On the combinatorial structure of Arthur packets: p-adic symplectic and orthogonal groups

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Global motivation

- k number field, $\mathbb{A}$ adele ring of k
- Γ_k absolute Galois group, W_k Weil group
- G split symplectic or special odd orthogonal group over k

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Definition

Automorphic representations of $G(\mathbb{A})$ are irreducible constituents of the regular representation on $L^2(G(k)\backslash G(\mathbb{A}))$.

$$L^2(G(k)\backslash G(\mathbb{A})) = L^2_{disc}(G) \oplus L^2_{cont}(G)$$

$$L^2_{disc}(G) = L^2_{cusp}(G) \oplus L^2_{res}(G)$$

Global Langlands Correspondence

$$\left\{ \begin{array}{c} \text{discrete automorphic} \\ \text{representations of } G(\mathbb{A}) \end{array} \right\}_{/\sim} \longleftrightarrow \left\{ \begin{array}{c} \text{discrete Arthur parameters} \\ \psi{:}L_k \times SL(2,\mathbb{C}) \to \widehat{G} \end{array} \right\}_{/\widehat{G}-conj}$$

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L_k is the hypothetical global Langlands group satisfying

$$1 \longrightarrow K_k \longrightarrow L_k \longrightarrow W_k \longrightarrow 1$$

where K_k is compact.

Arthur packet

- $\mathcal{A}_2(G)$: equivalence classes of discrete automorphic representations.
- $\Psi_2(G)$: equivalence classes of discrete Arthur parameters.

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- $\Psi_2(G)$: equivalence classes of discrete Arthur parameters.

Theorem (Arthur)

For each $\psi \in \Psi_2(G)$, there exits a "multi-set" Π_ψ of equivalence classes of irreducible admissible representations of $G(\mathbb{A})$ such that

1. $$\Pi_\psi = \otimes'_v \Pi_{\psi_v}$$
2. $$\mathcal{A}_2(G) \subseteq \bigsqcup_{\psi \in \Psi_2(G)} \Pi_\psi$$
3. *(Endoscopy theory): One can distinguish the automorphic representations in Π_ψ.*

Arthur packet

- Π_ψ is called global Arthur packet
- Π_{ψ_v} is a finite "multi-set" of equivalence classes of irreducible admissible representations of $G(k_v)$, called local Arthur packet.

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Example

1. $G = SO(3) \cong PGL(2)$: Π_ψ is a single automorphic representation.

2. $G = Sp(2) \cong SL(2)$: Π_ψ is the restriction of an automorphic representation of $GL_2(\mathbb{A})$ to $SL_2(\mathbb{A})$.

Global problem

How to distinguish the residue spectrum in Π_ψ?

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Mœglin:

- global condition: zeros (poles) of certain L-functions "related" to ψ.
- local condition: "fine" parametrization of Π_{ψ_v}.

Arthur parameter

Let F be a p-adic field, $L_F = W_F \times SL(2,\mathbb{C})$

G	$\widehat{G}$
$Sp(2n)$	$SO(2n+1,\mathbb{C})$
$SO(2n+1)$	$Sp(2n,\mathbb{C})$

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Let $\widehat{G} \xrightarrow{std.} GL_N(\mathbb{C})$ (N = 2n or 2n+1) be the standard representation.

$$\psi : W_F \times SL(2,\mathbb{C}) \times SL(2,\mathbb{C}) \to \widehat{G} \xrightarrow{std.} GL(N,\mathbb{C})$$

with bounded image on $\psi|_{W_F}$.

Jordan blocks

$$\psi = \oplus_i(\rho_i \otimes \nu_{a_i} \otimes \nu_{b_i})$$

- ρ_i equivalence class of unitary irreducible representation of W_F
- $a_i, b_i \in \mathbb{N}$
- ν_m is Sym^{m-1}-representation of $SL(2, \mathbb{C})$

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Define

$$Jord(\psi) = \{(\rho_i, a_i, b_i)\}$$

and

$$Jord_\rho(\psi) := \{(\rho', a', b') \in Jord(\psi) : \rho' = \rho\}.$$

Parity

For self-dual ρ: orthogonal type or symplectic type

$$(\rho, a, b) \text{ is orthogonal } \Leftrightarrow \begin{cases} a+b \text{ is even, if } \rho \text{ is orthogonal} \\ a+b \text{ is odd, if } \rho \text{ is symplectic} \end{cases}$$

$$(\rho, a, b) \text{ is symplectic } \Leftrightarrow \begin{cases} a+b \text{ is odd, if } \rho \text{ is orthogonal} \\ a+b \text{ is even, if } \rho \text{ is symplectic} \end{cases}$$

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$$\psi_p = \bigoplus_{\substack{(\rho,a,b)\in Jord(\psi) \\ \text{same parity as } \widehat{G}}} \rho \otimes \nu_a \otimes \nu_b$$

From now on, we will assume $\psi = \psi_p$.

