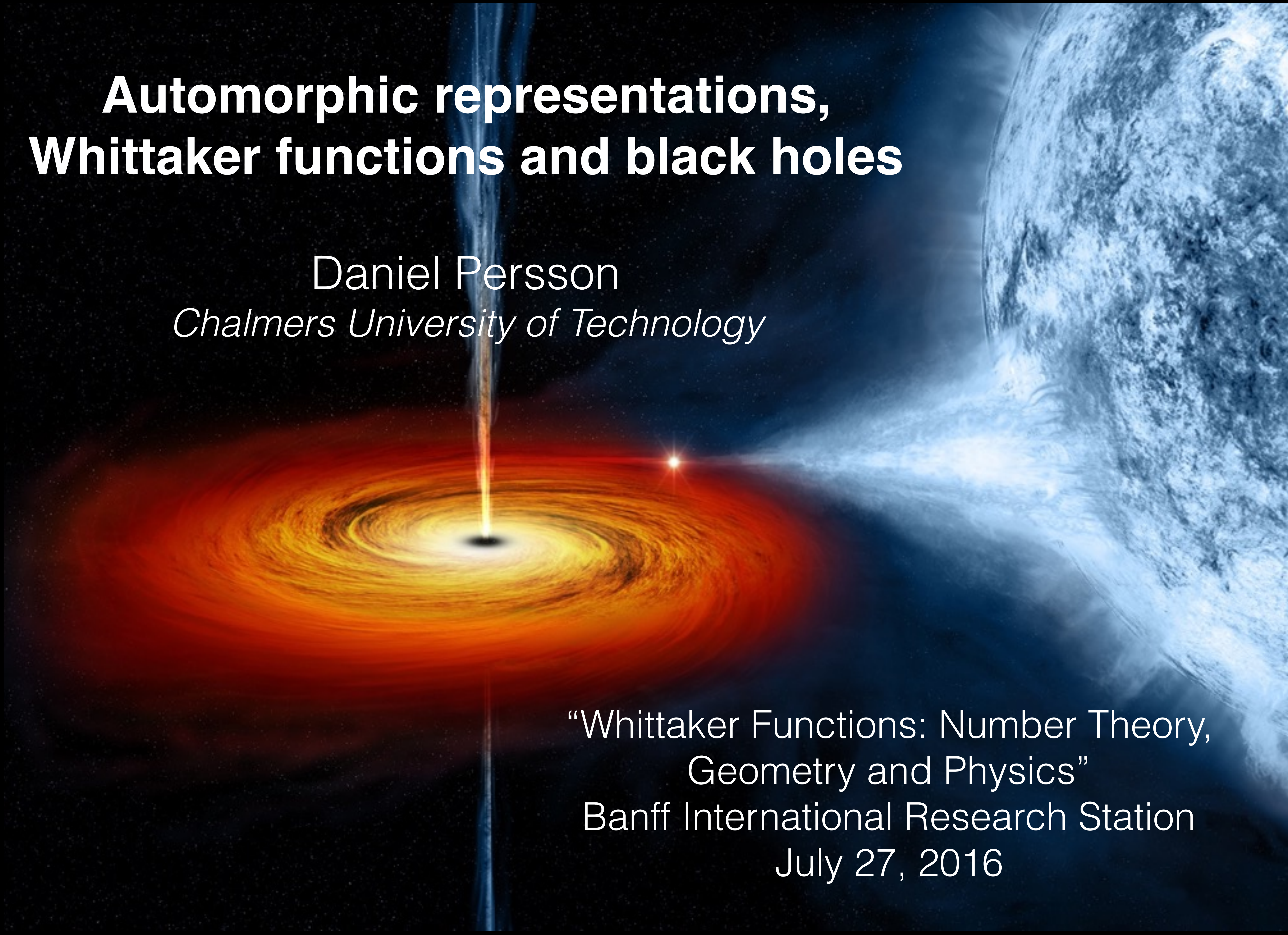

Automorphic representations, Whittaker functions and black holes

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"Whittaker Functions: Number Theory, Geometry and Physics"

Banff International Research Station

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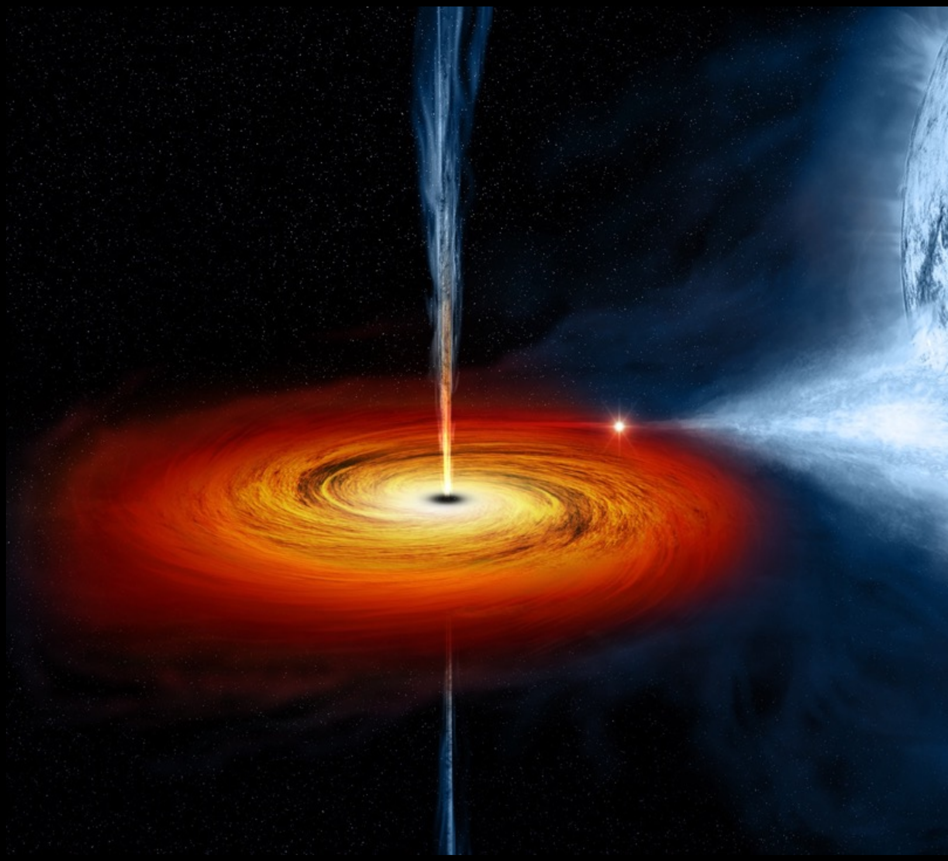
Outline



