Parameterized Morse theory in low-dimensional and symplectic topology

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1   Overview and highlights of workshop

Morse theory uses generic functions from smooth manifolds to $\mathbb{R}$ (Morse functions) to study the topology of smooth manifolds, and provides, for example, the basic tool for decomposing smooth manifolds into elementary building blocks called handles. Recently the study of parameterized families of Morse functions has been applied in new and exciting ways to understand a diverse range of objects in low-dimensional and symplectic topology, such as Morse 2–functions in dimension 4, Heegaard splittings in dimension 3, generating families in contact and symplectic geometry, and n–categories and topological field theories (TFTs) in low dimensions. Here is a brief description of these objects and the ways in which parameterized Morse theory is used in their study:

A **Morse** 2**–function** is a generic smooth map from a smooth manifold to a smooth 2–dimensional manifold (such as $\mathbb{R}^2$). Locally Morse 2–functions behave like generic 1–parameter families of Morse functions, but globally they do not have a "time" direction. The singular set of a Morse 2–function is 1–dimensional and maps to a collection of immersed curves with cusps in the base, the "graphic". Parameterized Morse theory is needed to understand how Morse 2–functions can be used to decompose and reconstruct smooth manifolds [20], especially in dimension 4 when regular fibers are surfaces, to understand uniqueness statements for such decompositions [21], and to use such decompositions to produce computable invariants.

A **Heegaard splitting** of a 3–manifold is a decomposition into two solid genus g handlebodies, and the existence of Heegaard splittings follows directly from Morse theory. Heegaard splittings are unique up to a certain stabilization procedure, a fact which is proved with standard Cerf theory. Parameterized Morse theory arises much more subtly when comparing two Heegaard splittings and asking how many stabilizations are needed to make them the same. The two Heegaard splittings are replaced with two Morse functions, or sweepouts, and these give a map to $\mathbb{R}^2$ which, generically, is a Morse 2–function as discussed above. For example, Johnson [27] built on ideas of Rubinstein and Scharlemann [36] and used a careful understanding of the graphic of critical values to get bounds on the number of stabilizations needed.

To understand **generating families**, consider a cotangent bundle with the standard symplectic structure, or a one-jet space with the standard contact structure. Using an N-dimensional Cerf theory, Viterbo showed how to generate any Lagrangian submanifold in the cotangent bundle with an N-parameter family of functions, where N might be arbitrarily large [48]. In this context, the N-parameter family is called a "generating family." For a similarly-defined generating family of any Legendrian submanifold in a one-jet space, a certain relative Morse homology is called the submanifold's *generating homology*. Traynor and her collaborators have a number of computations demonstrating the homology's applicability [33, 39, 40], while Fuchs

and Rutherford show that this homology is the same as a certain linearization of the Chekanov-Eliashberg differential graded algebra (DGA) for Legendrian knots in standard contact R^3 [18].

An n–**category** is something for which the prototypical example is given by cobordisms between cobordisms between cobordisms and so on, for n levels. For example, the 2–category of surfaces has objects which are points, morphisms which are 1–dimensional cobordisms between points, and 2–morphisms which are 2–dimensional cobordisms between 1–dimensional cobordisms between points. A **TFT** is a functor from a cobordism n–category to an n–category of algebraic objects, such as vector spaces, giving invariants of manifolds computable by cutting up into n–category cobordisms. Just as ordinary Morse functions give decompositions into elementary cobordisms, i.e. handles, n–families of Morse functions and Morse n–functions can give decompositions in the n–category sense, and introducing further parameters can give relations amongst such decompositions. Schommer-Pries [44] has carried out this program to get a complete set of generators for the 2–category of surfaces.

The study of each of these objects naturally uses very similar tools in parameterized Morse theory, but the broad fields in which the study of these objects live are often seen as quite far apart, so that researchers in these fields are not necessarily aware of the overlaps. This workshop brought together these researchers to learn from each other, develop common terminology, and share tools. As people working in one field gained a clearer picture of how parameterized Morse theory is being used in other fields, they came away with new techniques to use in their own fields as well as new ideas for how their own techniques may contribute to the other fields.

