g_\varepsilon}{\varepsilon} = 1.$



What happens in S_+?

$$\boxed{\int_{S_+} (\partial_s \psi_\varepsilon)^2 \, \rho_\varepsilon \, \gamma_0 \, ds \, dt + \int_{S_+} \frac{(\partial_t \psi_\varepsilon)^2}{\varepsilon^2} \, \rho_\varepsilon \, \gamma_0 \, ds \, dt \le C}$$

Writing

$$\psi_\varepsilon(s,t) = \varphi_\varepsilon(s) + \eta_\varepsilon(s,t) \qquad \text{with } \int_0^1 \eta_\varepsilon(s,t)dt = 0$$

we get that (up to a subsequence)

$$\begin{aligned} \sqrt{\rho_\varepsilon}\varphi_\varepsilon &\rightharpoonup \varphi_0 && \text{in } H^1((0,+\infty), d\gamma_0) \\ \sqrt{\rho_\varepsilon}\varphi_\varepsilon &\to \varphi_0 && \text{in } L^2((0,+\infty), d\gamma_0) \end{aligned}$$

Conclusions

Now, in the weak formulation of the eigenvalue equation, we consider test functions $\phi(s,t) = \varphi(s)$, where $\varphi \in C_0^\infty(\mathbb{R})$ and $\varphi' = 0$ on $[-1, 0]$, and take the limit as $\varepsilon \to 0$.

Conclusions

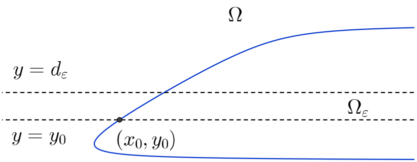
Now, in the weak formulation of the eigenvalue equation, we consider test functions $\phi(s,t) = \varphi(s)$, where $\varphi \in C_0^\infty(\mathbb{R})$ and $\varphi' = 0$ on $[-1,0]$, and take the limit as $\varepsilon \to 0$.

After many computations.....

as $\varepsilon \to 0$ only S_+ matters and

- $\lim_{\varepsilon \to 0} \int_{\Omega_\varepsilon} D\varphi \, D\psi_\varepsilon \, d\gamma_\varepsilon = \int_0^{+\infty} \varphi' \varphi_0' \gamma_0 \, ds$
- $\lim_{\varepsilon \to 0} \int_{\Omega_\varepsilon} \varphi \psi_\varepsilon d\gamma_\varepsilon = \int_0^{+\infty} \varphi \varphi_0 \gamma_0 ds$
- $||\psi_\varepsilon||_{L^2(\Omega_\varepsilon, d\gamma)} \to ||\varphi_0||_{L^2((0,+\infty), d\gamma_0)}$
- $\lim_{\varepsilon \to 0} \mu_k(\Omega_\varepsilon) = \nu_k = \mu_k(x_0, +\infty)$

Coming back to the uniqueness of optimal sets



$$1 = \lim_{\varepsilon \to 0} \mu_1(\Omega_\varepsilon) = \mu_1(x_0, +\infty) = 1 + \lambda_1(x_0, +\infty)$$

$$\Downarrow$$

$$\lambda_1(x_0, +\infty) = 0 \quad \Rrightarrow \quad x_0 = -\infty$$