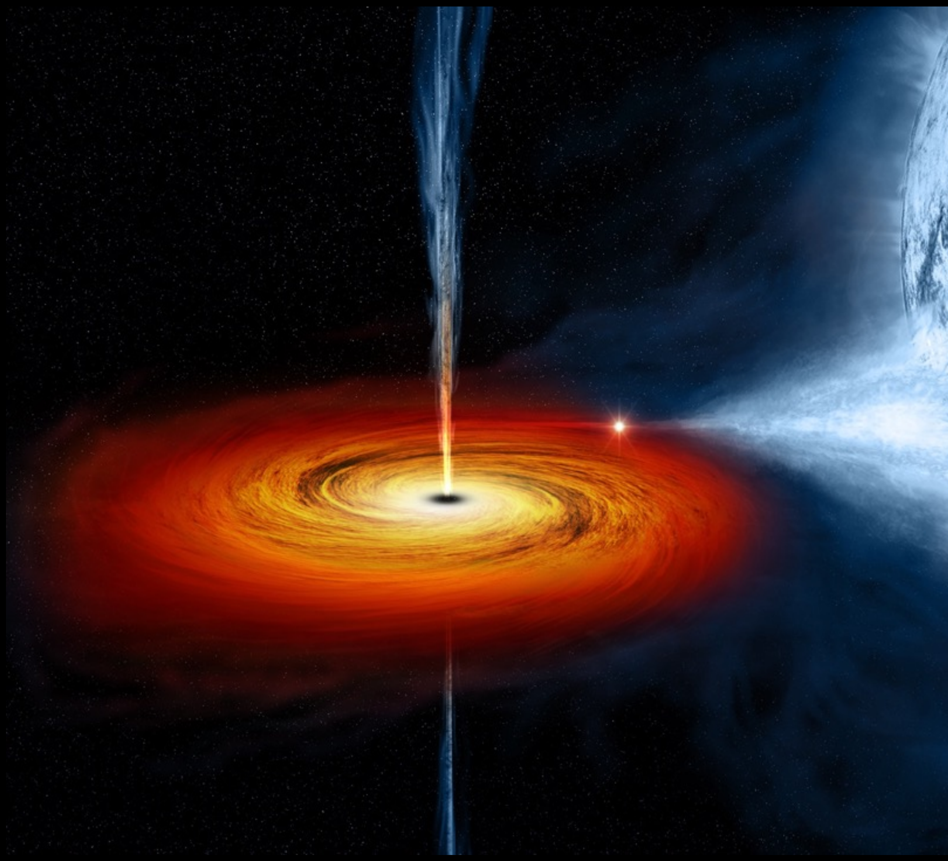
1. Prelude: Black holes

2. Fourier coefficients of Eisenstein series

3. Minimal representations of exceptional groups

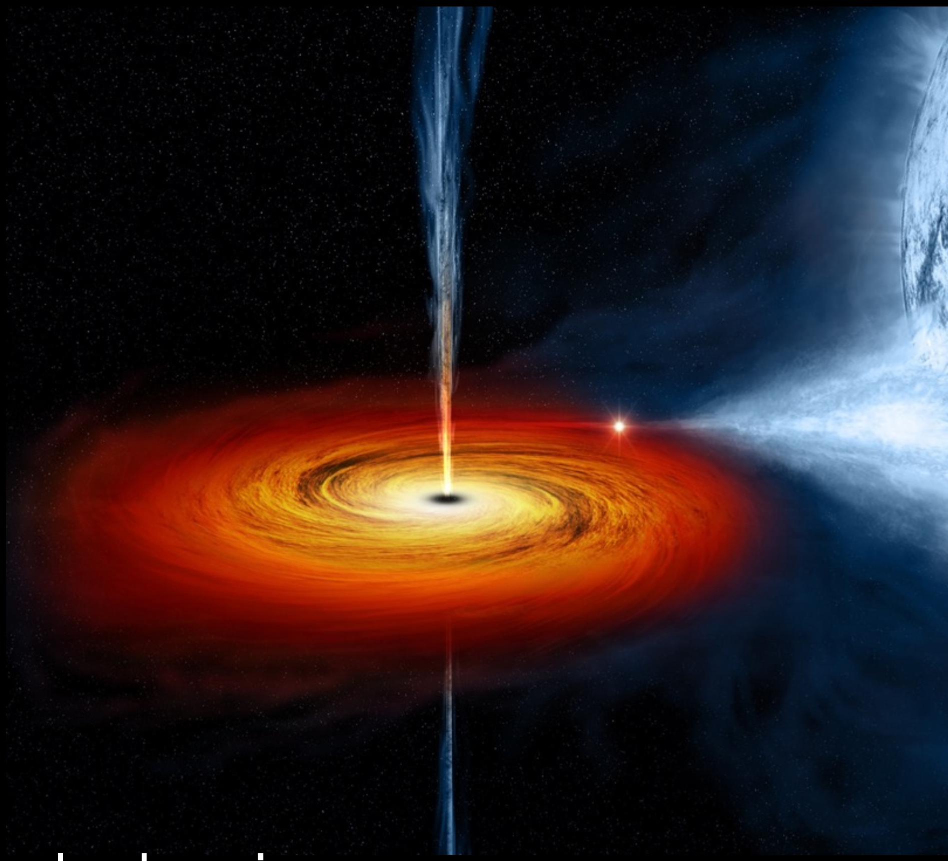
4. Next-to-minimal representations

5. Outlook: Conjectures and open problems



1. Prelude: Black holes

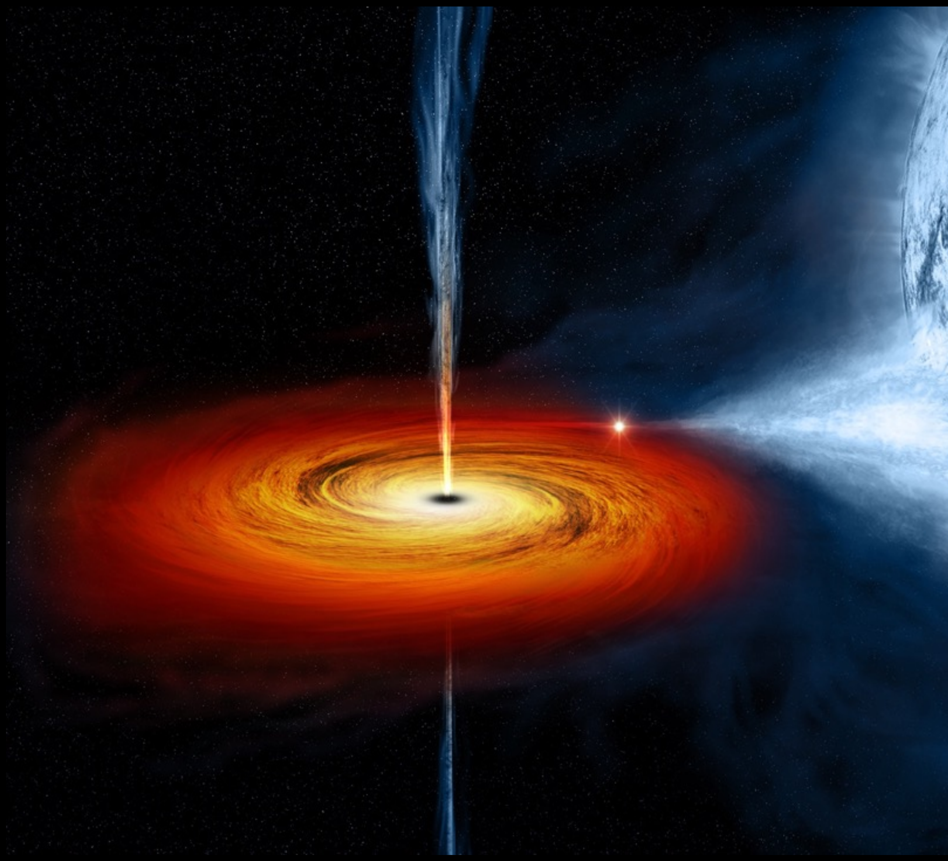
Black holes are some of the most fascinating objects in the universe