For those readers who would like to watch some of the recorded videos from the workshop, our week was structured as follows: Monday was devoted to four introductory talks, on 3–manifolds, 4–manifolds, TFTs, and symplectic topology. Tuesday morning focused on 3–manifolds, Tuesday afternoon focused on 4–manifolds, and Tuesday evening was devoted to a group discussion of open problems in 3– and 4–dimensional topology. Wednesday morning focused on TFTs, Wednesday afternoon was free, and Wednesday evening we discussed open problems related to TFTs. Thursday focused on symplectic and contact topology, ending with an open problem session Thursday evening. Friday morning featured one more 4–manifold talk and one more contact topology talk.

We would like to highlight two specific talks as representative of the impact and diversity of the workshop. Jamie Vicary introduced a room full of topologists to some fascinating ideas in computer science, ranging from quantum teleportation to encrypted communication, that are beautifully tied to low-dimensional topology and Morse theory. Yakov Eliashberg gave the first public announcement of a ground breaking result (with coauthors Emmy Murphy and Strom Borman) extending Eliashberg's well known existence and uniqueness results for overtwisted contact structures in dimension 3 to arbitrary odd dimensions.

The evening problem sessions were not recorded on video but careful notes were produced. The bulk of this report describes the problems which arose during these sessions.

We would like to thank Fréderic Bourgeois, Rob Kirby, Paul Melvin, Peter Teichner and Abby Thompson for running the evening problem sessions, Bruce Bartlett, Ryan Budney and Josh Sabloff for taking notes during those sessions, Rob Kirby, Paul Melvin and Abby Thompson for transcribing the notes into palatable format, and all the other workshop participants for contributing so much stimulating discussion and so many exciting questions.

2 Problems

This next section describes problems (many of which were open to the conference participants as of March 2014) posed during the evening problem sessions. The person listed at the beginning of each problem is the one who posed it to the other participants. After some of the problems follows a list of remarks that were discussed in response.

2.1 3– and 4–dimensional manifolds

Problem 1 *(Johnson and Baker) Characterize fibred tunnel number one knots.*

Remark: Rathbun [35] has shown the tunnel can be isotoped to lie in the fiber.

Problem 2 *(Baker) Does there exist an infinite family of fibred knots in S^3 such that the same surgery produces the same manifold?*

Remark: Osoinach [34] introduced a way to create infinite families of knots for which the same surgery produces the same manifold, but it appears only finitely many knots in such families will be fibered.

Problem 3 *(Baker) Classify tunnel number one knots with lens space surgeries.*

Remark: All knots in S^3 known to admit a lens space surgery (the Berge knots) have tunnel number one.

Problem 4 *(Baker) Must a knot in S^3 with a lens space surgery have tunnel number one?*

Remark: This is weaker than the Berge conjecture, which is that such knots are tunnel one and doubly-primitive.

Problem 5 *(Baker) Which strongly quasipositive knots (SQP) are fibred?*

Remarks: Given an SQP knot in an SQP presentation, is there an efficient way to determine if the knot is fibred? Is there a monotonic simplification within SQP presentation to easily determine if a knot is fibred?

A knot is strongly quasipositive if it has a braid presentation as a product of the braids $\sigma_{i,j} = (\sigma_i \ldots \sigma_{j-2})\sigma_{j-1}(\sigma_i \ldots \sigma_{j-2})^{-1}$ where σ_i is the standard braid generator.

Lee Rudolph [37] introduced quasipositive knots in his study of complex algebraic curves, and strongly quasipositive knots arise from links of singularities. Hedden [26] has shown that fibered SQP knots are precisely the fibered knots in S^3 whose open books support the tight contact structure, and any knot in S^3 with a positive surgery to a Heegaard Floer L-space must be a fibered SQP knot.

Problem 6 *(Baker) Let K_1 and K_2 be a pair of homotopic knots in a 3-manifold M and K_1^*, K_2^* be a pair of homotopic knots in M^* such that for some non-trivial slope r, $K_i(r)$ (filling) is homeomorphic to M^* with surgery dual K_i^*. Because they're homotopic, it makes sense to talk about the same surgery slope for both K_1 and K_2. Must one of the pairs K_1, K_2 or K_1^*, K_2^* be isotopic? Perhaps up to an orientation-preserving automorphism of M or M^*.*

Remark: Specializing to $M = S^3$, this asks the following. If r surgery on each of a pair of distinct knots produces the same manifold M^*, then must the surgery duals to these knots be in distinct free homotopy classes modulo orientation-preserving homeomorphisms of M^*? An affirmative answer would, for example, resolve the Berge Conjecture.

