Automorphic forms and lattice sums in exceptional field theory

Axel Kleinschmidt (Albert Einstein Institute, Potsdam)

Whittaker functions: Number theory, Geometry and Physics

Banff, July 27, 2016

Joint work with Guillaume Bossard [arXiv:1510.07859]
Also: [P. Fleig, H. Gustafsson, AK, D. Persson arXiv:1511.04265]

Motivation and goal

Physics aims:

- string theory effective action beyond supergravity approximation
- higher derivative corrections in $D = 11 - d$ dimensions with T^d
- non-perturbative effects and black hole physics

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Maths aims:

- wavefront sets of small automorphic representations of split real Lie groups
- alternative expressions for Eisenstein series
- beyond automorphic forms?

String theory scattering amplitudes

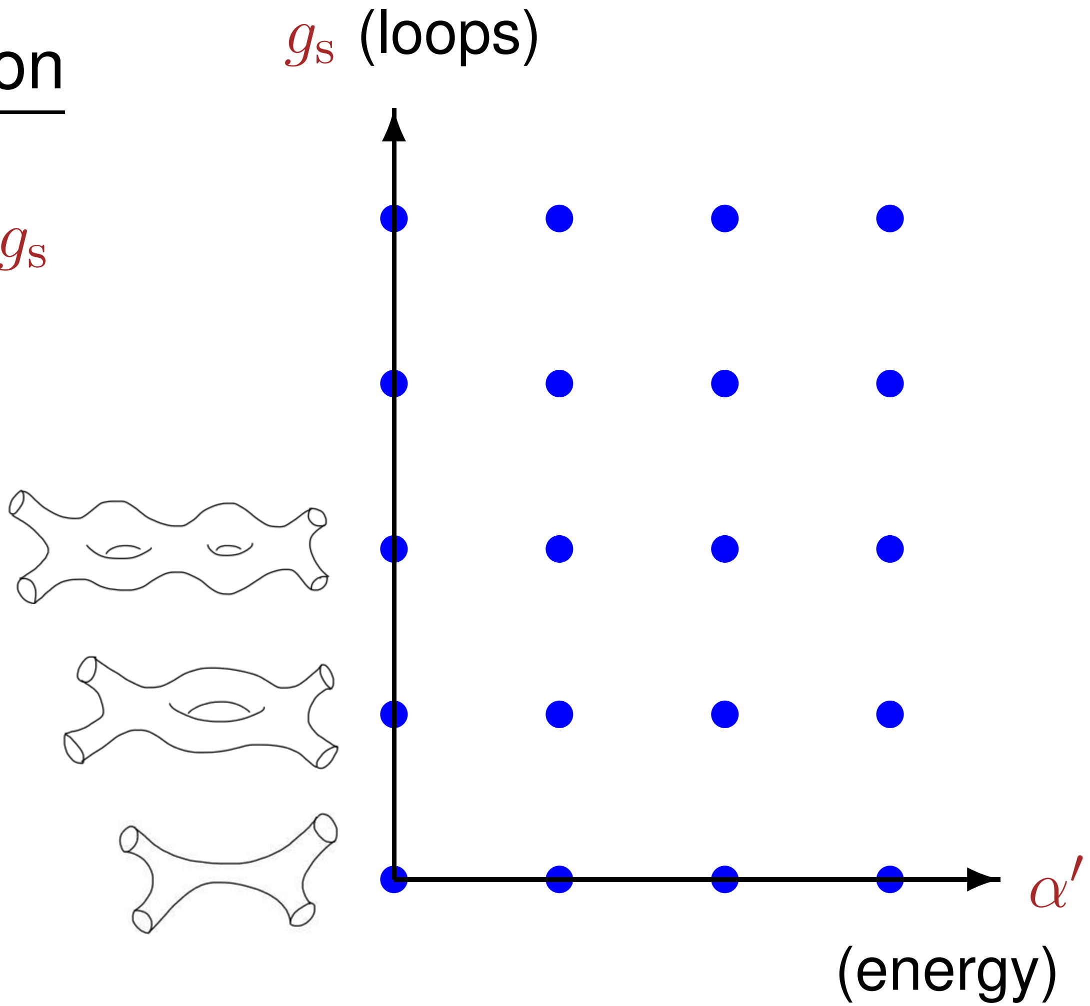
Scattering amplitudes of strings have a double expansion

- Perturbative loop expansion

  Diagram weighted by powers of string coupling g_{s}

- Energy expansion

  Energies involved in interaction measured in powers of string scale $\ell_{\mathrm{s}}^2 = \alpha'$



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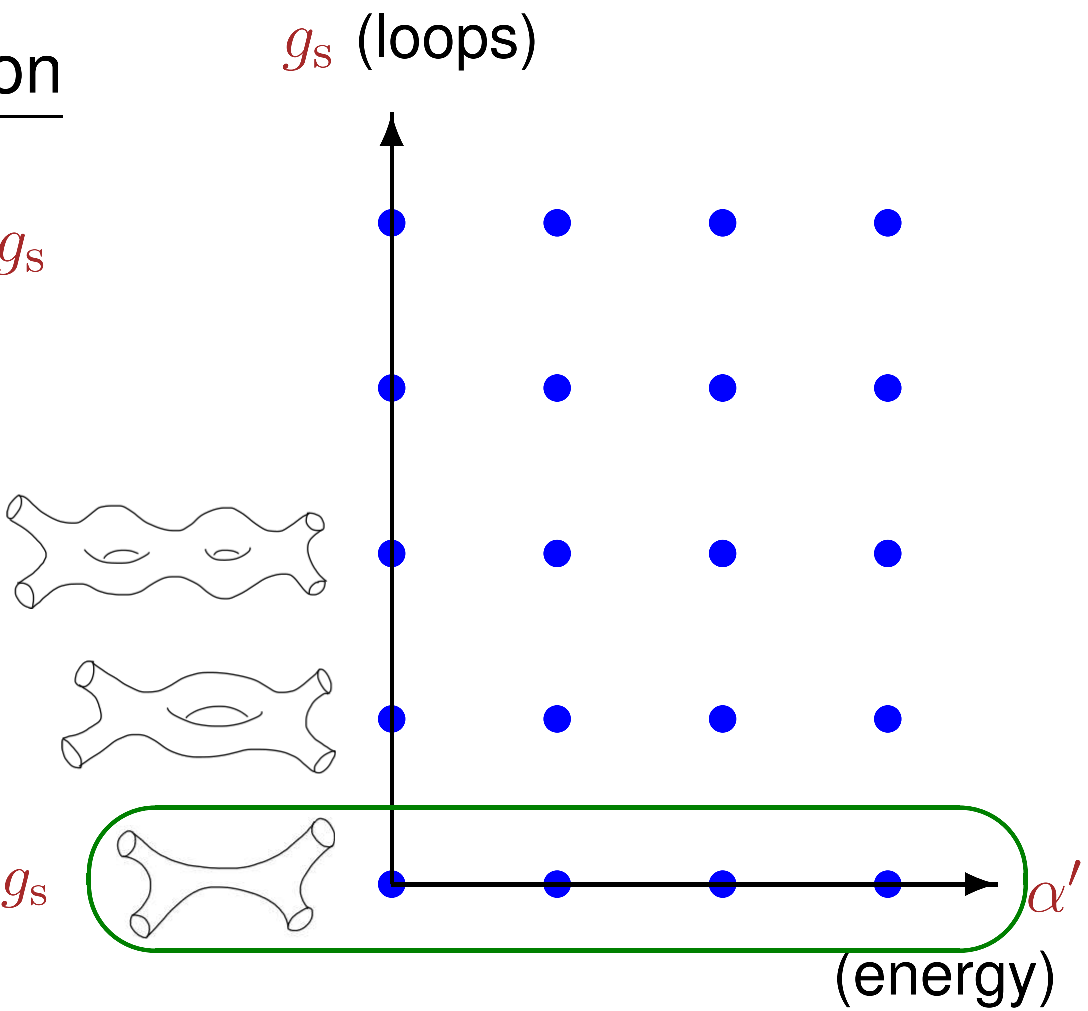

computed by integrals over moduli space of Riemann surfaces becomes hard after two loops [D'Hoker, Phong]