A black hole forms in the final stage of the collapse of a sufficiently large star

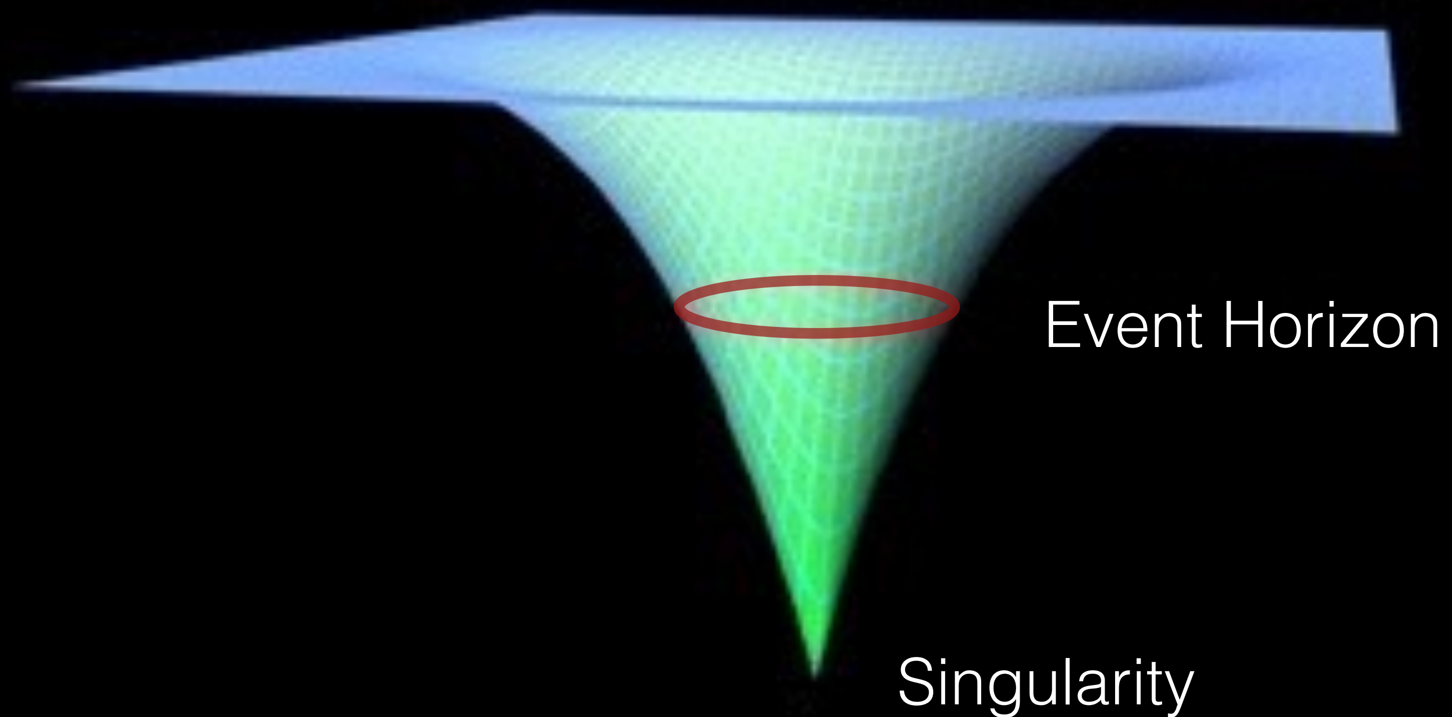
Nothing may escape after crossing the black hole horizon

Black holes are some of the most fascinating objects in the universe



A black hole forms in the final stage of the collapse of a sufficiently large star

At its centre space and time breaks down into a **singularity**



Quantum effects should resolve the singularity

But black holes are not black!

Hawking radiation!

Black holes behave like a black body with entropy

$$S = \frac{\text{Area}}{4 G_N \hbar}$$

Bekenstein-Hawking formula

Black holes behave like a black body with entropy

$$S = \frac{\text{Area}}{4 G_N \hbar}$$

Bekenstein-Hawking formula

This is an amazing equation!

$$\text{Statistical Physics} = \frac{\text{Gravity}}{\text{Quantum Mechanics}}$$

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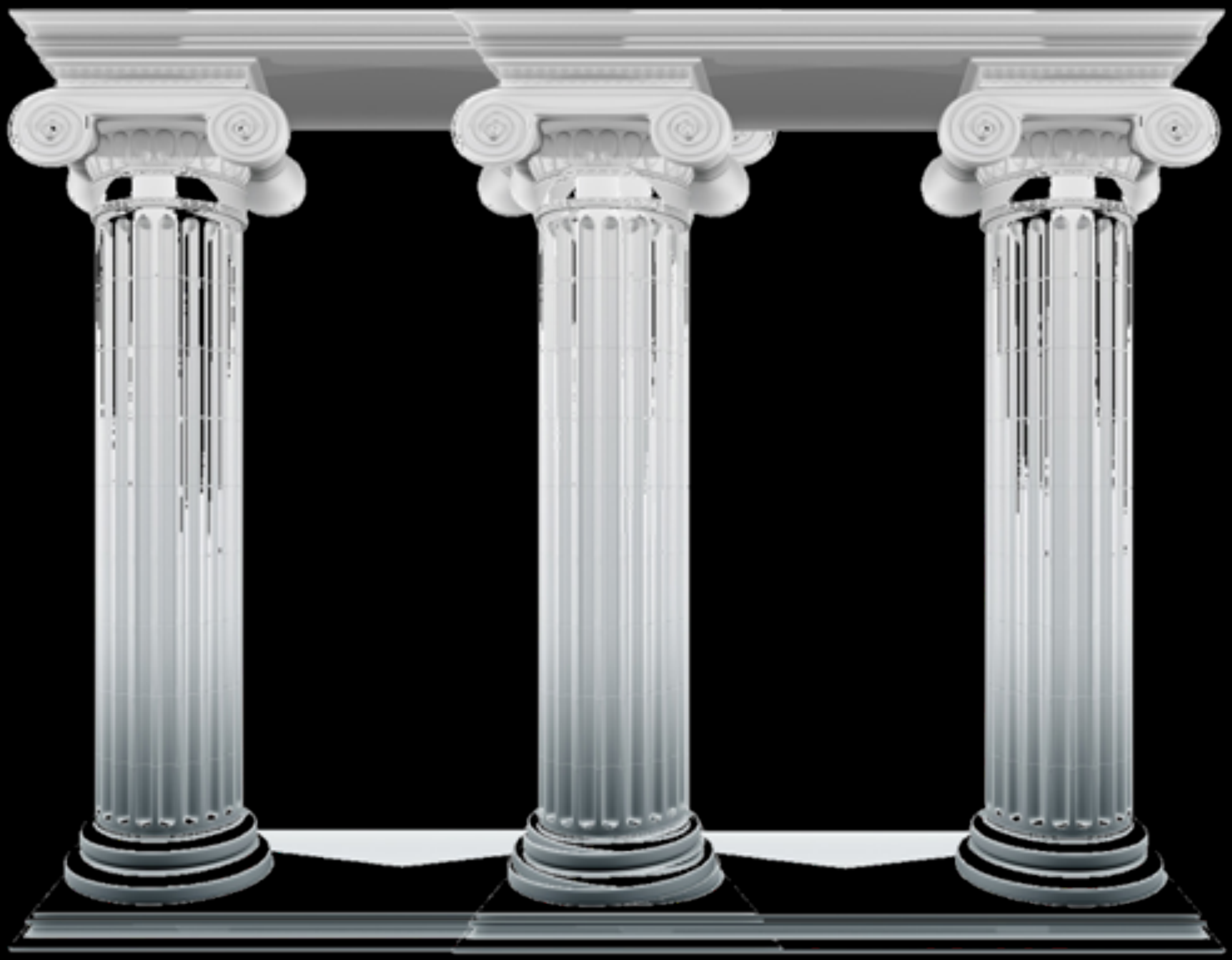
This relates all three
pillars of theoretical physics!