Problem 7 *(Budney) Let $K_n = \{f : \mathbb{R} \to \mathbb{R} \times \mathbb{R}^{n-1}, f(x) = (x, 0) \forall x \notin [-1, 1]\}$. The maps f are required to be C^∞-smooth embeddings, and we give the space K_n the C^∞-topology. The inclusion $\mathbb{R}^n \to \mathbb{R}^{n+1}$ gives an inclusion $K_n \to K_{n+1}$. A classical argument all topologists are familiar with is that all smooth embeddings $S^1 \to S^4$ are isotopic. In this context, that argument provides two null-homotopies of the inclusion $K_n \to K_{n+1}$. The first null-homotopy comes from perturbing the knot in the positive orthogonal direction $\mathbb{R}^n \times \{0\} \subset \mathbb{R}^{n+1}$ and then applying the straight-line homotopy in the $\mathbb{R}^n \times \{0\}$ factor. The second null-homotopy comes from using the negative of that bump function. The two maps together give a map*

$$K_n \to \Omega K_{n+1}$$

Is this map null-homotopic? i.e. are the two ways of trivializing knots from K_n in K_{n+1} distinct?

Remark: If it is null-homotopic, can you find two canonical ones? And does the map $K_n \to \Omega^2 K_{n+1}$ induce an isomorphism on the lowest-dimensional homotopy group? The first non-trivial homotopy group of K_n is in dimension $2n - 6$, and is isomorphic to $\mathbb{Z}$ for $n \geq 4$.

Problem 8 *(Gay) Given a Morse function $f : X^4 \to \mathbb{R}$ on a 4–manifold X, find a lower-bound on the complexity of the Cerf graphic connecting f to $-f$ in terms of the combinatorial data in f.*

Remark: In [25] it was shown that for some 3–manifolds M, g births and deaths were needed where g is the genus of the Heegaard surface for f; [27] for a combinatorial version.

Problem 9 *(Melvin) The surgery number of a 3-manifold Y is defined to be the smallest n such that the Y can be realized as integer surgery on an n component link in the 3-sphere. Compute the surgery numbers of lens spaces. In particular, which lens spaces have surgery number 2?*

Remark: Note that the lens spaces of surgery number 1 have been characterized by Josh Greene [24]; they are precisely those that arise from surgery on Berge knots.

Problem 10 *(Auckly) Given a cusped hyperbolic 3-manifold M, all but finitely-many fillings are hyperbolic, and for all but finitely many fillings, the filled manifold M' has an isometry group which embeds in the isometry group of M, i.e. $Isom(M') \subset Isom(M)$. Find examples of exceptional surgeries where $Isom(M')$ does not embed in $Isom(M)$.*

Let M and N be homotopy equivalent (or homeomorphic, after Freedman) closed oriented simply-connected smooth 4-manifolds. Since Wall [49], we know that M and N are stably-diffeomorphic, i.e., they become diffeomorphic after connect summing both M and N with n copies of $S^2 \times S^2$ for large enough n. However, there is no known a priori upper bound on n.

Furthermore, it was shown in [11] that all *known* ways to construct infinite families of exotic (i.e. pairwise homeomorphic but not diffeomorphic) simply-connected 4-manifolds up to date always produce 4-manifolds which become diffeomorphic after one stabilization, that is, $n = 1$. The same result for knot surgery (remove a $T^2 \times B^2$, and glue in a knot complement cross S^1) was obtained earlier by Auckly [5] and Akbulut [2] using Kirby calculus.

Problem 11 *(Auckly): Are there any pairs M and N as above for which at least n stabilizations are needed for them to become diffeomorphic, with $n > 1$? Are there pairs for which n is arbitrarily large?*

Remark: Similar stabilization problems and analogous results for embeddings of surfaces are studied in [6] and in [12]. In the latter paper, exotic embeddings (topologically isotopic but not smoothly) of surfaces in 4-manifolds are shown to be stably smoothly isotopic where stabilization means a pairwise connected sum with (S^4, T^2). Again, for all known constructions, one stabilization is enough to get smoothly isotopic surfaces.

Recall that a *logarithmic transform* is the surgery operation in dimension 4 analogous to Dehn surgery in dimension 3: take out the tubular neighborhood of an embedded self-intersection zero 2-torus and glue it back in possibly with a twisted boundary diffeomorphism.