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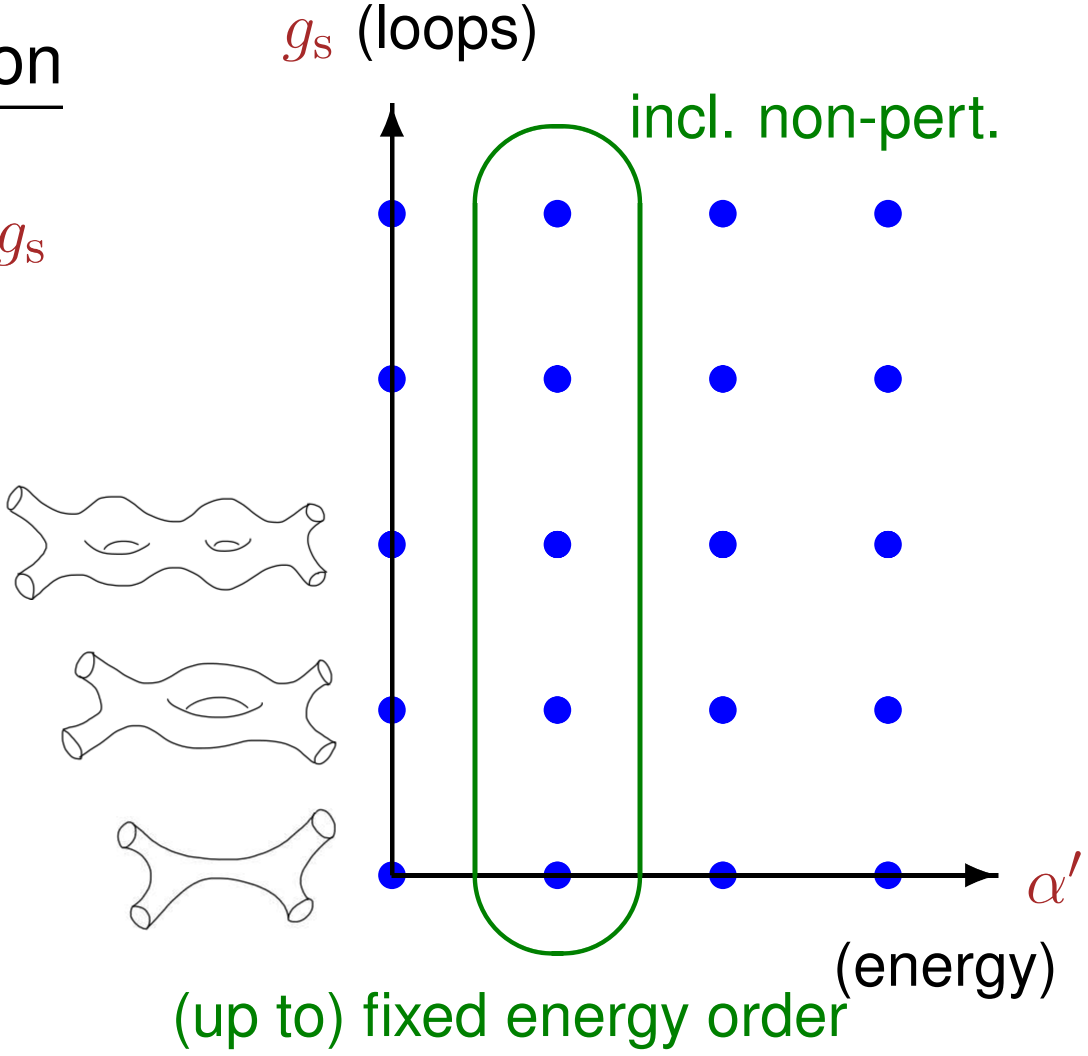
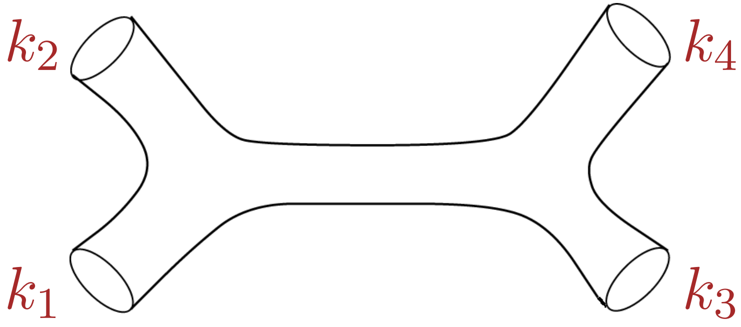


sometimes fixed by (discrete) symmetries/automorphy

String theory effective action

Consider four-graviton scattering amplitude (in $D = 10$ space-time dimensions) at tree level



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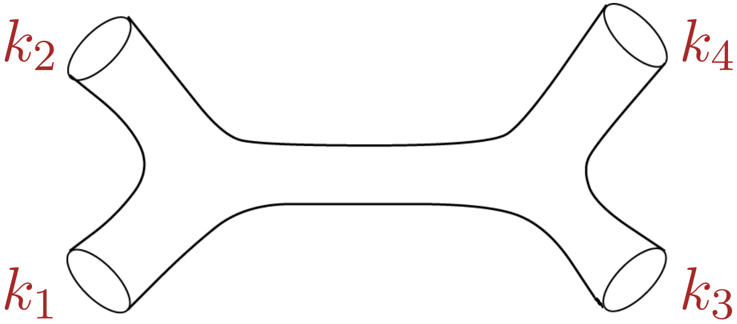

$$\mathcal{A}^{\text{tree}}(s,t,u) = g_{\text{s}}^{-2}\frac{4}{stu}\frac{\Gamma(1-\alpha's)\Gamma(1-\alpha't)\Gamma(1-\alpha'u)}{\Gamma(1+\alpha's)\Gamma(1+\alpha't)\Gamma(1+\alpha'u)}\mathcal{R}^4$$

