Things I'd Like to Know

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UNIVERSITY OF
CALGARY

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We are looking for exponentially many needles, but our haystack is superexponential, so that it becomes increasingly difficult to find even a single needle.

Building Barrycades

Barry Cipra noticed that the partial sums of the numbers $\{1, 2, \ldots, n\}$ were the triangular numbers $\{1, 3, 6, \ldots, \frac{1}{2}n(n+1)\}$ and asked if there were several different permutations of the numbers from 1 to n, whose partial sums, other than the complete sum $\frac{1}{2}n(n+1)$, form the set of all numbers from 1 to $\frac{1}{2}n(n+1) - 1$, each number appearing just once.

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For example, $\{1, 2, 3, 4\}$ has partial sums 1, 3, 6, 10, while $\{2, 3, 4, 1\}$ has partial sums 2, 5, 9, (10) and $\{4, 3, 1, 2\}$ has partial sums 4, 7, 8, (10), which between them include all the integers from 1 to 9 exactly once each.

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From now on we'll assume that a barrycade is always breakfree. The barrycade corresponding to the above example is shown in Figure 1.

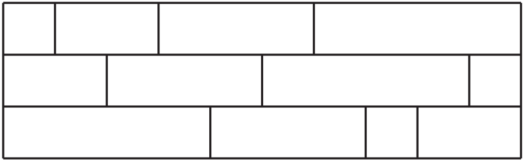

Figure 1: A barrycade for $n = 4$

Since the sum of the log-lengths is $\frac{1}{2}n(n+1)$ and the number of joins in each layer is $n-1$, the number of layers is

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Moreover, it also seems that the number of different barrycades, for a given even value of n, grows quite rapidly as n increases. We won't count them as different if they just have their layers in a different order. In fact we'll always build our barrycades with the leftmost logs having increasing lengths as we go from top to bottom.

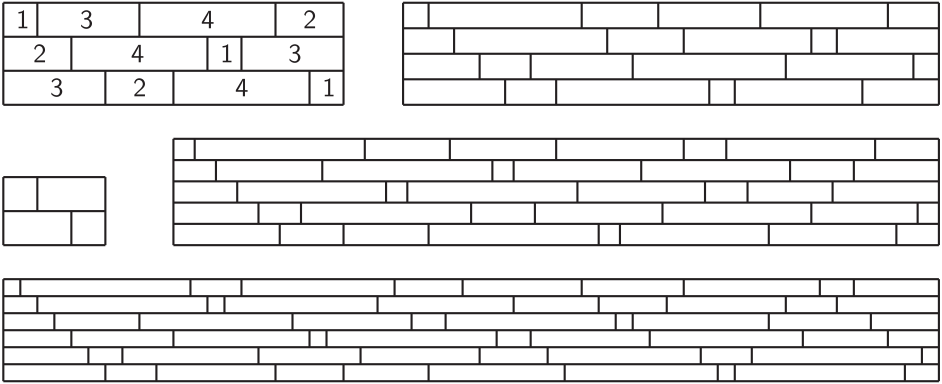

Figure 2: Breakfree barrycades for $n = 4, 6, 2, 8$ and 10

The barrycades in Figure 2 are not only breakfree, but they also satisfy the **Fink condition** (suggested by Alex Fink): that is, they are **balanced** in the sense that, if we look at them as being made up of $n-1$ sections of equal width, the $\frac{1}{2}(n+2)$ joins in each section occur just one in each layer.

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In Figure 3 the sections are separated by dashed vertical lines.

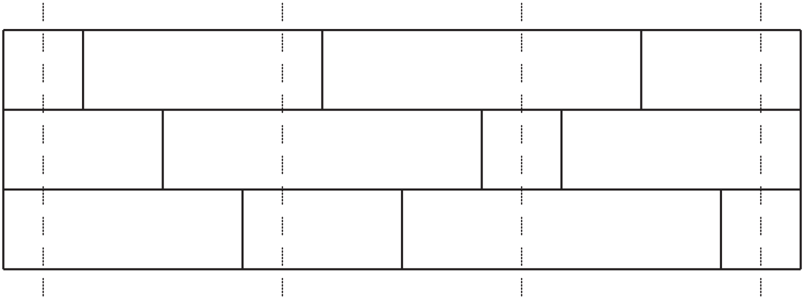

Figure 3: A balanced barrycade for $n = 4$

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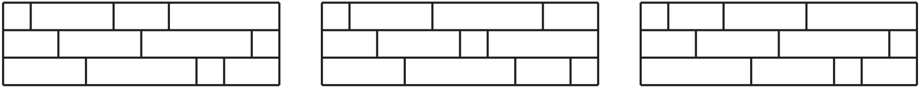