A **quantum theory of gravity** must be able to provide a microscopic account for the black hole entropy

$$S = \log\left(\sharp\,\text{microstates}\right)$$

A **quantum theory of gravity** must be able to provide a microscopic account for the black hole entropy

$$S = \log\left(\sharp\,\mathrm{microstates}\right)$$

$$= \log\,\Omega(Q)$$

$Q =$ electromagnetic charge

Enter number theory in physics

$$\log\,\Omega(Q) = \frac{A(Q)}{4G_N\hbar}$$

Enter number theory in physics

We package the microstates into a **partition function**:

$$Z(\beta) = \sum_{\text{states}} e^{-\beta E_{\text{states}}} \qquad \beta = 1/T$$

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In string theory the partition function is an **automorphic form!**

Enter number theory in physics

We package the microstates into a **partition function**:

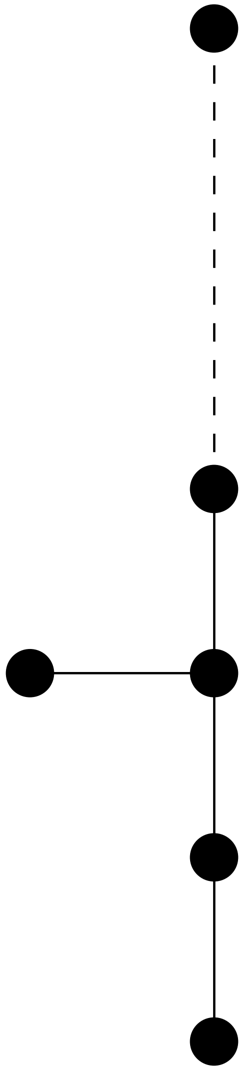
$$Z(\beta) = \sum_{\text{states}} e^{-\beta E_{\text{states}}} \qquad \beta = 1/T$$

In string theory the partition function is an **automorphic form!**

Finding the black hole microstate degeneracies corresponds to calculating **Fourier coefficients** of automorphic forms

Toroidal compactifications yield the famous chain of **U-duality groups**

[Cremmer, Julia][Hull, Townsend]



D	G	K	$G(\mathbb{Z})$
10	$\mathrm{SL}(2,\mathbb{R})$	$\mathrm{SO}(2)$	$\mathrm{SL}(2,\mathbb{Z})$
9	$\mathrm{SL}(2,\mathbb{R}) \times \mathbb{R}^+$	$\mathrm{SO}(2)$	$\mathrm{SL}(2,\mathbb{Z})$
8	$\mathrm{SL}(3,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$	$\mathrm{SO}(3) \times \mathrm{SO}(2)$	$\mathrm{SL}(3,\mathbb{Z}) \times \mathrm{SL}(2,\mathbb{Z})$
7	$\mathrm{SL}(5,\mathbb{R})$	$\mathrm{SO}(5)$	$\mathrm{SL}(5,\mathbb{Z})$
6	$\mathrm{Spin}(5,5,\mathbb{R})$	$\big(\mathrm{Spin}(5) \times \mathrm{Spin}(5)\big)/\mathbb{Z}_2$	$\mathrm{Spin}(5,5,\mathbb{Z})$
5	$\mathrm{E}_6(\mathbb{R})$	$\mathrm{USp}(8)/\mathbb{Z}_2$	$\mathrm{E}_6(\mathbb{Z})$
4	$\mathrm{E}_7(\mathbb{R})$	$\mathrm{SU}(8)/\mathbb{Z}_2$	$\mathrm{E}_7(\mathbb{Z})$
3	$\mathrm{E}_8(\mathbb{R})$	$\mathrm{Spin}(16)/\mathbb{Z}_2$	$\mathrm{E}_8(\mathbb{Z})$

Partition functions are given by **automorphic forms** on

$$G(\mathbb{Z})\backslash G(\mathbb{R})/K$$

Green, Gutperle, Sethi, Vanhove, Kiritsis, Pioline, Obers, Kazhdan, Waldron, Basu, Russo, Cederwall, Bao, Nilsson, D.P., Lambert, West, Gubay, Miller, Fleig, Kleinschmidt,…

What is known?

Certain partition functions are Eisenstein series attached to **small automorphic representations** of G .

[Green, Miller, Vanhove][Pioline]

minimal automorphic representation	next-to-minimal automorphic representation
π_{min}	π_{ntm}

Fourier coefficients of these functions reveal perturbative and **non-perturbative** quantum effects. Very hard to compute!

2. Fourier coefficients of Eisenstein series

Mainly based on our recent papers:

[1511.04265] w/ Fleig, Gustafsson, Kleinschmidt

[1412.5625] w/ Gustafsson, Kleinschmidt

[1312.3643] w/ Fleig, Kleinschmidt



and work in progress with Gourevitch and Sahi


Adelic Eisenstein series

- G split, simply-laced semisimple Lie group over $\mathbb{Q}$
- $B = AN$ **Borel subgroup**
- quasi-character: $\chi : B(\mathbb{A}) \to \mathbb{C}^\times$
- **induced representation:** $\mathrm{Ind}_{B(\mathbb{A})}^{G(\mathbb{A})}\chi = \prod_p \mathrm{Ind}_{B(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}\chi_p$
- $f_\chi \in \mathrm{Ind}_{B(\mathbb{A})}^{G(\mathbb{A})}\chi$ **unique spherical standard section**

$$f_\chi(g) = f_\chi(nak) = \chi(na) = \chi(a)$$

$$f_\chi = \prod f_{\chi_p}$$

Adelic Eisenstein series

Associated to this data we construct the **Eisenstein series**

$$E(f_\chi, g) = \sum_{\gamma \in B(\mathbb{Q})\backslash G(\mathbb{Q})} f_\chi(\gamma g) \qquad g \in G(\mathbb{A})$$

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$$E(f_\chi, g) = \sum_{\gamma \in B(\mathbb{Q}) \backslash G(\mathbb{Q})} f_\chi(\gamma g) \qquad g \in G(\mathbb{A})$$

It is also convenient to represent this in the following form:

$$E(\lambda, g) = \sum_{\gamma \in B(\mathbb{Q}) \backslash G(\mathbb{Q})} e^{\langle \lambda + \rho | H(\gamma g) \rangle} \qquad \lambda \in \mathfrak{h}^\star \otimes \mathbb{C}$$

It converges absolutely in the Godement range of λ.