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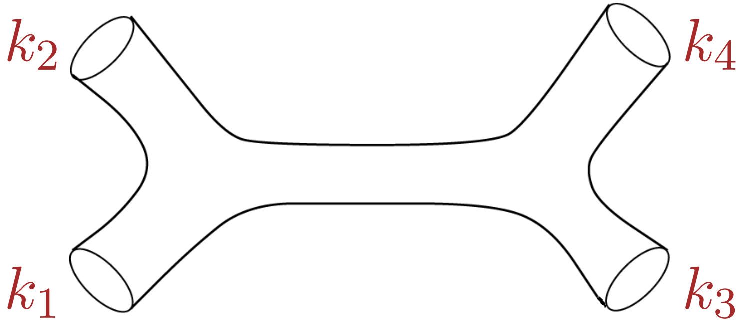

string scale

$\alpha' = \ell_{\rm s}^2$

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Mandelstam variables

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absorbs polarisation tensors

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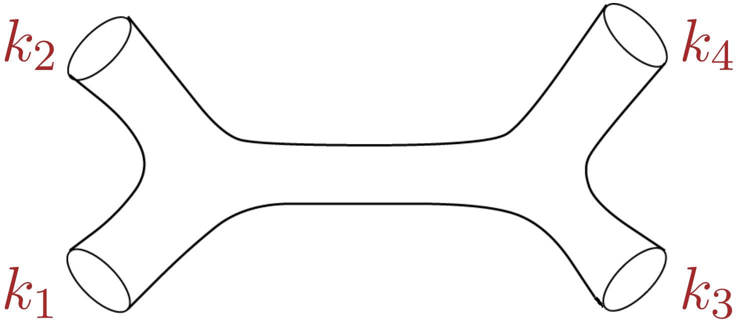

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$s = -(k_1 + k_2)^2$
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Expand for low energies

$\alpha' s << 1$, $\alpha' t << 1$ and $\alpha' u << 1$

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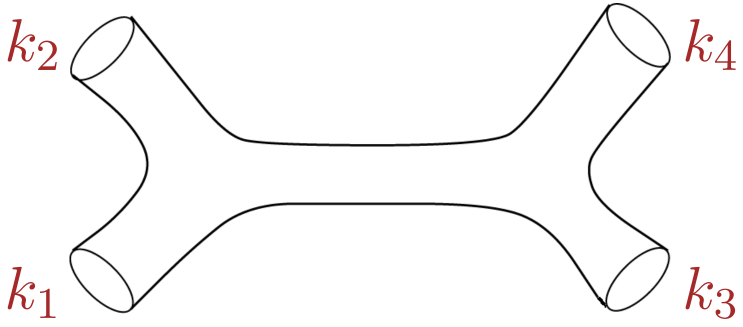

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$$= 4g_{\text{s}}^{-2}\mathcal{R}^4\left[\frac{1}{stu} + \underset{\uparrow\ \text{dimensionful}}{(\alpha')^3}\cdot 2\zeta(3) + (\alpha')^5(s^2+t^2+u^2)\cdot\zeta(5) + \ldots\right]$$

Low energy effective action

Gravitational interaction at lowest energies in D space-time dimensions normally described by general relativity (or supergravity) with Lagrangian

$$\mathcal{L} = \ell^{2-D} R \longleftarrow$$

Riemann scalar
curvature of space-time
two-derivatives

length scale $\sim \sqrt{\alpha'}$ $\nearrow$

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Higher orders in α' are related to higher derivative modifications. For gravity in D dimensions schematically from string tree level (Einstein frame)

$$e^{-1}\mathcal{L} = \ell^{2-D} R + \ell^{8-D} 2\zeta(3) g_{\mathrm{s}}^{-3/2} R^4$$

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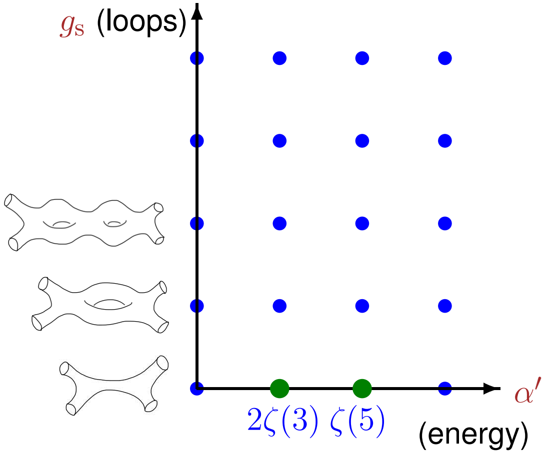

Moduli fields and U-duality (I)

The string coupling $g_{\rm s}$ is a modulus of string theory.

Moduli contain information of the background on which strings propagate.

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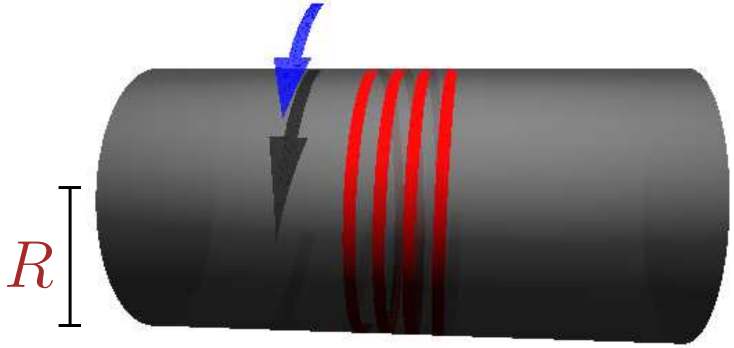
Other moduli: For toroidal backgrounds including $T^{d-1} = (S^1)^{d-1}$ the radii are also moduli



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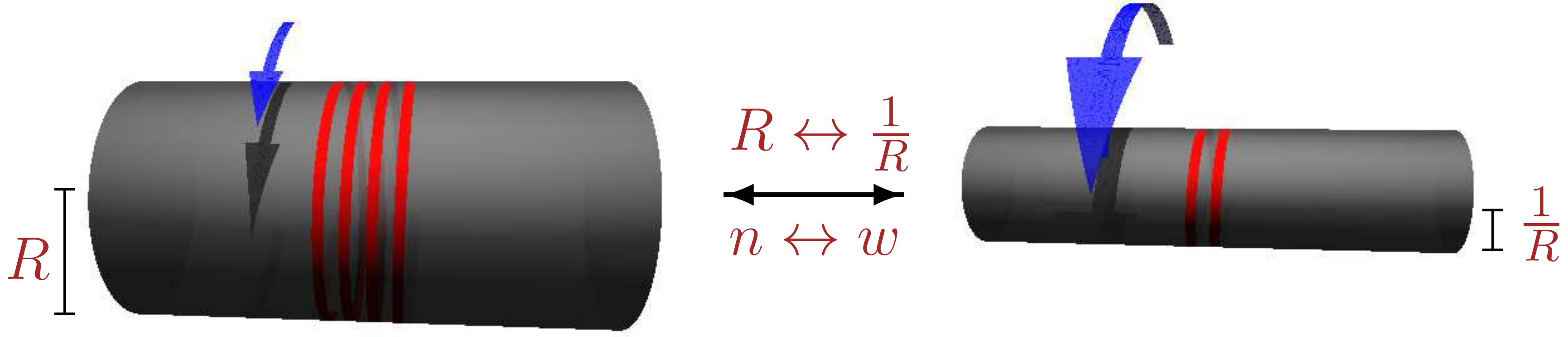