In [11], it was shown that between any two homeomorphic (in general, not necessarily simply-connected!) M and N as above, there is a cobordism from M to N which consists of round 2-handles only. As a corollary, one can pass from M to N after a sequence of n logarithmic transforms. If M and N are diffeomorphic after 1 stabilization with $S^2 \times S^2$, then one can pass from M to N after at most 2 integral logarithmic transforms, and conversely, if one can pass from M to N after 1 integral logarithmic transform, then M and N are diffeomorphic after 1 stabilization.

Hence, since *almost* all constructions of exotic 4-manifolds involve generalized logarithmic transforms along tori, it would be good to consider the following re-formulation of the problem above — with the obvious shift by 1 in mind.

Problem 12 *(Stern): Are there any pairs M and N as above for which at least n logarithmic transforms are needed to pass from M to N, with $n > 1$? Are there pairs for which n is arbitrarily large?*

Problem 13 *(Kirby) If a simply connected 4-manifold does not admit an almost-complex structure, is it a connected sum? Possible counterexamples are given in Problem 4.97 of [28].*

Problem 14 *(Budney) If a 3-manifold M embeds smoothly in S^4, it decomposes the 4-sphere into two 4-manifolds V_1 and V_2 having M as their common boundary. Can one ensure, after possibly re-embedding M in S^4 that $\pi_1 V_1$ and $\pi_1 V_2$ have solvable word problems?*

Remark: There are explicit examples of 3-manifolds in S^4 where either (or both) $\pi_1 V_1$ and $\pi_1 V_2$ have unsolvable word problems. See Dranishnikov and Repovs [14]. There are also examples due to Gompf for $M = S^3$ where V_1 has trivial fundamental group, yet the presentation from the standard height function on S^4 is not known to be Andrews-Curtis trivializable [23].

Problem 15 *(Teichner) Does every closed, smooth, oriented 4-manifold have a Heegaard splitting in the sense that it is a twisted double of a handlebody made with 0-, 1- and 2-handles?*

Problem 16 *(Baykur) Does every simply-connected smooth 4-manifold admit an involution with 2-dimensional fixed-point set?*

Problem 17 *(Budney) Is there an algorithm to recognize a triangulated S^4?*

Remark: In dimension 3 there is the Rubinstein-Thompson algorithm [46]. There is no algorithm to recognize all connected sums of $S^2 \times S^2$ [30]. In dimension $n \geq 5$ it is not an algorithmically-solvable problem (Nabutovsky).

Problem 18 *(Budney) Is there a reasonable theoretical criterion for a smooth 4-manifold to fibre over S^1?*

Remark: In dimension 3 there are two such theorems; Stallings' theorem states that all one needs is an epimorphism $\pi_1 M^3 \to \pi_1 S^1$ with finitely-presented kernel, and Schleimer's dissertation gives an algorithm to decide (assuming M has a triangulation). In high dimensions there is the Farrell fibering theorem, which uses the language of surgery theory.

Problem 19 *(Budney) If a 3-manifold embeds smoothly in a homotopy 4-sphere, does it embed smoothly in S^4?*

Remark: Replace 3 and 4 by n and $n + 1$ and the answer is affirmative for all $n \neq 3$. If the answer to this question appears to be negative, it could provide a strategy to recognize non-standard homotopy 4-spheres.

In the next four problems Morse 2-functions and trisections of 4-manifolds are mentioned. A Morse 2-function $f : X^n \to \Sigma^2$ is a generic smooth map to an orientable surface, often B^2 or S^2 [21]. When $\Sigma = B^2$, f can be homotoped so that if B^2 is a pie with three slices, then f^{-1} of each slice is diffeomorphic to a connected sum of k copies of $S^1 \times B^3$, whose boundary has a Heegaard splitting of genus g; thus $f^{-1}(0)$ is a surface of genus g [22]; When g is minimal, g is the trisection genus of X^4.

Problem 20 *(Zupan) Use Morse 2-functions [21] to find a notion of thin position for 4-manifolds.*

Problem 21 *(Gay) Compute a trisection genus of a 4-manifold that is 3 or larger.*

Remark: This represents the problem of trying to find computable lower-bounds on the trisection genus.

