Twisted Weyl group multiple Dirichlet series over the rational function field

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BIRS: Whittaker Functions: Number Theory, Geometry and Physics

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Big Picture

- Weyl group multiple Dirichlet series arise as Fourier-Whittaker coefficients of metaplectic Eisenstein series.
- When defined over algebraic function fields, they are rational functions in several variables.

Long-term goal: Understand how these rational functions encode arithmetic/geometric/combinatorial information.

Notation

$n \geq 1$	
$K = \mathbb{F}_q(t)$,	$q \equiv 1 \mod 2n$
$\mathcal{O} = \mathbb{F}_q[t]$	
Φ	reduced root system of rank r
Λ	root lattice of Φ
W	Weyl group of Φ
$\sigma_1, \ldots, \sigma_r$	simple reflections
$\alpha_1, \ldots, \alpha_r$	simple roots
$\varpi_1, \ldots \varpi_r$	fundamental weights
$\rho = \sum_{i=1}^{r} \varpi_i$	

Analogy between $\mathbb{Q}$ and $\mathbb{F}_q(t)$

$$
\begin{array}{ccc}
\mathbb{Q} & \longleftrightarrow & \mathbb{F}_q(t) \\
\hline
\mathbb{Z} \text{ integers} & \longleftrightarrow & \mathbb{F}_q[t] \text{ polynomials} \\
\mathbb{Z}_{>0} \text{ positive integers} & \longleftrightarrow & \mathbb{F}_q[t]_{mon} \text{ monic polynomials} \\
p \text{ prime} & \longleftrightarrow & P \text{ monic, irreducible} \\
|n|_{\mathbb{R}} & \longleftrightarrow & |c| := q^{\deg c}
\end{array}
.
$$

Weyl group multiple Dirichlet series

Let $\mathbf{s} = (s_1, \ldots, s_r)$, and let $\mathbf{m} = (m_1, \ldots, m_r) \in \mathcal{O}^r$.

$$Z(\mathbf{s}; \mathbf{m}; \Phi, n, \mathbb{F}_q(t)) = Z(\mathbf{s}; \mathbf{m}) := \sum_{\mathbf{c} \in (\mathcal{O}_{mon})^r} \frac{H(\mathbf{c}; \mathbf{m})}{|c_1|^{s_1} \cdots |c_r|^{s_r}}$$

- analytic continuation to $\mathbb{C}^r$
- functional equations form group isomorphic to W

We say $Z(\mathbf{s}; \mathbf{m})$ is untwisted when $\mathbf{m} = (1, \ldots, 1)$.

Let $Z^*(\mathbf{s}; \mathbf{m}) = \Xi(\mathbf{s}) Z(\mathbf{s}; \mathbf{m})$ be the normalized series, where $\Xi(\mathbf{s})$ is a product of zeta functions.

Coefficients

The coefficients $H(\mathbf{c};\mathbf{m})$ are complex numbers defined so that $Z(\mathbf{s};\mathbf{m})$ has desired analytic properties. Existence of **Gauss sums** gives a twisted multiplicativity:

The $H(\mathbf{c};\mathbf{m})$ are completely determined by

$$H(P^{k_1},\ldots,P^{k_r};P^{l_1},\ldots,P^{l_r})$$

for each prime P such that $P^{k_i}||c_i$ and $P^{l_i}||m_i$.

The p-parts are generating functions for the prime-power coefficients.

FACTS:

- **No** Euler product in general; “twisted” analogue.
- $Z^*(\mathbf{s};\mathbf{m})$ is rational in $X_1 = q^{-s_1}, \ldots, X_r = q^{-s_r}$.
- Denominator of $Z^*(\mathbf{s};\mathbf{m})$ is known.
- Numerator not well-understood.

Chinta–Gunnells construction

Fix $\ell = (l_1, \ldots, l_r) \in (\mathbb{Z}_{\geq 0})^r$, and let $\theta = \sum_{i=1}^r (l_i + 1)\varpi_i$.

- Define a Weyl group action on $\tilde{A}$ where $A = \mathbb{C}[\Lambda]$. (Identify $\mathbf{x}^{\lambda = \sum k_i \alpha_i}$ with $x_1^{k_1} \cdots x_r^{k_r}$.)
- Build a W-invariant function by averaging:

$$F(\mathbf{x}; \theta) = N(\mathbf{x}; \theta)/D(\mathbf{x})$$

Let $x_i = |P|^{-s_i}$. Then

$$\underbrace{N(\mathbf{x}; \theta) = \sum_{k_1, \ldots, k_r \geq 0} H(P^{k_1}, \ldots, P^{k_r}; P^{l_1}, \ldots, P^{l_r}) x_1^{k_1} \cdots x_r^{k_r}}_{p\text{-part}}.$$

p-PART FUNCTIONAL EQUATION

Invariance of $F(\mathbf{x};\theta)$ under W gives a functional equation.