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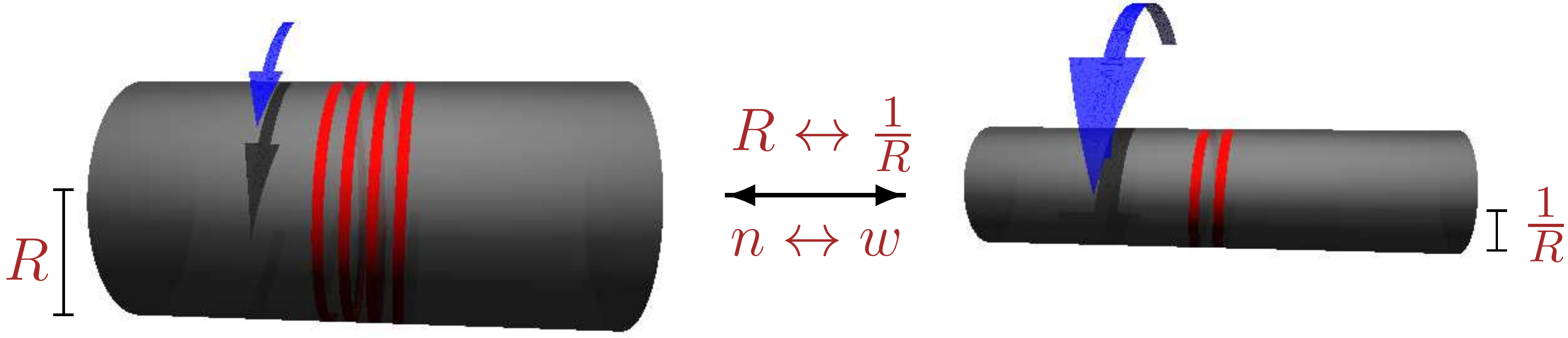

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Equivalent string theories! T-duality $SO(d-1, d-1, \mathbb{Z})$

Moduli fields and U-duality (II)

On g_s and (RR) axion χ action of $SL(2,\mathbb{Z})$ S-duality

$$z = \chi + i g_\mathrm{s}^{-1} \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}$$

giving equivalent string theories. $z \in SL(2,\mathbb{R})/SO(2)$

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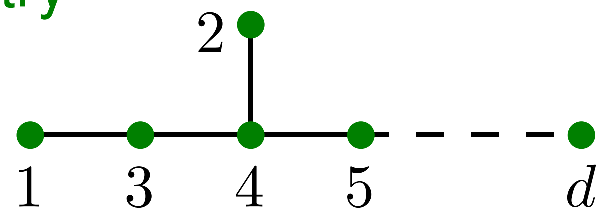
All moduli g together form moduli space $\mathcal{M}$ [Hull, Townsend 1995]

$$g \in \mathcal{M} = E_d(\mathbb{Z}) \backslash E_{d(d)} / K(E_d)$$

U-duality

Cremmer–Julia hidden symmetry

compact subgp



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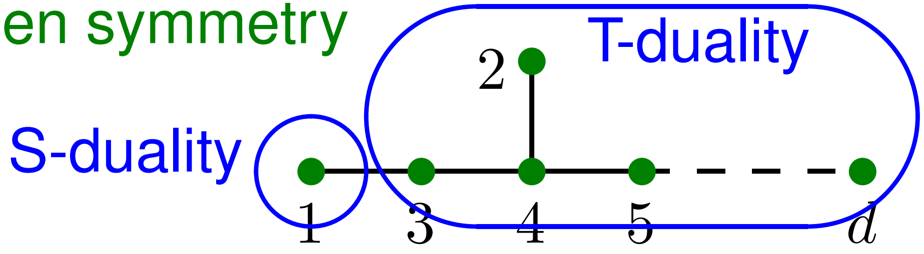
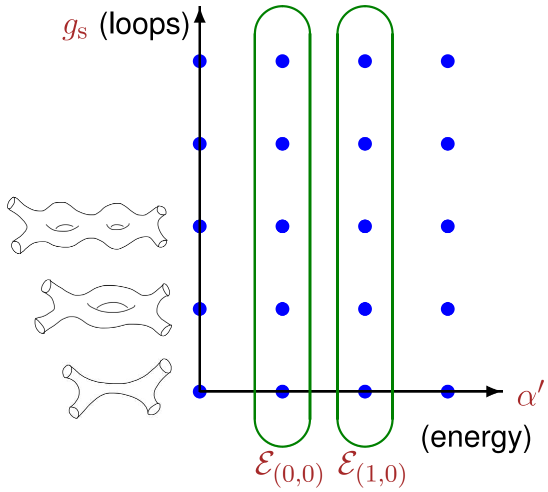


Coefficient functions in amplitude (I)

Expand the (analytic part of the) full scattering amplitude in energy direction

$$\mathcal{A}(s,t,u;g) = \mathcal{R}^4 \left(\frac{1}{stu} + \sum_{p,q\geq 0} \mathcal{E}_{(p,q)}(g)\sigma_2^p \sigma_3^q \right)$$



with $\sigma_n = \frac{(\alpha')^n}{4^n}(s^n + t^n + u^n)$ and $g \in \mathcal{M}$.

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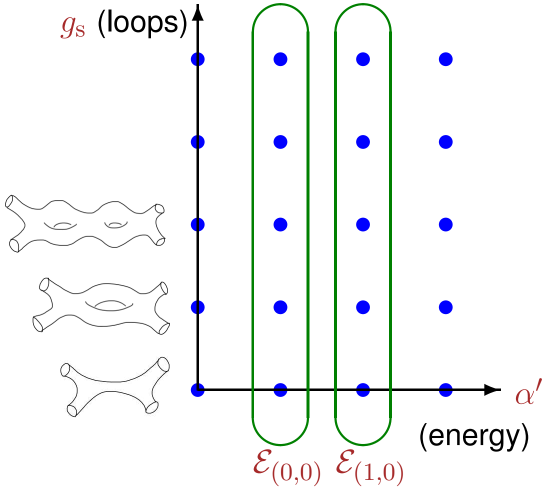

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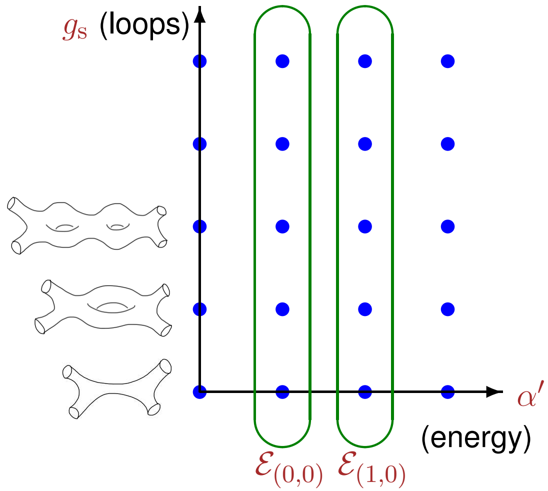

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⇒ Looking for (spherical) automorphic forms on E_d

Coefficient functions in amplitude (II)

A lot known for lowest $\mathcal{E}_{(p,q)}$ from supersymmetry and internal consistency [Green, Gutperle, Kiritsis, Miller, Obers, Pioline, Russo, Sethi, Vanhove, Waldron,...]