Adelic Eisenstein series

Associated to this data we construct the **Eisenstein series**

$$E(f_\chi, g) = \sum_{\gamma \in B(\mathbb{Q})\backslash G(\mathbb{Q})} f_\chi(\gamma g) \qquad g \in G(\mathbb{A})$$

For a **unitary character** $\psi : N(\mathbb{Q})\backslash N(\mathbb{A}) \to U(1)$

we have the **Whittaker-Fourier coefficient**

$$W_\psi(f_\chi, g) = \int_{N(\mathbb{Q})\backslash N(\mathbb{A})} E(f_\chi, ng)\overline{\psi(n)}dn$$

It is well-known that this is Eulerian: [Langlands]

$$W_\psi(f_\chi, g) = W_\infty(f_{\chi_\infty}, g_\infty) \times \prod_{p<\infty} W_p(f_{\chi_p}, g_p)$$

with $g_\infty \in G(\mathbb{R}),\ g_p \in G(\mathbb{Q}_p)$ **and**

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with $g_\infty \in G(\mathbb{R}),\ g_p \in G(\mathbb{Q}_p)$ and

$$W_\infty(f_{\chi_\infty}, g_\infty) = \int_{N(\mathbb{R})} f_{\chi_\infty}(ng_\infty)\overline{\psi_\infty(n)}dn$$

$$W_p(f_{\chi_p}, g_p) = \int_{N(\mathbb{Q}_p)} f_{\chi_p}(ng_p)\overline{\psi_p(n)}dn$$

can be computed using the **CS-formula**

More general Fourier coefficients

- $P = LU$ **standard parabolic** of G

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More general Fourier coefficients

- $P = LU$ **standard parabolic** of G
- **unitary character** $\psi_U : U(\mathbb{Q})\backslash U(\mathbb{A}) \to U(1)$
- We then have the U -**Fourier coefficient:**

$$F_{\psi_U}(f_\chi, g) = \int_{U(\mathbb{Q})\backslash U(\mathbb{A})} E(f_\chi, ug)\overline{\psi_U(u)}du$$

much less is known in general in this case…

$$F_{\psi_U}(f_\chi, g) = \int_{U(\mathbb{Q})\backslash U(\mathbb{A})} E(f_\chi, ug)\overline{\psi_U(u)}du$$

- These are not Eulerian in general, no CS-formula…
- It is sufficient to determine the coefficient for one representative in each Levi orbit of ψ_U
- Each Levi orbit is contained in some complex nilpotent G -orbit
- It is fruitful to restrict to small automorphic representations.

String theory limits - choices of unipotent subgroups

→ Decompactification limit

- perturbative parameter: radius of decompactified circle
- non-perturbative effects: KK-instantons, BPS-instantons

→ String perturbation limit

- perturbative parameter: string coupling
- non-perturbative effects: D-instantons, NS5-instantons

→ M-theory limit

- perturbative parameter: volume of M-theory torus
- non-perturbative effects: M2- & M5-instantons

3. Minimal representations of exceptional groups

Minimal automorphic representations

Definition: *An automorphic representation*

$$\pi = \bigotimes_{p\leq\infty} \pi_p$$

is minimal if each factor π_p *has smallest non-trivial Gelfand-Kirillov dimension.*

[Joseph][Brylinski, Kostant][Ginzburg, Rallis, Soudry][Kazhdan, Savin]....

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[Joseph][Brylinski, Kostant][Ginzburg, Rallis, Soudry][Kazhdan, Savin]....

Automorphic forms $\varphi \in \pi_{min}$ are characterised by having **very few non-vanishing Fourier coefficients**. [Ginzburg, Rallis, Soudry]

Maximal parabolic subgroups

Now consider the case when $P = LU$ is a **maximal parabolic**

This implies that U only contains **a single simple root** α

Now choose a representative in the Levi orbit which is only sensitive to this simple root:

$$\psi_U = \psi\Big|_U = \psi_\alpha$$

This is non-trivial only on the simple root space N_α

Theorem [Miller-Sahi]: *Let* G *be a split group of type* E_6 *or* E_7

Then any Fourier coefficient of $\varphi \in \pi_{min}$ *of* G *is completely determined by the maximally degenerate Whittaker coefficients*

$$W_{\psi_\alpha}(\varphi, g) = \int_{N(\mathbb{Q})\backslash N(\mathbb{A})} \varphi(ng)\overline{\psi_\alpha(n)}dn$$

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$$W_{\psi_\alpha}(\varphi, g) = \int_{N(\mathbb{Q})\backslash N(\mathbb{A})} \varphi(ng)\overline{\psi_\alpha(n)}dn$$

Can one use this to calculate

$$F_{\psi_U}(\varphi, g) = \int_{U(\mathbb{Q})\backslash U(\mathbb{A})} E(\varphi, ug)\overline{\psi_U(u)}du$$

in terms of W_{ψ_α} **?**

What is the relation between the **degenerate Whittaker coefficient**:

$$W_{\psi_\alpha}(\varphi, g) = \int_{N(\mathbb{Q})\backslash N(\mathbb{A})} \varphi(ng)\overline{\psi_\alpha(n)}dn$$

and the U **-coefficient**:

$$F_{\psi_U}(\varphi, g) = \int_{U(\mathbb{Q})\backslash U(\mathbb{A})} E(\varphi, ug)\overline{\psi_U(u)}du$$

?

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A priori they live on **different spaces!**

$$W_{\psi_\alpha}(nak) = \psi_\alpha(n)W_{\psi_\alpha}(a) \qquad F_{\psi_U}(ulk) = \psi_U(u)F_{\psi_U}(l)$$

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$$F_{\psi_U}(\varphi, g) = \int_{U(\mathbb{Q})\backslash U(\mathbb{A})} E(\varphi, ug)\overline{\psi_U(u)}du \qquad ?$$

Conjecture [Gustafsson, Kleinschmidt, D.P.]:

For $\varphi \in \pi_{min}$ these two functions are equal.

Proof: *In progress with [Gourevitch, Gustafsson, Kleinschmidt, D.P., Sahi]*

Example: Let $G = SL(3, \mathbb{A})$ [Gustafsson, Kleinschmidt, D.P.]

$$\psi_\alpha(x) = \psi_\alpha(e^{2\pi i(uE_\alpha + vE_\beta)}) = e^{2\pi i n u}, \qquad n \in \mathbb{Q},\ u \in \mathbb{A}$$

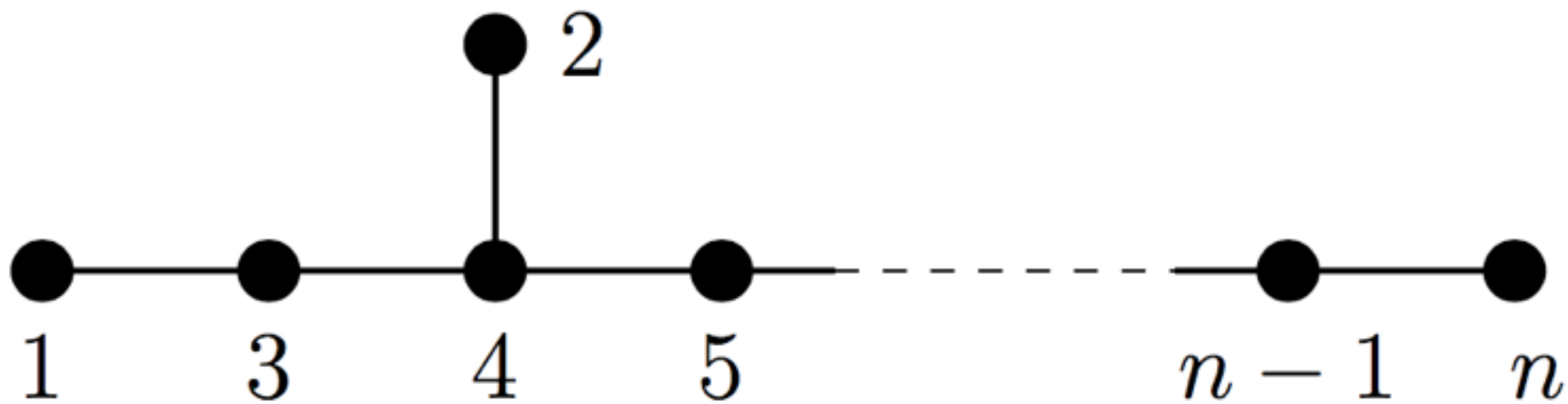
$$U = \left\{ \begin{pmatrix} 1 & u_1 & u_2 \\ & 1 & \\ & & 1 \end{pmatrix} : u_i \in \mathbb{A} \right\}$$

