A Hamiltonian approach to p-adic Whittaker functions

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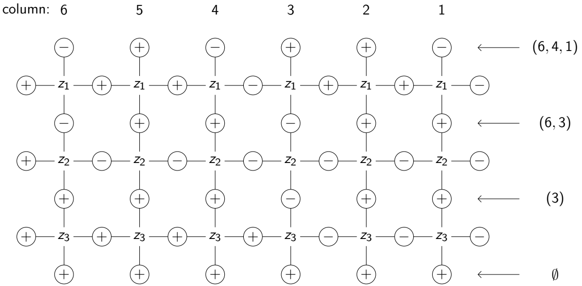
Some key ideas from the Previous Talk

- In Dan's talk, you saw that the values of the p-adic spherical Whittaker function for GL_r (and its metaplectic covers) could be represented by generating functions on 6-vertex models.

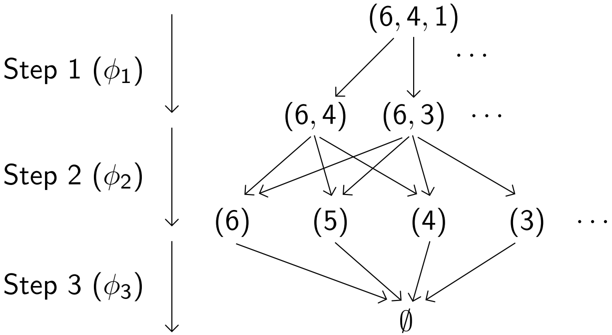
- Summands in the generating function correspond to assignments of edges on a lattice with fixed boundary conditions. E.g.:



- The Yang-Baxter equation was used to explore properties of this generating function, i.e., of the spherical Whittaker functions.

A new point of view: discrete time evolution

Instead, we may view the generating function as the result of (d)evolving from a partition $(6, 4, 1)$ down to the empty partition $\emptyset$ in which one (non-zero) part is lost with each application:



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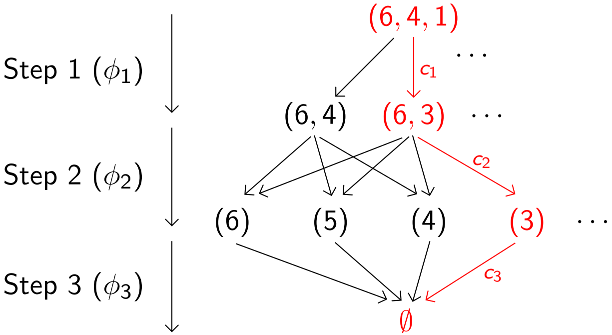


A physicist might write (using bra-ket notation):

$$\langle\emptyset|\,\phi_3\phi_2\phi_1\,|(6,4,1)\rangle = c_1c_2c_3$$

But how to find such operators ϕ_i?

Jimbo and Miwa's "Solitons and ∞-dim'l Lie algebras"

Form a Clifford algebra A over $\mathbb{C}$ using generators ψ_j and ψ_j^* for each half integer j, satisfying relations:

$$\psi_k\psi_j + \psi_j\psi_k = 0 = \psi_k^*\psi_j^* + \psi_j^*\psi_k^* \quad \text{and} \quad \psi_j\psi_k^* + \psi_k^*\psi_j = \delta_{kj}$$

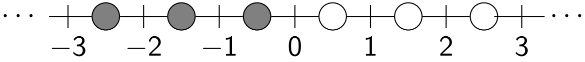
Consider the cyclic left A-module $\mathcal{F} := A/A\mathcal{W}_{\text{ann}}$ with generator $|\emptyset\rangle$ where

$$\mathcal{W}_{\text{ann}} := \oplus_{i<0}\mathbb{C}\psi_i^* \oplus \oplus_{i>0}\mathbb{C}\psi_i.$$

It is better to understand this module using PICTURES!

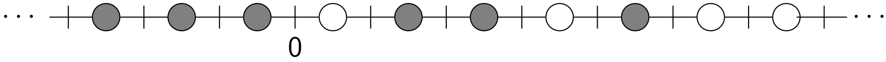
The module $\mathcal{F}$ in pictures

First, we represent the vacuum state $|\emptyset\rangle$ with a sea of particles (black dots) occupying each negative half-integer position:

$|\emptyset\rangle =$ 

Then ψ_j and ψ_j^* act as deletion and creation operators at the position j, respectively. Trying to create a particle where one already exists, or deleting one that isn't already there, produces 0. E.g., $\psi^*_{-\frac{1}{2}} |\emptyset\rangle = 0$.