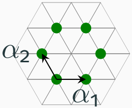
The functional equation gives a recurrence relation on the coefficients of $N(\mathbf{x};\theta)$. (See arXiv:1509.07400).

Write $N(\mathbf{x};\theta) = \sum a_\lambda \mathbf{x}^\lambda$. Recall for $\lambda = \sum k_i \alpha_i$, we have $a_\lambda = H(P^{k_1}, \ldots, P^{k_r}; P^{l_1}, \ldots, P^{l_r})$.

Theorem (Chinta–Friedberg–Gunnells, F.)

The support of $N(\mathbf{x};\theta)$ is contained in weight polytope Π_θ, defined as the convex hull $\theta - w\theta; w \in W$.

Example: $\Phi = A_2$, $n = 3$



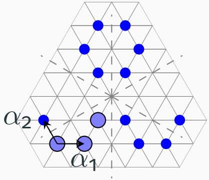
Support of $N(\mathbf{x};\rho)$

$$F(\mathbf{x};\rho) = \frac{1 + g(P)x_1 + g(P)x_2 + |P|^2 x_1^2 x_2 + |P|^2 x x_2^2 + g(P)|P|^2 x_1^2 x_2^2}{(1 - |P|^2 x_1^3)(1 - |P|^2 x_2^3)(1 - |P|^5 x_1^3 x_2^3)}$$

Theorem (Chinta–Friedberg–Gunnells, F.)

Define Θ^+ to be the set of all regular dominant weights in the irreducible representation of highest weight θ. Then up to the coefficients $a_{\theta-\xi}$ for $\xi \in \Theta^+$, $N(\mathbf{x};\theta)$ is completely determined by the functional equations of $F(\mathbf{x};\theta)$.

Example: $\Phi = A_2$, $n = 3$



Support of $N(\mathbf{x}; 4\varpi_1 + \varpi_2)$

$Z^*(\mathbf{s};\mathbf{m})$; Untwisted Case

Proposition (F.)

Let $\tilde{F}(\mathbf{X};\rho)$ *denote* $F(\mathbf{x};\rho)$ *after the variable change*

$$\nu : \begin{cases} s_i & \mapsto & 2 - s_i \\ |P| & \mapsto & 1/q \\ g_k^*(1,P) & \mapsto & \tau(\epsilon^k) \end{cases} .$$

Then $\tilde{F}(\mathbf{X};\rho) = Z^*(\mathbf{X};1,\ldots,1)$.

Example (Chinta 2008)

$\Phi = A_2$, $n = 3$

$$F(\mathbf{x}; \rho) = \frac{1 + g(P)x_1 + g(P)x_2 + |P|^2 x_1^2 x_2 + |P|^2 x x_2^2 + g(P)|P|^2 x_1^2 x_2^2}{(1 - |P|^2 x_1^3)(1 - |P|^2 x_2^3)(1 - |P|^5 x_1^3 x_2^3)}$$

$$Z^*(\mathbf{X}; 1, 1) = \frac{1 + q\tau X_1 + q\tau X_2 + q^4 X_1^2 X_2 + q^4 X_1 X_2^2 + q^5 \tau X_1^2 X_2^2}{(1 - q^4 X_1^3)(1 - q^4 X_2^3)(1 - q^7 X_1^3 X_2^3)}$$

Idea of Proof

The p-parts and the global series have the same polar behavior and satisfy the same functional equations (up to a variable change).

$Z^*(\mathbf{s}; \mathbf{m})$; TWISTED CASE

Theorem (F.)

Fix $\mathbf{m} \in \mathcal{O}^r$ *and put* $\theta = \sum(\deg m_i + 1)\varpi_i$. *Let* Θ^+ *be the set of regular dominant weights in the irreducible representation of highest weight* θ. *Then*

$$Z^*(\mathbf{X}; \mathbf{m}) = \sum_{\xi \in \Theta^+} M_{\theta - \xi} \tilde{F}(\mathbf{X}; \xi) \mathbf{X}^{\theta - \xi},$$

where for $\lambda = \sum_{i=1}^r \lambda_i \alpha_i$, *the coefficients* M_λ *are the character sums*

$$M_\lambda = \sum_{\substack{\mathbf{c} \in (\mathcal{O}_{mon})^r \\ \deg c_i = \lambda_i}} H(\mathbf{c}; \mathbf{m}).$$