In this case we find the following equality

$$F_{\psi_{U_{m,n}}}(\varphi, g) = W_{\psi_n}\left(\varphi, \begin{pmatrix} -1 & 0 & \\ 0 & 0 & -1 \\ 0 & -1 & m/n \end{pmatrix} g\right)$$

so the functions are **equal up to a Levi translate** of the argument!

Exceptional groups



Functional dimension of minimal representations:

$$\mathrm{GKdim}\,\pi_{min} = \begin{cases} 11, & E_6 \\ 17, & E_7 \\ 29, & E_8 \end{cases}$$

Automorphic realization

Consider the Borel-Eisenstein series on $G(\mathbb{A})$

$$E(\lambda, g) = \sum_{\gamma \in B(\mathbb{Q}) \backslash G(\mathbb{Q})} e^{\langle \lambda + \rho | H(\gamma g) \rangle}$$

Now fix the weight to

$$\lambda = 2s\Lambda_1 - \rho$$

where Λ_1 is the fundamental weight associated to node 1.

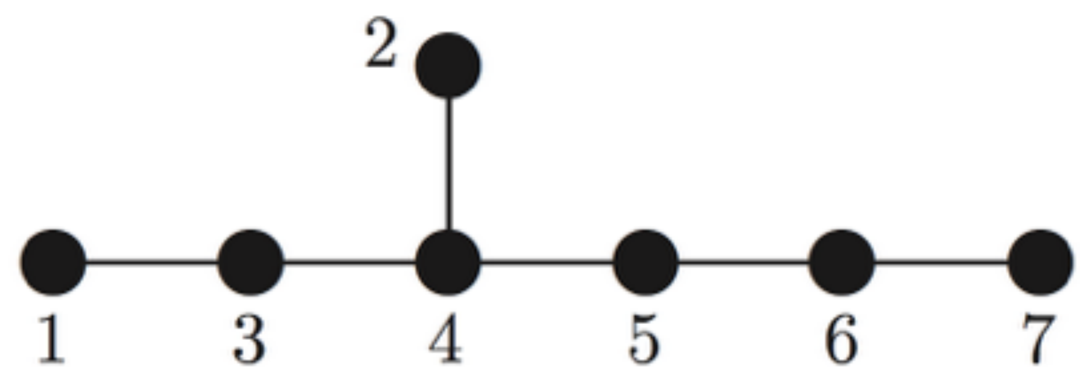
Theorem [Ginzburg,Rallis,Soudry][Green,Miller,Vanhove]

For $G = E_6, E_7, E_8$ *the Eisenstein series* $E(2s\Lambda - \rho, g)$ *evaluated at* $s = 3/2$ *is attached to the representation* π_{min} *with wavefront set* $WF(\pi_{min}) = \overline{\mathcal{O}_{min}}$.

This theorem yields an explicit automorphic realisation of the minimal representation.

Our aim is to use this to calculate Fourier coefficients associated with maximal parabolic subgroups.

Example: $G = E_7$



Consider the **3-grading** of the Lie algebra

$$\mathfrak{e}_7 \quad = \quad \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \quad = \quad \mathbf{27} \oplus (\mathfrak{e}_6 \oplus \mathbf{1}) \oplus \mathbf{27}$$

The space $\mathfrak{g}_0 \oplus \mathfrak{g}_1$ is the Lie algebra of a maximal parabolic $P = LU$ with 27-dim unipotent U and Levi $L = E_6 \times GL(1)$

The degenerate Whittaker vector associated with α_1 is given by:

[Fleig, Kleinschmidt, D.P.]

$$W_{\psi_k}(3/2, a) = |k|^{3/2}\sigma_{-3}(k)K_{3/2}(2\pi|k|a)$$

where $a \in A \subset E_7$ and

$$\sigma_s(k) = \sum_{d|k} d^s$$

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where $a \in A \subset E_7$ and

$$\sigma_s(k) = \sum_{d|k} d^s$$

We now want to relate this to the U - Fourier coefficient

$$F_{\psi_U}(3/2, g) = \int_{U(\mathbb{Q})\backslash U(\mathbb{A})} E(3/2, ug)\overline{\psi_U(u)}du$$

This captures **instantons in the decompactification limit** of string theory!

Conjecture: [Pioline][Gustafsson, Kleinschmidt, D.P.][Bossard, Verschinin]

$$F_{\psi_U}(3/2; h, r) = |k|^{3/2}\sigma_{-3}(k)K_{3/2}(2\pi r|k| \times ||h^{-1}\vec{x}||)$$

where $h \in E_6$, $r \in GL(1)$ and $\vec{x} \in \mathbb{Z}^{27}$

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Proof: ***To appear by*** *[Gustafsson, Gourevitch, Kleinschmidt, D.P., Sahi]*

This gives the **complete abelian Fourier expansion** of the minimal representation

Physically the vector $\vec{x}$ corresponds to the **instanton charges** of the 27 vector fields in D=5.

4. Next-to-minimal representations

Properties of π_{ntm}

No multiplicity one theorem known for π_{ntm} .

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Theorem [Green, Miller, Vanhove]: *Let* $G = E_6, E_7, E_8$
The Eisenstein series

$$E(s,g) = \sum_{\gamma \in B(\mathbb{Q})\backslash G(\mathbb{Q})} e^{\langle 2s\Lambda_1 | H(\gamma g)\rangle}$$

evaluated at $s = 5/2$ *is a spherical vector in* π_{ntm} .

Whittaker coefficients attached to π_{ntm}

Theorem **[Fleig, Kleinschmidt, D.P.]**:

The abelian term of the Fourier expansion of $E(5/2, g)$ *is given by*

$$\sum_{\substack{\psi : N(\mathbb{Q})\backslash N(\mathbb{A}) \to U(1) \\ \psi \neq 1}} W_\psi(5/2, na) = \sum_{\alpha \in \Pi} \sum_{\psi_\alpha} c_\alpha(a) W_{\psi_\alpha}(5/2, na)$$

$$+ \sum_{\substack{\alpha, \beta \in \Pi \\ [E_\alpha, E_\beta] = 0}} \sum_{\psi_{\alpha,\beta}} c_{\alpha,\beta}(a) W_{\psi_{\alpha,\beta}}(5/2, na)$$

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$$+ \sum_{\substack{\alpha,\beta\in\Pi\\ [E_\alpha,E_\beta]=0}}\sum_{\psi_{\alpha,\beta}} c_{\alpha,\beta}(a) W_{\psi_{\alpha,\beta}}(5/2,na)$$

Bala-Carter type $2A_1$ (product of two K-Bessel functions)

Conjecture [Gustafsson, Kleinschmidt, D.P.]:

Let G be a semisimple, simply laced Lie group.