Given a strict partition $\lambda = (5, 3, 2)$ we have the fermion representation $|\lambda\rangle := \psi^*_{5-\frac{1}{2}} \psi^*_{3-\frac{1}{2}} \psi^*_{2-\frac{1}{2}} |\emptyset\rangle =$



Note there are signs: $\psi^*_{5-\frac{1}{2}} \psi^*_{2-\frac{1}{2}} \psi^*_{3-\frac{1}{2}} |\emptyset\rangle = - |\lambda\rangle$

Back to algebra: $\mathcal{F}$ as a $\mathfrak{gl}(\infty)$ module

The Lie algebra $\mathfrak{gl}(\infty)$ inside A is built from quadratic elements:

$$\mathfrak{gl}(\infty) := \mathbb{C}\cdot 1 \oplus \left\{ \sum_{i,j} a_{i,j} : \psi_i^* \psi_j : \middle| \ \exists N \text{ such that } a_{i,j} = 0 \text{ if } |i-j| > N \right\},$$

where we take

$$: \psi_i^* \psi_j : = - : \psi_j \psi_i^* : \stackrel{def}{=} \begin{cases} \psi_i^* \psi_j & \text{if } i > 0 \\ -\psi_j \psi_i^* & \text{if } i < 0 \end{cases}$$

The element

$$J_0 = \sum_{i \in \mathbb{Z} - \frac{1}{2}} : \psi_i^* \psi_i :$$

is central, and its eigenspaces decompose $\mathcal{F}$ into irreducible representations $\mathcal{F}_\ell$ of $\mathfrak{gl}(\infty)$, indexed by integers ℓ. Their highest weight vectors are:

$$|\ell\rangle := \begin{cases} \psi_{\ell+\frac{1}{2}} \cdots \psi_{-\frac{1}{2}} |\emptyset\rangle & \text{if } \ell < 0 \\ |\emptyset\rangle & \text{if } \ell = 0 \\ \psi^*_{\ell-\frac{1}{2}} \cdots \psi^*_{\frac{1}{2}} |\emptyset\rangle & \text{if } \ell > 0. \end{cases}$$

Action of creation/annihilation operators

More generally, for $q \in \mathbb{Z}$ we define the operator J_q by

$$J_q = \sum_{r \in \mathbb{Z}+\frac{1}{2}} :\psi^*_{r-q}\psi_r: \quad \text{where} \quad :\psi^*_j\psi_k: := \begin{cases} \psi^*_j\psi_k & \text{when } j > 0 \\ -\psi_k\psi^*_j & \text{when } j < 0. \end{cases}$$

Let's do an example using our pictorial model...

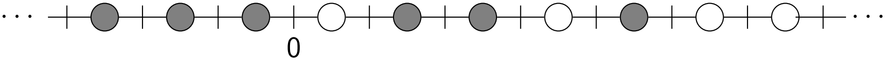
Computing $J_1 \left|(5,3,2)\right\rangle$ carefully...

Recall

$$J_1 = \sum_{r \in \mathbb{Z}+\frac{1}{2}} :\psi^*_{r-1}\psi_r: \quad \text{with} \quad :\psi^*_j\psi_k: := \begin{cases} \psi^*_j\psi_k & \text{when } j>0 \\ -\psi_k\psi^*_j & \text{when } j<0. \end{cases}$$

Up to sign, $:\psi^*_{r-1}\psi_r:$ deletes a particle at position $r-1/2$ and fills a particle one slot to the left at $r-3/2$.

Let's apply it to $\left|(5,3,2)\right\rangle := \psi^*_{5-\frac{1}{2}}\psi^*_{3-\frac{1}{2}}\psi^*_{2-\frac{1}{2}}\left|\emptyset\right\rangle$:



Computing $J_1 \, |(5,3,2)\rangle$ carefully...