Problem 22 *(Scharlemann) If a homology 4-ball with boundary S^3 can be described as $2/3$ of a trisection diagram, is this enough to conclude that the 4-manifold is the standard PL B^4?*

Remarks: One can view $2/3$ of a trisection as a *Heegaard union*, as introduced in [43][Section 3]: Two 4-dimensional handlebodies J_1, J_2, of genus $\rho_1 \leq \rho_2$, are glued together along a 3-dimensional handlebody H so that in each of $\partial J_1, \partial J_2$ the complement of H is also a 3-dimensional handlebody. In [43][Prop. 3.3] it is shown that if a homology 4-ball W with $\partial W = S^3$ is a Heegaard union, then the weak generalized Property R conjecture (for a link of ρ_1 components) implies $W \cong B^4$. (The proof shows slightly more, namely that we can restrict attention to links with tunnel number $\leq \rho_2$. This last observation may just be a distraction, but it was useful in [42], because at the time Property R had only been proven for knots of tunnel number 1 [41].)

The problem itself is motivated by one approach to the Schoenflies Conjecture: given a standard Kearton-Lickorish embedding of S^3 in S^4, try to iteratively reimbed the complementary components X and Y in a level-preserving way so that ultimately one or the other (say X) has middle level a 3-dimensional handlebody. If this can be accomplished (as Fox's reimbedding theorem gently suggests might be possible), then X has the structure of a Heegaard union, so a solution to the problem above would show that X is D^4.

Problem 23 *(Johnson) What 3-manifolds occur as boundaries of $2/3$ of a trisection diagram?*

Remark: "2/3 of a trisection diagram" means gluing together two 4-dimensional handlebodies along a 3-dimensional handlebody which is half of a Heegaard splitting of each boundary (which is the connect-sum of $S^1 \times S^2$'s).

The answer to this question is "all". Here is a sketch of a proof which relies on a good understanding of [22] where the argument is only implicit. Observe that if a sector (1/3 of a trisection) is B^4 with the genus 1 Heegaard splitting of the bounding S^3, then the sector has one fold curve with a cusp, and the cusp is associated with a Dehn twist in the T^2 along the $1 - 1$ curved which takes meridian to longitude. In general a cusp corresponds to a Dehn twist along a curve in the Heegaard surface. (It also corresponds to adding a 2-handle to one 3-dimensional handlebody to obtain another 3-dimensional handlebody.) Now write our closed, orientable 3-manifold as a Heegaard splitting with a diffeomorphism d of the Heegaard surface, realize d as a product of Dehn twists, and realize the Dehn twists as cusps in fold curves.

Problem 24 *(Baykur) Find an exotic family of simply-connected 4-manifolds X_n with broken genus of X_n arbitrarily large, as n gets large. Determine the broken genus of the remaining standard simply-connected 4-manifolds, i.e. connected sums of copies of $S^2 \times S^2$ and $K3$.*

Remark: A simplified broken Lefschetz fibration for X^4 is a Lefschetz fibration over S^2 but with at most one fold circle which is embedded with all Lefschetz singularities on one side. The broken genus is the minimal genus of a fiber (on the higher side of the fold circle) over all possible broken Lefschetz fibrations [9][10].

A related notion is an indefinite Morse 2-function to $X^4 \to S^2$ with one indefinite embedded fold with cusps [50]; the cusps can be replaced by Lefschetz singularities which turns it into a simplified broken Lefschetz fibration.

Problem 25 *(Auckly) Find more accurate lower-bounds on the Martelli complexity [31][4] of smooth 4-manifolds.*

Problem 26 *(Eliashberg) Given an almost-symplectic M^4, $\omega \in H^2 M$ with $\omega^2 \neq 0$ with an almost-complex structure, does there exist a surface $\Sigma \subset M$ such that $\Sigma^* \in H^2 M$ is a multiple of ω, and a branched cover of (M, Σ) admits a symplectic structure such that Σ is a symplectic surface in the cover?*

Remark: This is the algebraically symplectic implies virtually symplectic problem.

Problem 27 *(Baker) Is there a useful generalization of A'Campo's links of divides to higher dimensions?*

Remark: Studying real Morsifications of complex plane curve singularities led A'Campo [1] to the notion of a divide and the link of a divide. A divide P is the image of a generic relative immersion of a disjoint union of r arcs into the unit disk in $\mathbb{R}^2$. Viewing S^3 as the unit sphere in $T\mathbb{R}^2$, the link of a divide P is the link in S^3 of r components $L(P) = \{(x, v) \in T\mathbb{R}^2 : x \in P, v \in T_x P, |x|^2 + |v|^2 = 1\}$. Links of connected divides with δ double points are oriented fibered links with fibers of genus $r - 1 + \delta$ whose monodromy is a product of $r - 1 + 2\delta$ positive Dehn twists. If $r = 1$ so that $L(P)$ is a knot, then both its gordian number and 4–ball genus is δ. There are several generalizations of links of divides that encompass a greater variety of knots and links in 3–manifolds.