Visualize Jordan blocks

For $(\rho, a, b) \in Jord(\psi)$,

$$A = (a+b)/2 - 1 \qquad B = |a-b|/2$$

and

$$\zeta = \begin{cases} \text{Sign}(a-b), & \text{if } a \neq b \\ \text{arbitrary}, & \text{otherwise.} \end{cases}$$

So we can also denote (ρ, a, b) by (ρ, A, B, ζ).

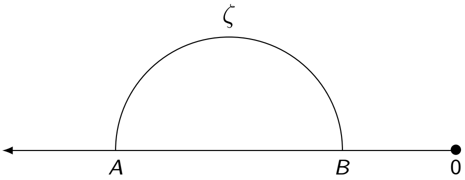


Figure: ρ

Admissible order

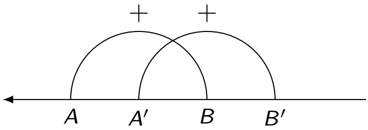
A total order $>_\psi$ on $Jord_\rho(\psi)$ is called admissible if

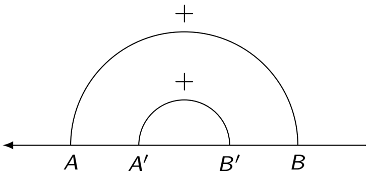
$\forall(\rho, A, B, \zeta), (\rho, A', B', \zeta') \in Jord_\rho(\psi)$ satisfying

$$A > A', B > B' \text{ and } \zeta = \zeta'$$

we have $(\rho, A, B, \zeta) >_\psi (\rho, A', B', \zeta')$.

Example

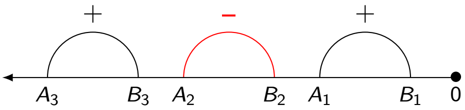




Discrete diagonal restriction

Definition

We say ψ has discrete diagonal restriction if for each ρ the Jordan blocks in $Jord_\rho(\psi)$ are "disjoint".



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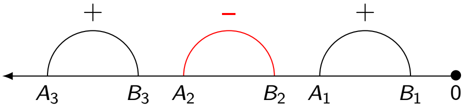

In this case, $Jord_\rho(\psi)$ has a natural order $>_\psi$, namely

$$(\rho, A, B, \zeta) >_\psi (\rho, A', B', \zeta') \text{ if and only if } A > A'.$$

Mœglin's parametrization I

Theorem (Mœglin)

Suppose ψ has discrete diagonal restriction, $>_\psi$ is the natural order,

$$\Pi_\psi = \bigoplus_{\{(\underline{l},\underline{\eta}) : \prod_{(\rho,a,b)\in Jord(\psi)} \varepsilon_{\underline{l},\underline{\eta}}(\rho,a,b)=1\}/\sim} \pi_{M,>_\psi}(\psi,\underline{l},\underline{\eta}).$$

where $\pi_{M,>_\psi}(\psi,\underline{l},\underline{\eta})$ is irreducible.

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- $(\underline{l},\underline{\eta})$ are integral valued functions over $Jord(\psi)$, such that

$$\underline{l}(\rho,A,B,\zeta) \in [0,[(A-B+1)/2]] \text{ and } \underline{\eta}(\rho,A,B,\zeta) \in \{\pm 1\},$$

- $\varepsilon_{\underline{l},\underline{\eta}}(\rho,A,B,\zeta) := \underline{\eta}(\rho,A,B,\zeta)^{A-B+1}(-1)^{[(A-B+1)/2]+\underline{l}(\rho,A,B,\zeta)}$
- $(\underline{l},\underline{\eta}) \sim (\underline{l}',\underline{\eta}')$ if and only if $\underline{l}=\underline{l}'$, and

$$(\underline{\eta}/\underline{\eta}')(\rho,A,B,\zeta)=1$$

unless $\underline{l}(\rho,A,B,\zeta)=(A-B+1)/2$.