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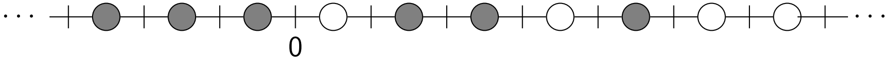
Let's apply it to $|(5,3,2)\rangle := \psi^*_{5-\frac{1}{2}}\psi^*_{3-\frac{1}{2}}\psi^*_{2-\frac{1}{2}}\,|\emptyset\rangle$:



$$\begin{aligned} J_1\,|(5,3,2)\rangle &= \left(\psi^*_{4-\frac{1}{2}}\psi_{5-\frac{1}{2}} + \psi^*_{1-\frac{1}{2}}\psi_{2-\frac{1}{2}}\right)|(5,3,2)\rangle \\ &= \psi^*_{4-\frac{1}{2}}\psi_{5-\frac{1}{2}}\psi^*_{5-\frac{1}{2}}\psi^*_{3-\frac{1}{2}}\psi^*_{2-\frac{1}{2}}\,|\emptyset\rangle + \psi^*_{1-\frac{1}{2}}\psi_{2-\frac{1}{2}}\psi^*_{5-\frac{1}{2}}\psi^*_{3-\frac{1}{2}}\psi^*_{2-\frac{1}{2}}\,|\emptyset\rangle \\ &= \psi^*_{4-\frac{1}{2}}\psi^*_{3-\frac{1}{2}}\psi^*_{2-\frac{1}{2}}\,|\emptyset\rangle + (-1)^2\psi^*_{1-\frac{1}{2}}\psi^*_{5-\frac{1}{2}}\psi^*_{3-\frac{1}{2}}\,|\emptyset\rangle \\ &= \psi^*_{4-\frac{1}{2}}\psi^*_{3-\frac{1}{2}}\psi^*_{2-\frac{1}{2}}\,|\emptyset\rangle + (-1)^2(-1)^2\psi^*_{5-\frac{1}{2}}\psi^*_{3-\frac{1}{2}}\psi^*_{1-\frac{1}{2}}\,|\emptyset\rangle \\ &= |(4,3,2)\rangle + |(5,3,1)\rangle\,. \end{aligned}$$

Time evolution of fermions

We calculated $J_1 |(5,3,2)\rangle = |(4,3,2)\rangle + |(5,3,1)\rangle$. In general...

Proposition

If q is positive, then J_q acts on states by considering all ways to move one particle q units leftward.

We prove a more general identity for the following power series. Let

$$H[\mathbf{t}] := \sum_{q=1}^{\infty} t_q J_q, \quad \text{with } \mathbf{t} = (t_1, t_2, \dots)$$

and

$$e^{H[\mathbf{t}]} := \sum_{k=1}^{\infty} \frac{H[\mathbf{t}]^k}{k!}.$$

Then one can prove

$$e^{H[\mathbf{t}]} \left(\sum_{k \in \mathbb{Z} - \frac{1}{2}} \psi_{-k} z^{k-\frac{1}{2}} \right) e^{-H[\mathbf{t}]} = e^{\sum_{q=1}^{\infty} t_q z^q} \sum_{k \in \mathbb{Z} - \frac{1}{2}} \psi_{-k} z^{k-\frac{1}{2}}$$

Tau functions

Boson-Fermion Correspondence

The following map is an isomorphism of vector spaces from $\mathcal{F}_\ell$ to V_ℓ, where each $V_\ell \simeq \mathbb{C}[\mathbf{t}]$:

$$a\,|\emptyset\rangle \longmapsto \langle \ell \mid e^{H[\mathbf{t}]} a\,|\emptyset\rangle\,.$$

Moreover, choosing a with $a\,|\emptyset\rangle = |\lambda;\ell\rangle$ ($|\lambda\rangle$ with a shifted vacuum at ℓ),

$$\langle \ell|\, e^{H[\mathbf{t}]}\,|\lambda;\ell\rangle =: s_\lambda[\mathbf{t}],$$

where the variables t_q in $\mathbf{t} = (t_1, t_2, ...)$ are (up to a simple factor of $\frac{1}{q}$) equal to the power sum symmetric functions. Thus changing variables:

$$t_q = \frac{1}{q}\sum_{i=1}^{n} x_i^q$$

gives the Schur function s_λ as a symmetric polynomial in $x_1, \ldots, x_n$.

Bra-kets of this form are called "tau functions."

Everything we've discussed so far...