Then all Fourier coefficients of $\varphi \in \pi_{ntm}$ are completely determined by degenerate Whittaker vectors of the form

$$W_{\psi_\alpha}(\varphi, g) = \int_{N(\mathbb{Q})\backslash N(\mathbb{A})} \varphi(ng)\overline{\psi_\alpha(n)}dn$$

$$W_{\psi_{\alpha,\beta}}(\varphi, g) = \int_{N(\mathbb{Q})\backslash N(\mathbb{A})} \varphi(ng)\overline{\psi_{\alpha,\beta}(n)}dn$$

where (α, β) are commuting simple roots.

Proof. *In progress with [Gustafsson, Gourevitch, Kleinschmidt, D.P., Sahi]*

5. Outlook: Conjectures and open problems

Spherical vectors for Kac-Moody groups

Let $G = E_9, E_{10}, E_{11}$. The Eisenstein series $E(3/2, g)$ has a partial Fourier expansion [Fleig, Kleinschmidt, D.P.]

$$E(3/2, g) = E_0 + \sum_{\alpha \in \Pi} \sum_{\psi_\alpha} c_\alpha(a) W_{\psi_\alpha}(3/2, na) + \text{“non-ab”}$$

where $W_{\psi_\alpha}(3/2, na) = \prod_{p \leq \infty} W_p(3/2, na)$.

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where $W_{\psi_\alpha}(3/2, na) = \prod_{p \leq \infty} W_p(3/2, na)$.

Conjecture: *The minimal representation of* E_9, E_{10}, E_{11} *exists, factorizes*

$$\pi_{min} = \otimes_p \pi_{min,p}$$

satisfies a uniqueness property, and $W_p(3/2, na)$ *is (the abelian limit of) a spherical vector in* $\pi_{min,p}$

String theory on Calabi-Yau 3-folds

In general, very little is known about the duality group in this case.

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However, consider the case of X a **rigid** CY3-fold. $(h_{2,1}(X) = 0)$

Intermediate Jacobian of X is an **elliptic curve**:

$$H^3(X, \mathbb{R})/H^3(X, \mathbb{Z}) = \mathbb{C}/\mathcal{O}_d$$

ring of integers: $\mathcal{O}_d \subset \mathbb{Q}(\sqrt{-d})$ ($d > 0$ and square-free)

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ring of integers: $\mathcal{O}_d \subset \mathbb{Q}(\sqrt{-d})$ ($d > 0$ and square-free)

Conjecture: [Bao, Kleinschmidt, Nilsson, D.P., Pioline]

String theory on X is invariant under the **Picard modular group**

$$PU(2,1;\mathcal{O}_d) := U(2,1) \cap PGL(3,\mathcal{O}_d)$$

Theorem: [Bao, Kleinschmidt, Nilsson, D.P., Pioline]

The Borel Eisenstein series

$$E(\chi_s, P, g) = \sum_{\gamma \in P(\mathcal{O}_d)\backslash PU(2,1;\mathcal{O}_d)} \chi_s(\gamma g)$$

has Fourier coefficients

$$F_{\psi_U}(s, g) = \int_{U(\mathcal{O}_d)\backslash U} E(\chi_s, P, ug)\overline{\psi_U(u)}du$$

Theorem: [Bao, Kleinschmidt, Nilsson, D.P., Pioline]

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$$E(\chi_s, P, g) = \sum_{\gamma \in P(\mathcal{O}_d)\backslash PU(2,1;\mathcal{O}_d)} \chi_s(\gamma g)$$

has Fourier coefficients

$$F_{\psi_U}(s,g) = \int_{U(\mathcal{O}_d)\backslash U} E(\chi_s, P, ug)\overline{\psi_U(u)}du$$

$$= F_{\psi_U,\infty}(s,g) \times \prod_{p<\infty} F_{\psi_U,p}(s,1)$$

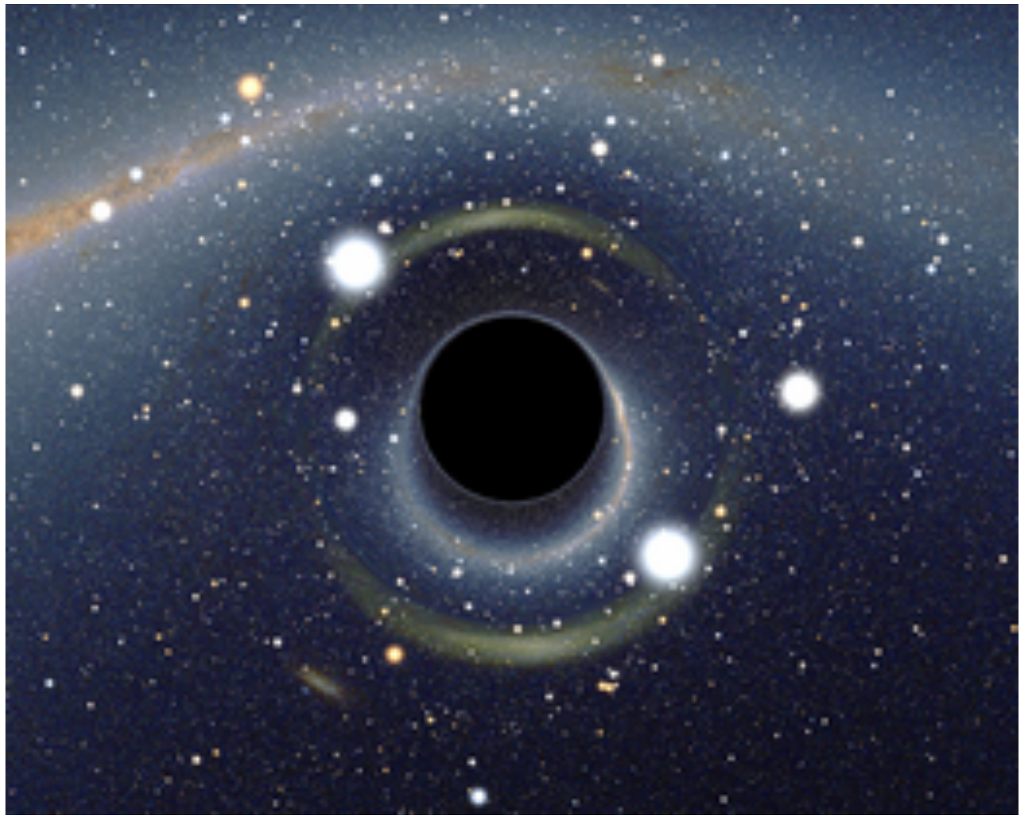
where

$$\prod_{p<\infty} F_{\psi_U,p}(s,1) = \sum_{\substack{\omega\in\mathcal{O}_d \\ \gamma/\omega\in\mathcal{O}_d^\star}} \left|\frac{\gamma}{\omega}\right|^{2s-2} \sum_{\substack{z\in\mathcal{O}_d \\ \gamma/(z\omega)\in\mathcal{O}_d^\star}} |z|^{4-4s}$$