$\mathcal{E}_{(0,0)}(g) = 2\zeta(3)E_{\alpha_1,3/2}(g)$	R^4 correction, $\frac{1}{2}$-BPS, min-rep
$\mathcal{E}_{(1,0)}(g) = \zeta(5)E_{\alpha_1,5/2}(g)$	$\nabla^4 R^4$ correction, $\frac{1}{4}$-BPS, ntm-rep
$\mathcal{E}_{(0,1)}(g) =$ later	$\nabla^6 R^4$ correction, $\frac{1}{8}$-BPS

in terms of (maximal parabolic) Eisenstein series

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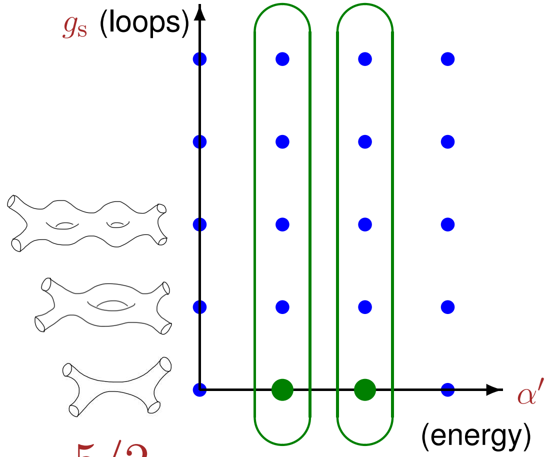

Consistency with tree-level results

$$\mathcal{E}_{(0,0)}(g) = 2\zeta(3)g_s^{3/2} + \ldots, \quad \mathcal{E}_{(1,0)}(g) = \zeta(5)g_s^{5/2} + \ldots,$$

Different viewpoint: Field theory

Instead of reviewing Fourier expansions and consistency of answers above [Green, Miller, Russo, Vanhove; Obers, Pioline;...]

$\Rightarrow$ use that four-graviton process is very special. Low order corrections R^4, $\nabla^4 R^4$ and $\nabla^6 R^4$ are partially BPS

$\Longrightarrow$ Only BPS states contribute; no other string theory states visible at low energies

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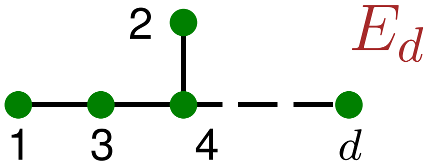
Aim: Investigate $\mathcal{E}_{(p,q)}$ for $D < 10$ by similar methods in manifestly U-duality covariant formalism

$\Longrightarrow$ Exceptional field theory loops

Exceptional field theory

[de Wit, Nicolai; Hull; Waldram et al.;
Hohm, Samtleben; West; ...]



Formalism to make hidden $E_d(\mathbb{R})$ (continuous!) manifest.

Consider extended space-time ($D = 11 - d$)

$$\mathcal{M}^D \times \mathcal{M}^{d(\alpha_d)}$$

Coordinates x^μ, y^M with $\mu = 0, ..., D-1$ and $M = 1, ..., d(\alpha_d)$.

$d(\alpha_d) = \dim \mathbf{R}_{\alpha_d}$: hst. weight rep. on node α_d

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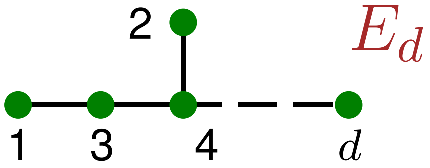

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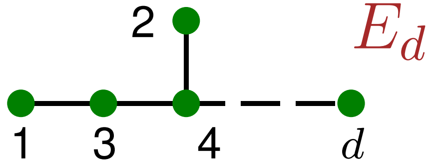
$\mathbf{R}_{\alpha_d}$ decomposes under 'gravity line' $GL(d,\mathbb{R}) \subset E_d(\mathbb{R})$

$$y^M = (y^m, y_{[mn]}, y_{[m_1 \ldots m_5]}, \ldots) \qquad (m, n, ... = 1, ..., d)$$

KK momenta ↗ M2 wrappings ↖

Generalised coordinates $y^M \in \mathbf{R}_{\alpha_d}$

E_d	$\mathbf{R}_{\alpha_d}$	
$SO(5,5)$	$\mathbf{16}$	
E_6	$\mathbf{27}$	
E_7	$\mathbf{56}$	
E_8	$\mathbf{248}$	



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E_d	$\mathbf{R}_{\alpha_d}$	$\mathbf{R}_{\alpha_1}$
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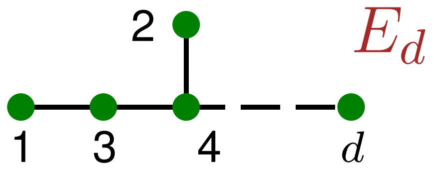
E_6	$\mathbf{27}$	$\overline{\mathbf{27}}$
E_7	$\mathbf{56}$	$\mathbf{133}$
E_8	$\mathbf{248}$	$\mathbf{3875} \oplus \mathbf{1}$



Generalised coordinates y^M have to obey section constraint

$$\left.\frac{\partial A}{\partial y^M}\frac{\partial B}{\partial y^N}\right|_{\mathbf{R}_{\alpha_1}} = 0$$

for any two fields $A(x^\mu, y^M)$, $B(x^\mu, y^M)$. LHS belongs to

$$\mathbf{R}_{\alpha_d} \otimes \mathbf{R}_{\alpha_d} = \mathbf{R}_{\alpha_1} \oplus \ldots$$

Section constraint

$$\left.\frac{\partial A}{\partial y^M}\frac{\partial B}{\partial y^N}\right|_{\mathbf{R}_{\alpha_1}} = 0$$

Possible solution: ‘M-theory’: $y^M = (y^m, \cancel{y_{mn}}, \cancel{y_{m_1\ldots m_5}}, \ldots)$

Alternative: Type IIB [Blair, Malek, Park]. These are the only two vector space solutions [BK]

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Here: ‘Toroidal’ extended space for y^M. Conjugate momenta are quantised charges

$$\Gamma_M = (n_m, n^{m_1 m_2}, n^{n_1\ldots n_5}, \ldots) \in \mathbb{Z}^{d(\alpha_d)}$$

Section constraint becomes $\frac{1}{2}$-BPS constraint on charges

$$\Gamma \times \tilde{\Gamma}\Big|_{\mathbf{R}_{\alpha_1}} = 0 \qquad \Rightarrow \qquad \text{write} \quad \Gamma \times \tilde{\Gamma} = 0 \text{ for brevity}$$

⟶ One loop

Amplitudes in EFT (I)

Exceptional field theory is mainly a classical theory. QFT treatment complicated due to section constraint.

Consider 3-point vertex in EFT $\phi\,\partial\phi\,\partial\phi$

$$\int_{\mathbb{R}^{11-d}} dx \int_{\mathbb{R}^{d(\alpha_d)}/\text{section}} dy\, \phi(x,y)\, (\nabla\phi(x,y)\cdot\nabla\phi(x,y))$$

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y-Fourier expand $\phi(x,y)=\sum\limits_{\Gamma\in\mathbb{Z}^{d(\alpha_d)}}\phi_\Gamma(x)e^{i\ell^{-1}\Gamma\cdot y}$. Vertex

$$\sum_{\substack{\Gamma_1,\Gamma_2\in\mathbb{Z}^{d(\alpha_d)}\\ \Gamma_1\times\Gamma_2=0}}\int\limits_{\mathbb{R}^{11-d}} dx\,\phi_{-\Gamma_1-\Gamma_2}(x)\Big[\partial_\mu\phi_{\Gamma_1}\partial^\mu\phi_{\Gamma_2}-\ell^{-2}\left\langle Z(\Gamma_1)|Z(\Gamma_2)\right\rangle\phi_{\Gamma_1}\phi_{\Gamma_2}\Big]$$

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Exceptional field theory is mainly a classical theory. QFT treatment complicated due to section constraint.