- We use configurations of fermions to model this module for $\mathfrak{gl}(\infty)$
- Out of this analysis, we produce a Hamiltonian operator in three steps:
    1. $J_q := \sum_{r\in\mathbb{Z}+\frac{1}{2}} :\psi^*_{r-q}\psi_r:$
    2. $H[\mathbf{t}] := \sum_{q=1}^{\infty} t_q J_q, \quad \text{with } \mathbf{t} = (t_1, t_2, \ldots)$
    3. $e^{H[\mathbf{t}]} := \sum_{k=1}^{\infty} \frac{H[\mathbf{t}]^k}{k!}$.
- The so-called "tau functions" $\langle \ell |\, e^{H[\mathbf{t}]}\, |\lambda; \ell\rangle$ are Schur functions upon setting

$$t_q = \frac{1}{q}\sum_{i=1}^{n} x_i^q$$

In light of this last change of variables we could factor

$$e^{H[\mathbf{t}]} = \prod_{i=1}^{n} e^{\phi_+(x_i)} \quad \text{with} \quad \phi_+(x_i) = \sum_{q\geq 1} \frac{x_i^q}{q} J_q$$

And the deformation is...

The operator

$$e^{\phi_+(x_i)} \quad \text{with} \quad \phi_+(x_i) = \sum_{q \geq 1} \frac{x_i^q}{q} J_q \quad \text{and} \quad J_q := \sum_{r \in \mathbb{Z} + \frac{1}{2}} : \psi^*_{r-q} \psi_r :$$

gives one step in the evolution of our fermionic system leading to Schur functions.

Its states are in bijection with those of a five-vertex model.

Can we deform this operator and simultaneously arrive at the six-vertex model?

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YES!

$$\text{Use} \quad e^{\phi_+(x;v)}\psi_{-\frac{1}{2}} \quad \text{with} \quad \phi_+(x;v) := \sum_{q\geq 1} \frac{x^q}{q}(1-(-v)^q)J_q$$

Let's do an example!

Now consider, in our continuing example with $\lambda = (5,3,2)$:

$$e^{\phi_+(x;v)}\psi_{-\frac{1}{2}}\,|(5,3,2)\rangle = e^{\phi_+(z;t)}\psi_{-\frac{1}{2}}\psi^*_{5-\frac{1}{2}}\psi^*_{3-\frac{1}{2}}\psi^*_{2-\frac{1}{2}}\,|\emptyset\rangle$$

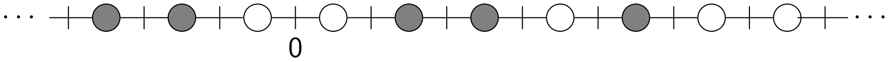
where

$$e^{\phi_+(x;v)} := \sum_{k=0}^{\infty}\frac{1}{k!}\left(\sum_{q\geq 1}\frac{(1+(-1)^{q+1}v^q)x^q}{q}J_q\right)^k .$$

and where, as before,

$$J_q = \sum_r :\psi^*_{r-q}\psi_r:$$

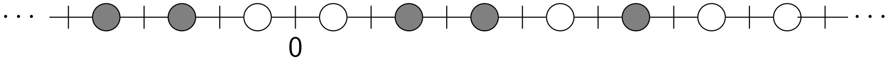
Remember that $\psi_{-\frac{1}{2}}\,|(5,3,2)\rangle =$



When we expand the powers of k, which products of $J_i's$ can occur?

Answer: Lots of J's

$\psi_{-\frac{1}{2}} \left|(5,3,2)\right\rangle =$



J_1, J_2, J_3, J_4, J_5, J_1J_1, J_1J_3, J_2J_1, J_2J_2, J_2J_3, J_2J_5, J_3J_1, J_3J_2, J_3J_4, J_4J_2, J_4J_3, J_5J1, J_5J_2, $J_1J_1J_1$, $J_1J_1J_2$, $J_1J_1J_3$, $J_1J_1J_5$, $J_1J_2J_1$, $J_1J_2J_2$, $J_1J_2J_3$, $J_1J_2J_4$, $J_1J_3J_1$, $J_1J_3J_2$, $J_1J_3J_3$, $J_1J_4J_1$, $J_1J_4J_2$, $J_2J_1J_1$, $J_2J_1J_2$, $J_2J_1J_3$, $J_2J_1J_4$, $J_2J_2J_1$, $J_2J_2J_2$, $J_2J_2J_3$, $J_2J_3J_1$, $J_2J_3J_2$, $J_2J_4J_1$, $J_3J_1J_1$, $J_3J_1J_2$, $J_3J_1J_3$, $J_3J_2J_2$, $J_4J_1J_1$, $J_4J_1J_2$, $J_4J_2J_1$, $J_5J_1J_1$