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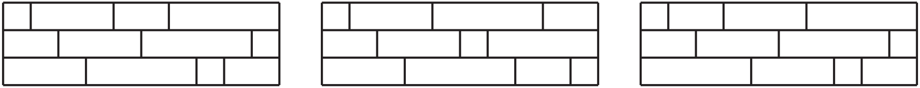

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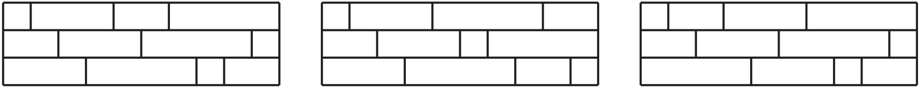

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For $n = 6$, Sam Benner finds 1120 barrycades.

For $n = 8$, he has counted no fewer than 28432700 barrycades.

In Figure 5 there is a rotary solution of the original problem for $n = 6$.

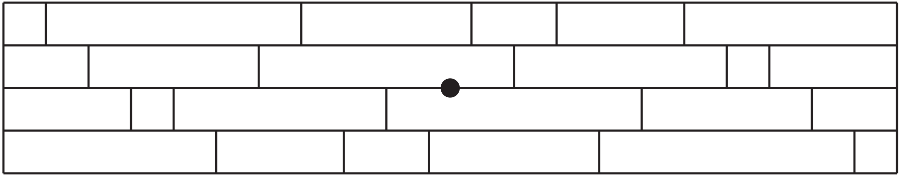

Figure 5: A rotary barrycade for $n = 6$

Stan Wagon used the problem of building barrycades as his Problem of the Week, and Rob Pratt found examples for $n = 2, 4, 6, \ldots, 26$ and rotary examples for $n = 26, 30, 34$ and 38.

Fibonacci Plays Biliards

At the July, 2002 Combinatorial Games Conference in Edmonton, Berlekamp & I found Yoshiyuki Kotani looking for values of n which would enable him to arrange the numbers 1 to n in a chain so that adjacent links summed to a perfect cube. Part of such a chain might be

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$$(16 \rightarrow) 9 \rightarrow 7 \rightarrow 2 \leftarrow 14 \rightarrow 11 \rightarrow 5 \rightarrow 4 \leftarrow 12 \leftarrow 13 \rightarrow 3 \leftarrow 6 \leftarrow 10 \leftarrow 15 \rightarrow 1 \leftarrow 8 (\leftarrow 17)$$

Figure 6: Solution(s) to Recaman's problem for $n = 15, 16, 17$.

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4	21	28	8	1	15	10	26	23
32								2
17								14
19								22
30								27
6								9
3								16
13								20
12	24	25	11	5	31	18	7	29

Figure 7: A necklace with adjacent pairs of beads adding to squares.

The corresponding problem with neighbors summing to Fibonacci numbers, $F_0 = 0$, $F_1 = 1$, $F_{k+1} = F_k + F_{k-1}$, instead of squares, has a better balanced solution.

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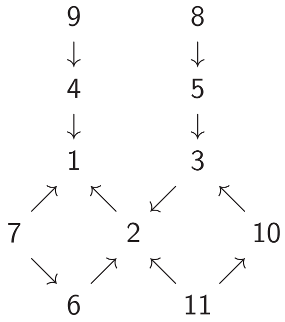
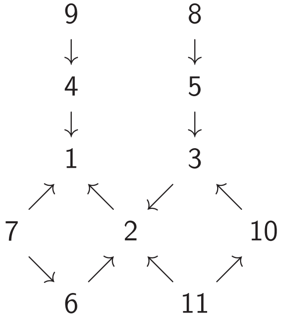


Figure 8: Graph whose adjacencies are Fibonacci sums



The arrows are drawn from the larger to the smaller number: the larger number is not part of the graph unless the smaller is already present.
From the graph we can read off 1 2; 1 2 3; 4 1 2 3;
4 1 2 3 5; 4 1 7 6 2 3 5; 4 1 7 6 2 3 5 8; 9 4 1 7 6 2 3 5 8
and 9 4 1 7 6 2 11 10 3 5 8.