Dominating parameter

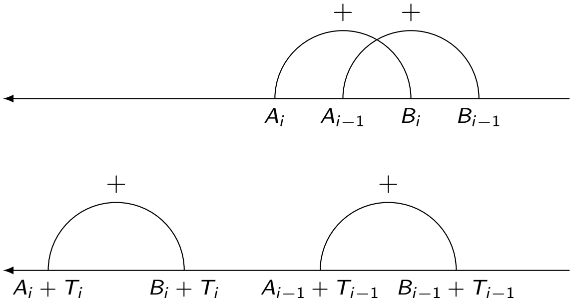
For ψ and admissible $>_\psi$, we can index $Jord_\rho(\psi)$ such that

$$(\rho, A_i, B_i, \zeta_i) >_\psi (\rho, A_{i-1}, B_{i-1}, \zeta_{i-1}).$$

We say $\psi_\gg$ dominates ψ with respect to $>_\psi$ if $Jord_\rho(\psi_\gg)$ consists of

$$(\rho, A_i + T_i, B_i + T_i, \zeta_i) \text{ for } T_i \geqslant 0,$$

with the same admissible order $>_\psi$ under the natural identification.



Mœglin's parametrization II

For ψ and admissible $>_\psi$, we choose a dominating parameter $\psi_\gg$ with discrete diagonal restriction. Then we define

$$\pi_{M,>_\psi}(\psi,\underline{l},\underline{\eta}) := \circ_{\rho;(\rho,A_i,B_i,\zeta_i)\in Jord_\rho(\psi)} \mathrm{Jac}_{X_i} \pi_{M,>_\psi}(\psi_\gg,\underline{l},\underline{\eta}).$$

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Proposition (Mœglin)

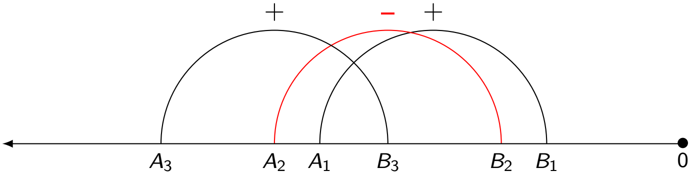
1. *$\pi_{M,>_\psi}(\psi,\underline{l},\underline{\eta})$ is either irreducible or zero.*
2. *If $\pi_{M,>_\psi}(\psi,\underline{l},\underline{\eta}) = \pi_{M,>_\psi}(\psi,\underline{l}',\underline{\eta}') \neq 0$, then $(\underline{l},\underline{\eta}) \sim (\underline{l}',\underline{\eta}')$.*
3. $$\Pi_\psi = \bigoplus_{\{(\underline{l},\underline{\eta}):\, \prod_{(\rho,a,b)\in Jord(\psi)} \varepsilon_{\underline{l},\underline{\eta}}(\rho,a,b)=1\}/\sim} \pi_{M,>_\psi}(\psi,\underline{l},\underline{\eta}).$$

   where $\pi_{M,>_\psi}(\psi,\underline{l},\underline{\eta})$ is irreducible or zero.

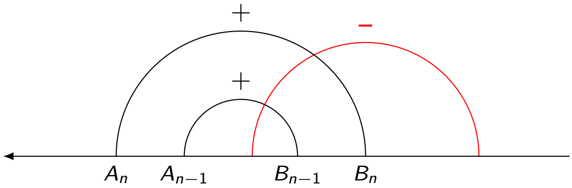
Local problem

What are the conditions on $(\underline{l}, \underline{\eta})$ for $\pi_{M,>_\psi}(\psi, \underline{l}, \underline{\eta}) \neq 0$?

Example



Pull



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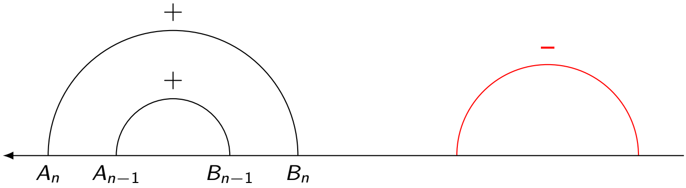

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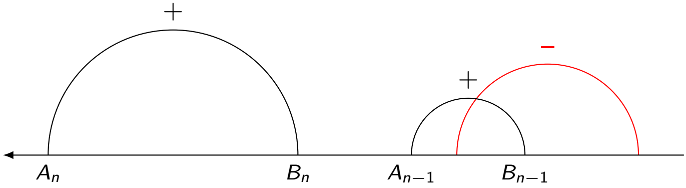