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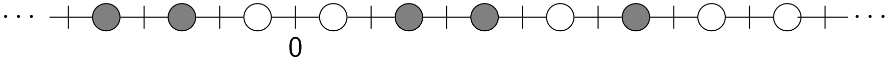

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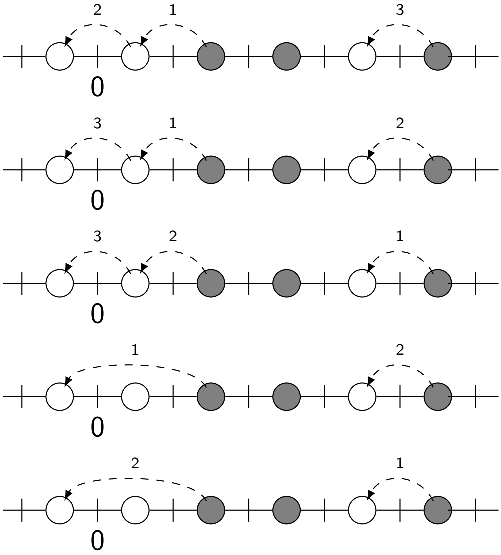
More refined question – what is

$$\left\langle (4,3) \middle| e^{\phi_+(x;v)}\psi_{-\frac{1}{2}} \middle| (5,3,2) \right\rangle ? \quad \text{(w/ orthonormal inner product)}$$

It comes from the action with J_1^3, J_2J_1, and J_1J_2 which produces...

Migrations from $\psi_{-\frac{1}{2}} |(5,3,2)\rangle$ to $|(4,3)\rangle$ under J_q's



Matching rows of the six vertex model...

Analyzing the contribution from each of these yields a coefficient of

$$-3\frac{1}{3!}(1+v)^3x^3 - 2\frac{1}{2!}\frac{1}{2}(1-v^2)(1+v)x^3 = -x^3(1+v)^2.$$

In other words, this is the value of

$$\left\langle (4,3) \,\middle|\, e^{\phi_+(x;v)}\psi_{-\frac{1}{2}} \,\middle|\, (5,3,2) \right\rangle$$

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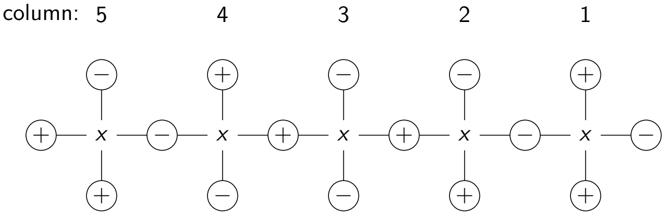
One can check that, using the Boltzmann weights in Dan's talk, this matches the contributions of a single row of ice with top boundary given by $(5,3,2)$ and bottom boundary given by $(4,3)$:



Connecting to Tokuyama's formula

This point of view can be used to give a completely different proof of Tokuyama's formula, whose output is the Shintani-Casselman-Shalika formula in type A.

Theorem (B.-Schultz)

Given a partition λ with n parts, then

$$\langle \emptyset; 0| \prod_{i=1}^{n} \left[e^{\phi_+(x_i;v)} \psi_{-1/2} \right] |\lambda; n\rangle$$

matches Tokuyama's generating function term by term. Moreover, we can prove independently that it equals the Shintani-Casselman-Shalika formula.

We also treat the cases of Cartan type C and a double cover of type B in the paper.

Final Remarks

- In the case of one variable, the Hamiltonian $e^{\phi_+(x;v)}$ appears in the super Boson-Fermion correspondence for Lie superalgebras (Kac-van de Leur)

- This gives a connection with skew super-symmetric Schur functions:

$$\langle \mu; n-1 \mid e^{\phi_+(x;v)}\psi_{-1/2} \mid \lambda; n\rangle = (-1)^n s_{\lambda/\mu}(x \mid vx).$$

  But we lose this connection upon applying the operator multiple times.

- This gives a point of contact with the results of B-Buciumas-Bump and one hopes that a common generalization to the metaplectic case exists.

- These Hamiltonians are operators on rows. We can think of Hecke operators (or Gerasimov-Lebedev-Oblezin's Baxter operators) as acting on the columns. Is there a link between them? Or a unifying theory here?