Consider 3-point vertex in EFT $\phi\,\partial\phi\,\partial\phi$

$$\int\limits_{\mathbb{R}^{11-d}} dx \int\limits_{\mathbb{R}^{d(\alpha_d)}/\text{section}} dy\,\phi(x,y)\left(\nabla\phi(x,y)\cdot\nabla\phi(x,y)\right)$$

y-Fourier expand $\phi(x,y) = \sum\limits_{\Gamma\in\mathbb{Z}^{d(\alpha_d)}} \phi_\Gamma(x) e^{i\ell^{-1}\Gamma\cdot y}$. Vertex

$$\sum_{\substack{\Gamma_1,\Gamma_2\in\mathbb{Z}^{d(\alpha_d)}\\ \Gamma_1\times\Gamma_2=0}} \int\limits_{\mathbb{R}^{11-d}} dx\,\phi_{-\Gamma_1-\Gamma_2}(x)\Big[\partial_\mu\phi_{\Gamma_1}\partial^\mu\phi_{\Gamma_2} \underbrace{-\ell^{-2}\left\langle Z(\Gamma_1)|Z(\Gamma_2)\right\rangle}_{\text{charge dependent mass}}\phi_{\Gamma_1}\phi_{\Gamma_2}\Big]$$

↑

Section constraint on y^M turned into constraint on charges

Amplitudes in EFT (II)

$\langle Z(\Gamma)|Z(\Gamma)\rangle$ like BPS-mass. In M-theory frame

$$ds_{11}^2 = e^{\frac{9-d}{3}\phi} M_{mn} dy^m dy^n + e^{-\frac{d}{3}\phi} \eta_{\mu\nu} dx^\mu dx^\nu$$

ϕ now dilaton; M_{mn} uni-modular metric on T^d.

$$|Z(\Gamma)|^2 = e^{-3\phi} M^{mn} n_m n_n + \frac{1}{2} e^{(6-d)\phi} M_{m_1 n_1} M_{m_2 n_2} n^{m_1 m_2} n^{n_1 n_2} + \ldots$$

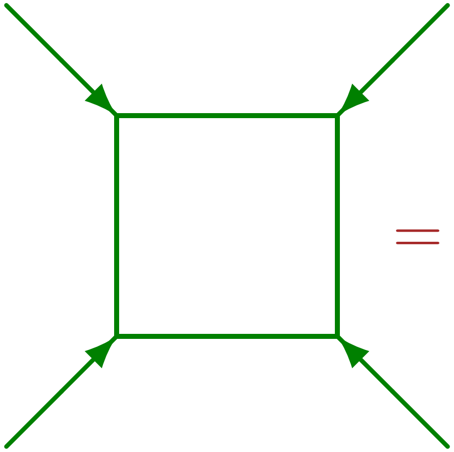
From form of vertex see that momenta in propagators are effectively shifted by Kaluza–Klein mass

$$p^2 \quad \longrightarrow \quad p^2 + \ell^{-2} |Z(\Gamma)|^2$$

and section constraint $\Gamma_i \times \Gamma_j = 0$ at every vertex.

One-loop in EFT (I)

Four-graviton amplitude reduces to scalar box



$$= \underbrace{\left[\tfrac{i\kappa^2}{2} t_8 t_8 \prod_{A=1}^{4} k_A R(k_A, e_A)\right]}_{\mathcal{R}^4} A^{\text{1-loop}}(k_1, k_2, k_3, k_4)$$

Pull out kinematic part

$$A^{\text{1-loop}}(k_1, k_2, k_3, k_4) = \kappa^2 \int \frac{d^{11-d}p}{(2\pi)^{11-d}} \sum_{\substack{\Gamma \in \mathbb{Z}^{d(\alpha_d)} \\ \Gamma\times\Gamma=0}} \frac{1}{((p-k_1)^2 + \ell^{-2}|Z|^2)}$$

$$\times \frac{1}{(p^2 + \ell^{-2}|Z|^2)((p-k_1-k_2)^2 + \ell^{-2}|Z|^2)((p+k_4)^2 + \ell^{-2}|Z|^2)}$$

$$+ \text{perms.}$$

One-loop in EFT (II)

$\Gamma = 0$ term corresponds to SUGRA in $D = 11 - d$; usual log threshold contribution $\Rightarrow$ remove for analytic eff. action

Treat loop integral over $d^{11-d}p$ with usual Schwinger and Feynman techniques:

$$A^{\text{1-loop}}(k_1, k_2, k_3, k_4) = 4\pi \ell^{9-d} \sum_{\substack{\Gamma \in \mathbb{Z}^{d(\alpha_d)} \\ \Gamma \times \Gamma = 0}} \int_0^\infty \frac{dv}{v^{\frac{d-1}{2}}} \int_0^1 dx_1 \int_0^{x_1} dx_2 \int_0^{x_2} dx_3$$

$$\times \exp\left[\frac{\pi}{v}\left((1-x_1)(x_2-x_3)s + x_3(x_1-x_2)t - \ell^{-2}|Z|^2\right)\right] + \text{perms.}$$

Low energy from expanding in Mandelstam variables

$$s = -(k_1+k_2)^2, \quad t = -(k_1+k_4)^2, \quad u = -(k_1+k_3)^2.$$

Low energy correction terms

For lowest two orders

$$A^{\text{1-loop}}(s,t,u) = \pi\ell^6\left(\xi(d-3)E_{\alpha_d,\frac{d-3}{2}} + \frac{\pi^2\ell^4(s^2+t^2+u^2)}{720}\xi(d+1)E_{\alpha_d,\frac{d+1}{2}} + \ldots\right)$$

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R^4 correction

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$\nabla^4 R^4$ correction

Notation

- $\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$ [completed Riemann zeta]
- $E_{\alpha_d,s} = \frac{1}{2\zeta(2s)} \sum_{\substack{\Gamma\neq 0\\ \Gamma\times\Gamma=0}} |Z(\Gamma)|^{-2s}$ [Eisenstein series]

Restricted lattice sum rewritable as single U-duality orbit!

$\longrightarrow$ Two loops $\longrightarrow$ Beyond

Remarks

Expressions converge for $\nabla^{2k}R^4$ term on T^d when $k > \frac{3-d}{2}$

- For $k = 0$ (R^4) and $d > 3$ ($D < 8$) find after using Langlands' functional relation the correct correction function $\mathcal{E}^{D}_{(0,0)}$ (including numerical coefficient).
  For $d = 3$ one has to regularise; related to known one-loop R^4 divergence in SUGRA.
- For $k = 2$ ($\nabla^4 R^4$) expressions converge. For $d \leq 5$ one obtains only one supersymmetric invariant of `[Bossard, Verschinin]`; for $7 \leq d < 5$ full (unique) invariant with correct coefficient. For $d = 8$ ancestor of 3-loop divergence `[BK]`.