This theory has a lattice of **electric magnetic charges**

$$\Gamma = H_3(X, \mathbb{Z}) \cong \mathbb{Z}^2$$

There are **black hole states** with charge $\gamma \in \Gamma$



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"small" (non-generic) representations of the super-Poincaré algebra

(BPS = Bogomol'nyi–Prasad–Sommerfeld)

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BPS-index: $\Omega(\gamma) \equiv$ number of BPS states of charge γ

$$\Omega \,:\, \Gamma \rightarrow \mathbb{Z}$$

Black hole entropy: $S(\gamma) = \log\,\Omega(\gamma)$

Conjecture: [Bao, Kleinschmidt, Nilsson, D.P., Pioline]

The counting of BPS-black holes in string theory on X *with charges* $\gamma \in H_3(X,\mathbb{Z})$ *is given by the Fourier coefficient*

$$\Omega(\gamma) = \sum_{\substack{\omega \in \mathcal{O}_d \\ \gamma/\omega \in \mathcal{O}_d^\star}} \left|\frac{\gamma}{\omega}\right|^{2s-2} \sum_{\substack{z \in \mathcal{O}_d \\ \gamma/(z\omega) \in \mathcal{O}_d^\star}} |z|^{4-4s}$$

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The function $\Omega(\gamma)$ *counts the number of special Lagrangian submanifolds of* X *in the homology class* $[\gamma] \in H_3(X, \mathbb{Z})$.

Quaternionic discrete series

For string theory on Calabi-Yau 3-folds with $h_{1,1}(X) = 1$ we expect that the duality group is the **exceptional Chevalley group** $G_2(\mathbb{Z})$.

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This is precisely the constraint satisfied by **Fourier coefficients of automorphic forms** attached to the **quaternionic discrete series** of $G_2(\mathbb{R})$. [Wallach][Gan, Gross, Savin]

Quaternionic discrete series

The **quaternionic discrete series** can be realised as [Gross, Wallach]

$$\pi_k = H^1(\mathcal{Z}, \mathcal{O}(-k)) \qquad\qquad k \geq 2$$

where $\mathcal{Z}$ is the twistor space:

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Open problem: Can one construct explicit **automorphic forms attached** to π_k in terms of **holomorphic functions** on $\mathcal{Z}$?

Black hole counting in string theory

Consider string theory on torus T^6

Moduli space: $\mathcal{M} = E_7(\mathbb{Z})\backslash E_7(\mathbb{R})/(SU(8)/\mathbb{Z}_2)$

This theory has a lattice of **electric magnetic charges**

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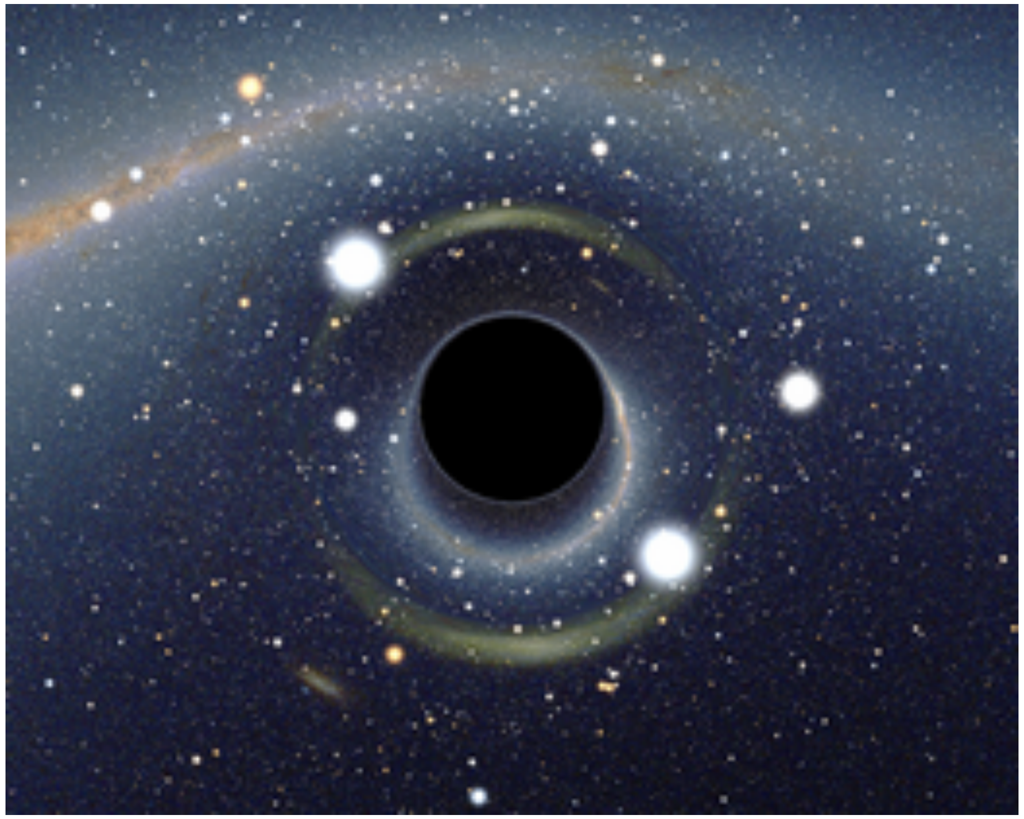

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For 1/2 BPS-states only charges in a 28-dimensional subspace $\mathcal{C} \subset \mathbb{Z}^{56}$ are realised.

Constraint: $\Omega(\gamma) = 0 \quad$ if $\quad \gamma \notin \mathcal{C}$

Symmetry: $\Omega(\gamma)$ must be $E_7(\mathbb{Z})$-invariant

A **generating function** of these states takes the form

$$Z(l,u) = \sum_{\gamma=(x_1,\ldots,x_{56})\in\mathbb{Z}^{56}} \Omega(\gamma) c_\gamma(l) e^{2\pi i(x_1 u_1 \cdots x_{56} u_{56})}$$

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This is precisely the structure of the abelian Fourier coefficients of an automorphic form φ on E_8 with respect to the **Heisenberg unipotent radical** $Q \subset E_8$

$$\sum_{\psi:Q(\mathbb{Q})\backslash Q(\mathbb{A})\to U(1)} F_{\psi_Q}(\varphi,l)\psi_Q(u)$$

If we take $\varphi \in \pi_{min}$ so $\mathrm{GKdim}(\pi_{min}) = 29$ then

$$F_{\psi_Q}(\varphi, g) = \int_{Q(\mathbb{Q})\backslash Q(\mathbb{A})} \varphi(ug)\overline{\psi_Q(u)}du = \prod_{p\leq\infty} F_{\psi,p}(\varphi, g)$$

vanishes unless ψ_Q lies in a 28-dimensional subspace of $\mathfrak{g}_1(\mathbb{Q})$.

[Kazhdan, Polishchuk]

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[Kazhdan, Polishchuk]

Conjecture:

The 1/2 BPS-states are counted by the p-adic spherical vectors in the minimal representation of E_8 *:*

$$\Omega(\gamma) = \prod_{p<\infty} F_{\psi_Q,p}(\pi_{min}, 1)$$

[Pioline][Gunaydin, Neitzke, Pioline, Waldron][Fleig, Gustafsson, Kleinschmidt, D.P.]

Final question: [Moore]

Is there a <u>natural</u> role for automorphic L-functions in BPS-state counting problems?