Are there higher dimensional analogues? For example, with S^5 as the unit sphere in $T\mathbb{R}^3$, one may ask about the link of a generic relative immersion of a disk in the unit 3-ball. This however leads to an immersion of S^3 into S^5 rather than an embedding.

2.2 Topological Field Theories

An introduction by Paul Melvin, one of the facilitators for the TFT evening session:

- I worked on quantum invariants in the early days. Several at this conference have asked: *What is the point of Topological Quantum Field Theory* (TQFT)? We've seen at this conference how it relates to mathematics, computer science and physics. A good starting point for understanding the latter is Baez's paper on its role in framing a theory of quantum gravity [8].

- On page 136 of Turaev's book [47], he formulates the properties of an arbitrary quantum invariant $\tau(M)$ of closed n-manifolds needed for it to be promoted to a TQFT T : nCob → Hilb. The main condition is the 'splitting axiom'. Maybe this is a 'cheat' but I always found it interesting.

Problem 28 *(Auckly) Are all Chern-Simons theories for $(G, level\ k)$ determined by finitely many BPS states?*

Remark: See Auckly and Koshkin's monograph [3] for further details. It appears that there might exist a 3-manifold invariant taking the form $Z_M(a, q)$ such that:

- It gives the rank N level k Chern-Simons invariant of M for $a = q^N$, $q = e^{2\pi i/(k+N)}$.

- It has the structure $Z_M(a, q) = \sum Z_d(q)a^d$

- The coefficients of Taylor expansion of $Z_d(q)$ about $q = 0$ are integers.

- The function $Z_d(-e^{iu})$ has an asymptotic expansion at $u = 0$ along the positive reals.

- The functions $Z_d(q)$ satisfy some funky modularity properties as evidenced by Hikami and others.

- It is determined by a finite number of integer BPS invariants.

Problem 29 *(Not recorded) Is there a 4-dimensional TFT (necessarily not unitary by the work of Mike Freedman et al [17]) that distinguishes smooth structures?*

Remark: Known invariants that distinguish smooth structures, such as Seiberg-Witten invaraints, are not known to be TFT's. To date we have used PDE's to distinguish smooth structures. But we know in dim 4 that Diff = PL, so we know there should be something combinatorial. An application of the cobordism hypothesis would be a combinatorial model.

Problem 30 *(Gay) What are the obstructions to finding generators and relations for $Bord_{1,2,3,4}$?*

Remark: According to Douglas and Schommer-Pries, there is no fundamental obstruction, just hard work; they did a bunch of calculations on this, working out 3-Morse theory singularities $M^4 \to \mathbb{R}^3 \to \mathbb{R}^2 \to \mathbb{R}$. It seems to give a whole stack of relations and it may be too difficult to simplify these relations. Better would be to have tools to say, for example, "These singularities are not necessary," perhaps a version of Igusa's theorem. Teichner points out that if you just do 2-3-4 then it's much harder, you need to throw in the surface with all its diffeomorphisms; not just the mapping class group but π_0 and π_1. You can't cut the surface up anymore.

Problem 31 *(Not recorded) What are the possible applications of the presentation of $Bord_{123}$ to the study of 3-dimensional manifolds?*

Remark: Suggestions include: recognizing S^3; proving some known restrictions on π_1 of 3–manifolds (eg. show that no 3-manifold has fundamental group $\mathbb{Z}_2 \times \mathbb{Z}_2$); showing that an orientable 3-manifold is parallelizable; deriving complexity measures on 3-manifolds; deciding if a 3-manifold has an essential torus or not. One difficulty is that Funar has recently proved [19] that Reshetikhin-Turaev TQFT's cannot distinguish certain torus bundles over the circle, and [13] shows that all TQFT's are of RT type. However, perhaps in a different target 2-category they may be distinguishable. There is also the information flow from topology to algebra. For instance, [13] shows how the Radford theorem on the square of the antipode in a Hopf algebra follows from $\pi_1 SO(3) = \mathbb{Z}_2$. This uses a trivial fact from 3-manifold topology to prove something interesting in algebra. Maybe more nontrivial 3-manifold facts will lead to even more interesting results in algebra.