Expressions also ok for $d > 8$; Kac–Moody case `[Fleig, AK]`

Two loops in EFT (I)

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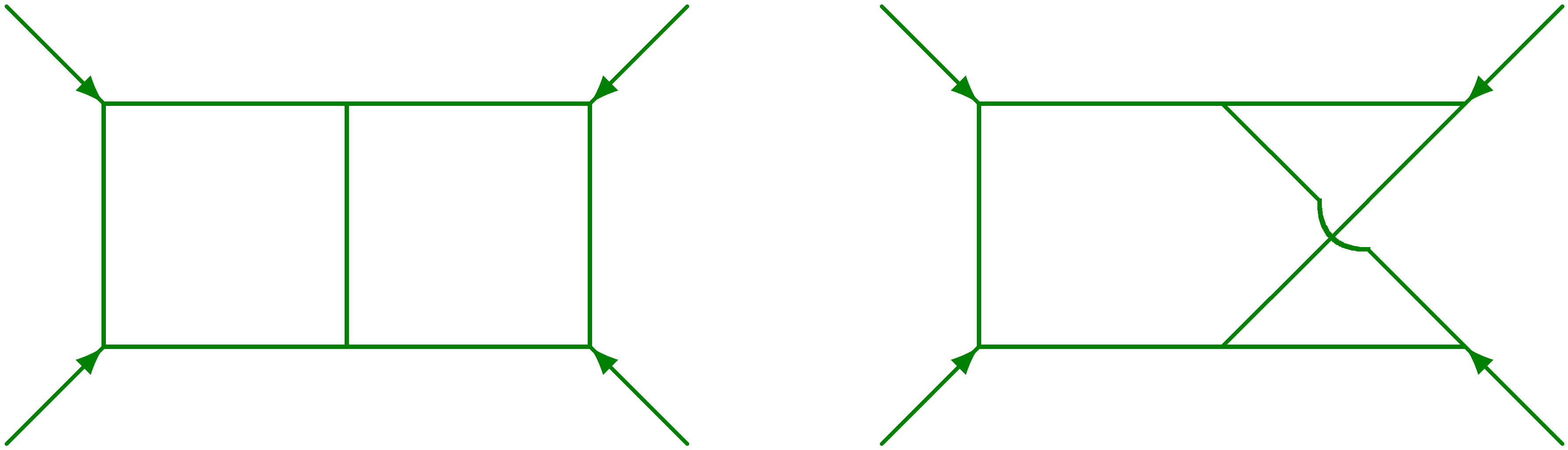
`[Bern et al.]`: combination of planar and non-planar scalar diagram at $L = 2$



After a few pages of calculation

$$A^{\text{2-loop}}(s,t,u) \sim \ell^6 \sum_{\substack{\Gamma_1,\Gamma_2\\ \Gamma_i\times\Gamma_j=0}} \int_0^\infty \frac{d^3\Omega}{(\det\Omega)^{\frac{7-d}{2}}} e^{-\Omega^{ij}\langle Z(\Gamma_i)|Z(\Gamma_j)\rangle}$$

$$\times \left[\frac{\ell^4(s^2+t^2+u^2)}{6} + \frac{\ell^6(s^3+t^3+u^3)}{72}\Phi_{(0,1)}(\Omega) + \ldots \right]$$

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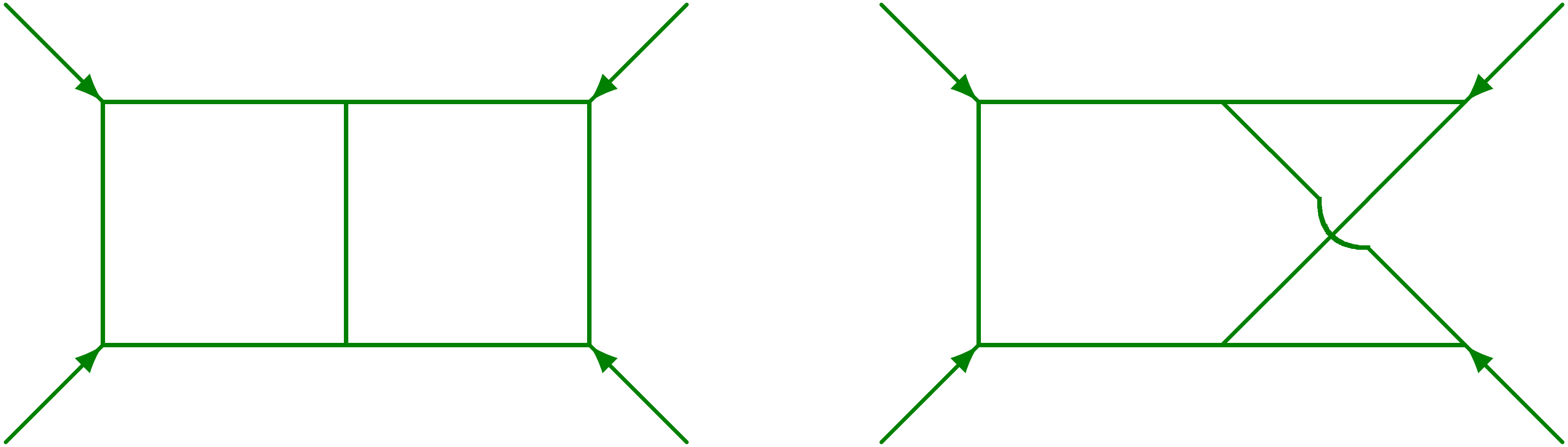

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Two loops in EFT (II)

Focus first on $\nabla^4 R^4$ contribution. Need to understand

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where $\Omega^{ij} = \Omega = \begin{pmatrix} L_1+L_3 & L_3 \\ L_3 & L_2+L_3 \end{pmatrix}$

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Sum is restricted to pairs of charges Γ_1, Γ_2 satisfying

$$\Gamma_i \times \Gamma_j|_{\mathbf{R}_{\alpha_1}} = 0$$

Solutions can be parametrised by suitable parabolic decompositions [BK].

Two loops in EFT (III)

Putting everything together

$$A^{\text{2-loop},\nabla^4 R^4}(s,t,u) = 8\pi\ell^{10}\xi(d-4)\xi(d-5)E_{\alpha_{d-1},\frac{d-4}{2}}$$

- This gives the correct function and coefficient for $3 \leq d \leq 8$ with the right coefficient. Case $d = 5$ ($D = 6$) trickier due to IR divergences.
- Certain doubling of contributions from one loop and two loops. Corrected if one-loop result renormalised.
- Other orbits of M subdominant at low energies except $d = 5$.

Beyond Eisenstein series (I)

Consider $\nabla^6 R^4$ term $\mathcal{E}_{(0,1)}$. Inhomogeneous equation [Green, Vanhove]

$$(\Delta - \lambda)\mathcal{E}_{(0,1)} = -\mathcal{E}_{(0,0)}^2$$

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For other dimensions can write Poincaré series form `[Ahlén, AK in progress]` that needs to be studied further.