Example 1

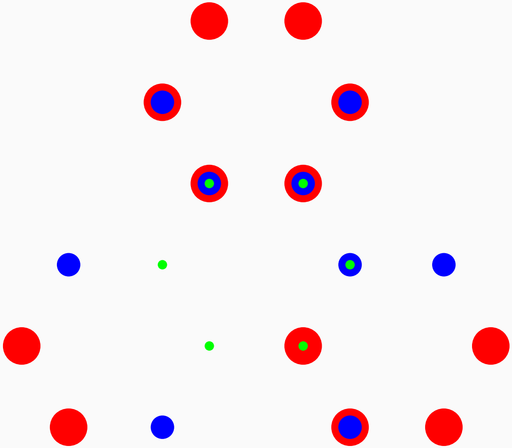
- $\Phi = A_2$, $n = 3$, $q = 7$
- $\mathbf{m} = (t^3 + 5t + 2, 1)$, $\theta = 4\varpi_1 + \varpi_2$
- $\Theta^+ = \{4\varpi_1 + \varpi_2, 2\varpi_1 + 2\varpi_2, \rho\}$
- $\{\theta - \xi : \xi \in \Theta^+\} = \{0, \alpha_1, 2\alpha_1 + \alpha_2\}$

$$\begin{aligned} Z^*(\mathbf{X}; \mathbf{m}) = & M_0 \tilde{F}(\mathbf{X}; 4\varpi_1 + \varpi_2) \\ & + M_{\alpha_1} \tilde{F}(\mathbf{X}; 2\varpi_1 + 2\varpi_2) X_1 \\ & + M_{2\alpha_1+\alpha_2} \tilde{F}(\mathbf{X}; \rho) X_1^2 X_2 \end{aligned}$$

Where

$M_0 = 1, M_{\alpha_1} \approx \tau(\epsilon)(-0.5-2.598i), M_{2\alpha_1+\alpha_2} =\approx \tau(\epsilon)^3(6.5-2.598i).$

Example 1



Support of the numerator of $Z^*(\mathbf{X};\mathbf{m})$, in terms of the $\tilde{N}(\mathbf{X};\xi)$.

Example 2

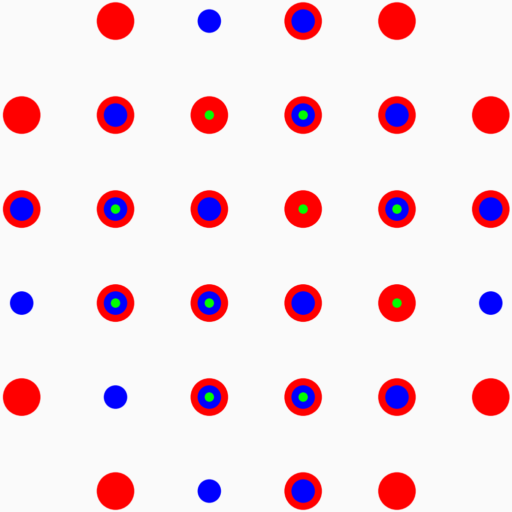
- $\Phi = B_2$, $n = 2$, $q = 5$
- $\mathbf{m} = (1, t^2 + 2)$, $\theta = \varpi_1 + 3\varpi_2$
- $\{\theta - \xi : \xi \in \Theta^+\} = \{0, \alpha_2, \alpha_1 + 2\alpha_2\}$

$$\begin{aligned} Z^*(\mathbf{X};\mathbf{m}) &= M_0 \tilde{F}(\mathbf{X}; \varpi_1 + 3\varpi_2) \\ &\quad + M_{\alpha_2} \tilde{F}(\mathbf{X}; 2\varpi_1 + \varpi_2) X_2 \\ &\quad + M_{\alpha_1 + 2\alpha_2} \tilde{F}(\mathbf{X}; \rho) X_1 X_2^2 \end{aligned}$$

Where

$$M_0 = 1, \qquad M_{\alpha_2} = -\tau(\epsilon), \qquad M_{\alpha_1+2\alpha_2} = q^2$$

Example 2

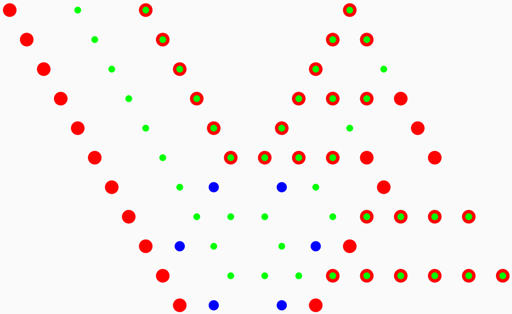


Support of the numerator of $Z^*(\mathbf{X}; \mathbf{m})$, in terms of the $\tilde{N}(\mathbf{X}; \xi)$.