We can also verify that 6 and 10 can't be included in a chain unless some larger number is also present (in the former case 4, 5 and 6 are monovalent vertices and all three can't be ends of the chain; in the latter case, 8, 9 and 10).

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Theorem. (Berlekamp, G.) There is a chain formed with the numbers 1 to n with each adjacent pair adding to a Fibonacci number, just if $n = 9$, 11, or F_k or $F_k - 1$, where F_k is a Fibonacci number with $k \geq 4$. The chain is essentially unique.

Square necklaces for $n = 32, 33, 34, \ldots, 245$ have been found, and it appears that as n increases, the number of different necklaces for a given value of n increases, too.

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But no-one has been able to prove that there are square necklaces for all $n \geq 32$.

Don't Try to Solve these Problems !

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If odd, treble and add one; if even, halve.

$$7 \to 22 \to 11 \to 34 \to 17 \to 52 \to 26 \to 13 \to 40 \to 20 \to 10 \to 5 \to 16$$

$$\to 8 \to 4 \to 2 \to 1 \to 4 \to 2 \to 1 \ \cdots$$

Is the following problem just as recalcitrant??

Conway's Subprime Fibonacci Sequences

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An SFS is formed in the same way as the Fibonacci sequence, but before we accept a composite number we divide it by its smallest prime factor: 0, 1, 1, 2, 3, 5. Now not 8, but $8/2 = 4$. 5 and 4 make 9, but we record $9/3 = 3$. 4 and 3 make 7, which is prime. 3 and 7 give $10/2 = 5$. 7 and 5 give $12/2 = 6$. 5 and 6 give 11. 6 and 11 give 17. 11 and 17 give $28/2 = 14$. And so on ....

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Recall that if a sequence is formed by starting with $u_0 = 0$ and $u_1 = 1$ and continuing with $u_n = au_{n-1} + bu_{n-2}$, then we have a divisibility sequence; i.e., if m divides n, then u_m divides u_n. In particular, if p is a prime, then p divides $u_{p-\left(\frac{\Delta}{p}\right)}$, where $\left(\frac{\Delta}{p}\right)$ is the Legendre symbol, and $\Delta = a^2 + 4b$ is the discriminant.

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For example, for the Fibonacci numbers, $\Delta = 5$.
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And infinitely many similar questions !

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ARE THERE INFINITELY MANY FIBONACCI PRIMES ??

Are there infinitely many Jacobsthal ($a = 1$, $b = 2$) primes ?

And infinitely many similar questions !

Hugh Williams is interested in the corresponding problems for fourth and higher order divisibility sequences.

Diophantine Equations

It is surprising that there are quadratic Diophantine equations for which we do not know if there are solutions

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An infinity of solutions have been found in each of the cases where we drop the condition of rationality for one edge, or for one face diagonal, or for the body diagonal.

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We also want $16\Delta^2 = (a+b+c)(b+c-a)(c+a-b)(a+b-c)$ to have integer solutions.

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If we also require the area to be rational, then Buchholz & Rathbun [4, 5] have shown that any rational point on the curve $(xy + 2)(x - y + 1) = 3$ with $0 < x, y < 1$ and $2x + y > 1$ corresponds to a triangle with rational edges, rational area, and **two** rational medians.

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There are infinitely many points at rational distances from **three** of the four corners.

There are five configurations of four rational triangles covering the unit square: delta, nu, kappa, lambda, and chi. An infinity of solutions is known in each case except the last.

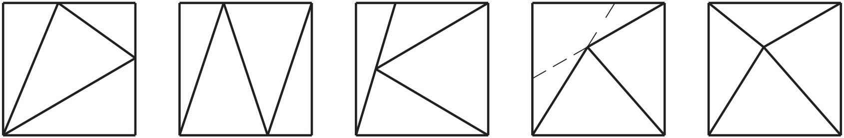

Figure 9: Rational(?) tilings of the square.

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Behavior of aliquot sequences is dominated by **guides**, which include the **downdriver**, 2, and **updrivers**, such as $2 * 3$ and $2^2 * 7$. The longer the guide has been a downdriver, the **less** likely it is to persist. The longer it has been an updriver, the **more** likely it is to persist.

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An important paper by Pollack & Pomerance [13] has recently been published. The fact that their formulas often contain three and four times iterated logarithms does not bode well for being able to find computer evidence.

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