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$$\mathcal{E}_{(0,1)}(g) = \sum_{\gamma \in P_1 \backslash E_d} \sigma(\gamma g)$$

with $\sigma(g)$ not a character on P_1 but depends on unipotent part through Bessel functions. $\longrightarrow$ yonder

Beyond Eisenstein series (II)

Using exceptional field theory can also find a solution

$$\mathcal{E}^{\text{2-loop}}_{(0,1)} = \frac{2\pi^{5-d}}{9} \sum_{\substack{\Gamma_i\in\mathbb{Z}_*^{2d(\alpha_d)}\\ \Gamma_i\times\Gamma_j=0}} \int_{\mathbb{R}_+^{\times 3}} \frac{d^3\Omega}{(\det\Omega)^{\frac{7-d}{2}}} \left(L_1+L_2+L_3-5\frac{L_1L_2L_3}{\det\Omega}\right) e^{-\Omega^{ij}\langle Z(\Gamma_i)|Z(\Gamma_j)\rangle}$$

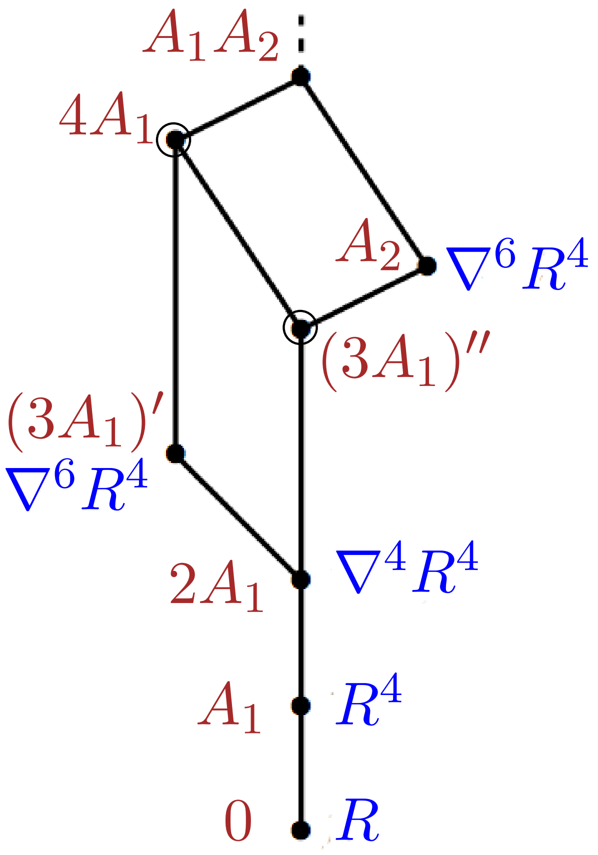
Resembles an independent string theory answer based on the Zhang–Kawazumi invariant [Pioline].

More general questions

- Space of functions required for solving inhomogeneous Laplace equation?
- Automorphic distributions?
- Fourier expansion and wavefront set?
- Automorphic representations? Global picture?

Summary and outlook

- Explicitly evaluated loop amplitudes in EFT
- Reproduced known $\mathcal{E}_{(p,q)}$ in manifestly U-duality covariant form
- Useful tools for dealing with section constraint
- Analysis of differential equation for higher order corrections and their wavefront sets



Hasse diagram for $E_{7(7)}$

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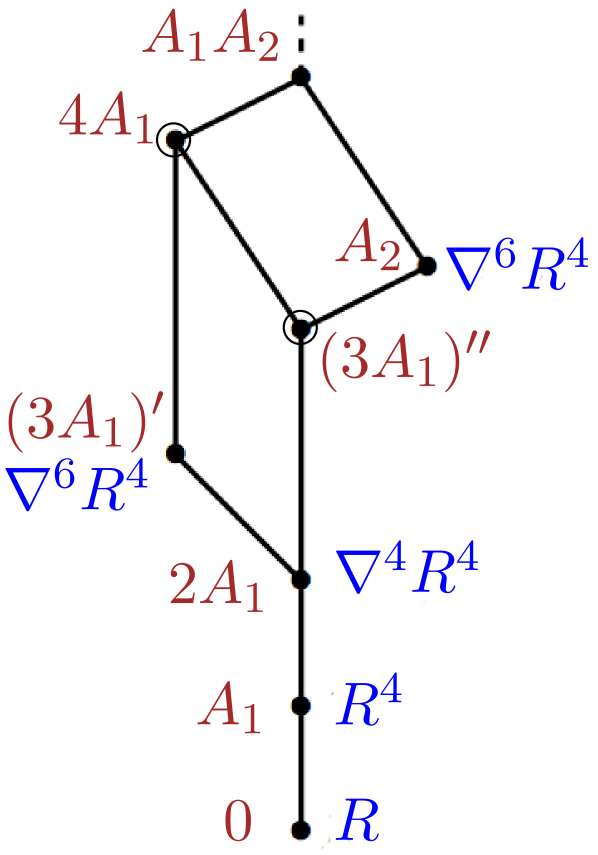
Thank you for your attention!



Hasse diagram for $E_{7(7)}$

Beyond Eisenstein (III)

Solve $(\Delta - 12)f(z) = -4\zeta(3)E_{3/2}(z)^2$: [Green, Miller, Vanhove]

$$f(z) = \sum_{\gamma\in\Gamma_\infty\backslash SL(2,\mathbb{Z})} \sigma(\gamma z),$$ where ($z = x + iy$) and

$$\sigma(z) = 2\zeta(3)^2 y^3 + \frac{1}{9}\pi^2 y + \sum_{n\neq 0} c_n(y) e^{2\pi i n x}$$

$$c_n(y) = 8\zeta(3)\sigma_{-2}(n)y\left[\left(1+\frac{10}{\pi^2 n^2 y^2}\right)K_0(2\pi|n|y)\right.$$

$$\left.+\left(\frac{6}{\pi|n|y}+\frac{10}{\pi^3|n|^3y^3}\right)K_1(2\pi|n|y) - \frac{16}{\pi(|n|y)^{1/2}}K_{7/2}(2\pi|n|y)\right]$$

For higher rank U-dualities (in progress with Olof Ahlén).

Kac–Moody questions

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At the level of Lie algebras $\mathfrak{k} \subset \mathfrak{g}$ over $\mathbb{R}$.

(1) ∞-dim'l fixed point Lie algebra of (Chevalley) involution.
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For $\mathfrak{k}$ of hyperbolic $\mathfrak{g} = \mathfrak{e}_{10}$ one has irreducible (spinor) representations of dimensions [Damour, AK, Nicolai]

$$\mathbf{32}, \quad \mathbf{320}, \quad \mathbf{1728}, \quad \mathbf{7040}$$

with quotients $\mathfrak{so}(32), \quad \mathfrak{so}(288, 32), \quad ?, \quad ?$

K-types

(Some of) these representations can be lifted to the Weyl group W and (covers of) K [Ghatei, Horn, Köhl, Weiss].

Question: Can they arise as K-types of some G representations?

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For other Kac–Moody groups, e.g. $\begin{pmatrix} 2 & -2 & \\ -2 & 2 & -1 \\ & -1 & 2 \end{pmatrix}$ other quotients possible, also with $U(1)$ factors
$\Rightarrow$ holomorphic discrete series?

Question: Spherical vectors for Kac–Moody reps?