Idea of Proof

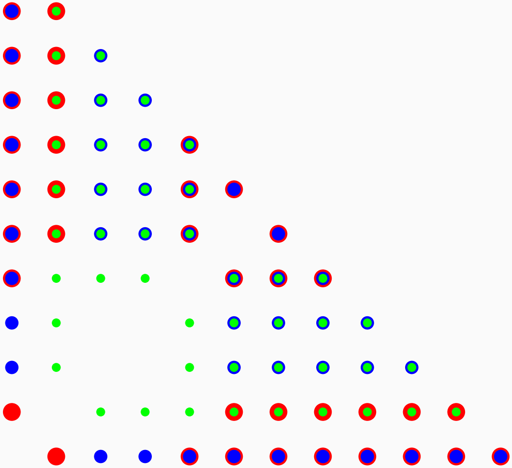
1. The $\tilde{F}(\mathbf{X};\xi)\mathbf{x}^{\theta-\xi}$, $\xi \in \Theta^+$ satisfy the global functional equations $\implies Z^*(\mathbf{X};\mathbf{m}) = \sum c_\xi \tilde{F}(\mathbf{X};\xi)\mathbf{X}^{\theta-\xi}$.
2. The support of $\Xi(s)^{-1}\tilde{F}(\mathbf{X};\xi)$ lies outside of Π_ξ. The coefficients $c_\xi = M_\lambda$ are the $\mathbf{X}^{\theta-\xi} = \mathbf{X}^\lambda$ coefficients of $Z(\mathbf{s};\mathbf{m})$ (no normalizing factors), written as a power series in q^{-s_i}.

Support of $\Xi(s)^{-1}\tilde{F}(\mathbf{X};\xi)$



$\Phi = A_2$, $n = 2$, support of $\Xi[\mathbf{x}]^{-1}\tilde{F}(\mathbf{X};\xi)\mathbf{X}^{\theta-\xi}$

Support of $\Xi(s)^{-1}\tilde{F}(\mathbf{X};\xi)$



$\Phi = B_2$, $n = 2$, support of $\Xi[\mathbf{x}]^{-1}\tilde{F}(\mathbf{X};\xi)\mathbf{X}^{\theta-\xi}$

Questions

1. What do the $M_{\theta-\xi}$, $\xi \in \Theta^+$ tell us?
2. Is there a connection between $Z_{\mathbb{F}_q(t)}(\mathbf{s};\mathbf{m})$ and $Z_K(\mathbf{s};\mathbf{1})$ for K higher genus?

The End

Thank you!

Local functional equations

For $\alpha \in \Phi$, let $n(\alpha) = \dfrac{n}{\gcd(n, \|\alpha\|^2)}$ and let $\Lambda' \subset \Lambda$ be the sublattice generated by the set $\{n(\alpha)\alpha\}_{\alpha\in\Phi}$.

Define $\tilde{A}_\beta$ as the set of functions f/g such that $\operatorname{supp} g$ lies in the kernel of the map $\nu : \Lambda \to \Lambda/\Lambda'$ and ν maps $\operatorname{supp} f$ to β.

Write $F(\mathbf{x}, \ell) = \sum_{\beta\in\Lambda/\Lambda'} f_\beta(\mathbf{x})$ so that $f_\beta(\mathbf{x}) \in \tilde{A}_\beta$. Then

$$f_\beta(\mathbf{x}) = \mathcal{P}_{\beta,\ell,k}(x_k) f_\beta(\sigma_k\mathbf{x}) + \mathcal{Q}_{\beta,\ell,k}(x_k) f_{\sigma_k\bullet\beta}(\sigma_k\mathbf{x}).$$

Global functional equation

Let $l = (l_1, \ldots, l_r)$ for $0 \leq l_j < n(\alpha_j) - 1$, and define

$$Z_l^*(\mathbf{s};\mathbf{m}) = \Xi(s) \sum_{\substack{\mathbf{c} \in \mathcal{O}_{mon} \\ \deg c_j \equiv l_j \mod n(\alpha_j)}} \frac{H(\mathbf{c};\mathbf{m})}{|c_1|^{s_1} \cdots |c_r|^{s_r}}.$$

Then

$$Z_l^*(\mathbf{s};\mathbf{m}) = |m_i|^{1-s_i} \left(P_{l_i, J_i(m,l)}^{\|\alpha_i\|^2}(s_i) Z_l^*(\sigma_i \mathbf{s};\mathbf{m}) + Q_{l_i, J_i(m,l)}^{\|\alpha_i\|^2}(s_i) Z_{\sigma_i \bullet l}^*(\sigma_i \mathbf{s};\mathbf{m}) \right)$$