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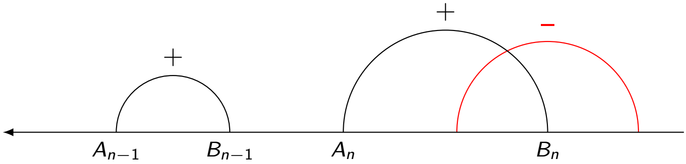
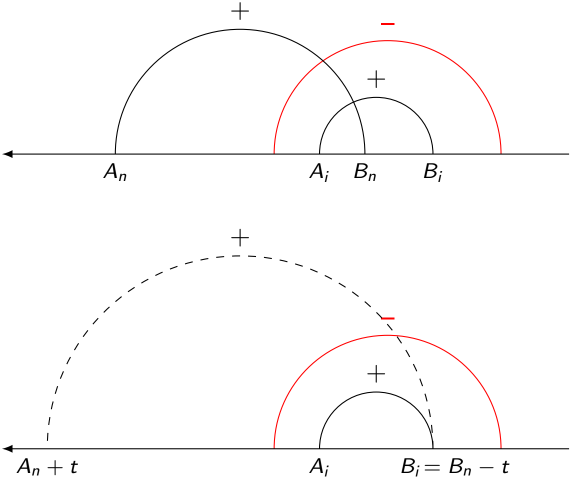
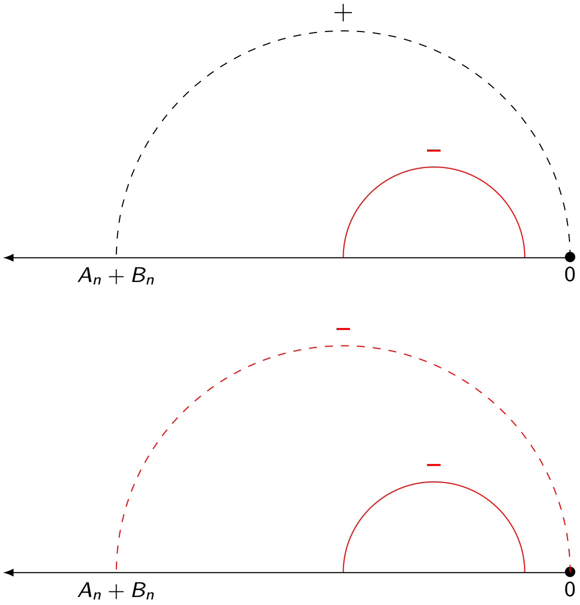


Expand



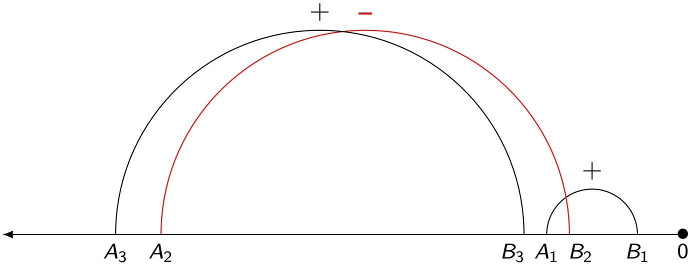
Change sign



Example

Let $\psi = \rho \otimes \nu_{51} \otimes \nu_{31} \, \oplus \, \rho \otimes \nu_{31} \otimes \nu_{45} \, \oplus \, \rho \otimes \nu_{13} \otimes \nu_5$. Then

$$[A_3, B_3] = [40, 10] \quad [A_2, B_2] = [37, 7] \quad [A_1, B_1] = [8, 4]$$



Example

$0 \leqslant l_1 \leqslant 2, 0 \leqslant l_2 \leqslant 15, 0 \leqslant l_3 \leqslant 15$, and $(-1)^{l_1+l_2+l_3}\eta_1\eta_2\eta_3 = 1$.

Example

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$\eta_3 = \eta_1$ and $\eta_2 = \eta_1$	$-5 \leqslant l_3 - l_2 + 2l_1 \leqslant 15$
$\eta_3 = \eta_1$ and $\eta_2 \neq \eta_1$	$l_3 + l_2 + 2l_1 > 25$
$\eta_3 \neq \eta_1$ and $\eta_2 = \eta_1$	$l_3 - l_1 < 11 + l_1$ and $l_3 + l_2 - 2l_1 > 15$
$\eta_3 \neq \eta_1$ and $\eta_2 = \eta_1$	$l_3 - l_1 \geqslant 11 + l_1$ and $-36 \leqslant -l_3 - l_2 + 2l_1 \leqslant -16$
$\eta_3 \neq \eta_1$ and $\eta_2 \neq \eta_1$	$l_3 - l_1 < 11 + l_1$ and $-15 \leqslant l_3 - l_2 - 2l_1 \leqslant 5$
$\eta_3 \neq \eta_1$ and $\eta_2 \neq \eta_1$	$l_3 - l_1 \geqslant 11 + l_1$ and $-l_3 + l_2 + 2l_1 > -6$

Each case gives rise to a polytope, and by counting the integral points in them we get $|\Pi_\psi| = 1651$.