Problem 32 *(Douglas) Do all 3D quantum invariants come from quantum groups?*

Problem 33 *(Kirby) What is the nicest reference for understanding the cobordism hypothesis of Baez and Dolan?*

Remark: The survey by Dan Freed [16].

2.3   Symplectic geometry

Problem 34 *(Sabloff) Characterize the smooth knot types that have a Legendrian representative with a Lagrangian filling.*

Remark: A sufficient condition is that the knot type is positive. A necessary condition is that it is quasi-positive. One conjectured necessary and sufficient condition is quasi-positive and sharp HOMFLY.

Problem 35 *(Sabloff) Classify the geography for non-loose Legendrian n-spheres in standard contact R^{2n+1}. That is, what range of tb and r can be realized by such?*

Remark: Murphy showed that there is no restriction for loose Legendrians [32]. If the non-loose Legendrian has a Lagrangian filling, then probably $r = 0$.

Problem 36 *(Sullivan) If a k-dimensional sphere of loose Legendrian submanifolds in standard contact R^{2n+1} has no formal obstruction to being contractible, can it be contracted to a single Legendrian?*

Remark: Murphy proves that obstructions to isotoping loose Legendrians are purely formal [32]. She does this using an h-principal and removing certain wrinkle and fold singularities. The k-parametric h-principal exists [15]. The removal of singularities, while not immediate, should be doable.

Problem 37 *(Eliashberg) Consider a Weinstein manifold constructed by surgery along a loose knot. Does it have a presentation using surgery along only non-loose knots? Similarly, can attaching along a single non-loose knot yield a flexible Weinstein manifold?*

Problem 38 *(Eliashberg) Characterize flexible Weinstein manifolds in terms of Lefschetz fibrations for dimension ≥ 6.*

Problem 39 *(Not recorded) Characterize all Legendrian submanifolds that have generating families.*

Remark: They must be non-loose, but that is not a sufficient condition.

Problem 40 *(Traynor) Compare generating homology (Morse theory) with linearized Legendrian contact homology (pseudo-holomorphic curves) for a Legendrian submanifold in one-jet spaces.*

Remark: For Legendrian knots in R^3, every generating homology is some linearization of contact homology. Lisa Traynor, Josh Sabloff and Paul Melvin are trying to work on the other direction. For Legendrian surfaces, Dan Rutherford and Mike Sullivan have work in progress that shows that a Legendrian has a generating homology if and only if it has a linearized contact homology, although the homologies a priori might be different. Frederic Bourgeois is developing a larger hybrid theory in any dimension which should "degenerate" in two ways to produce the two homology theories.

Problem 41 *(Sullivan) There are two maps from the homotopy groups of the based space of Legendrian submanifolds $\mathcal{L}$ (in one-jet spaces) to a certain group of endomorphisms: Sabloff and Sullivan construct one using generating homology $\pi_k(\mathcal{L}; L) \to End_{k-1}(GH(L))$ [38]; Bourgeois and Bronnle construct using linearized Legendrian contact homology $\pi_k(\mathcal{L}; L) \to End_{k-1}(LCH(L))$. Are these the same? As a follow-up, do any of these preserve structures such as the Whitehead product $\pi_k \times \pi_l \to \pi_{k+l-1}$?*

Remark: A comparison of the two maps may result from Bourgeois' work in the previous problem. As for the follow-up, it seems likely when l (or k) is 1, and unlikely otherwise, from dimensionality reasons. (Somehow, these maps see more at the homological level, where such a product, if defined, is uninteresting if $l, k > 1$.)

Problem 42 *(Sullivan) Do Legendrian fronts constitute a bordism category? Can Lurie's cobordism hypothesis apply and simplify computations of generating homology, for example?*

Remark: Here objects would be Legendrian points, 1-morphisms would be Legendrain tangles, etc. It seems promising, given that Schommer-Pries' Cerf-theory approach [44] to the cobordism hypothesis for 2-categories resembles Legendrian front projections.

Problem 43 *(Gay) When constructing generating families, why use R^N as the fiber? Are there benefits to changing the fiber?*

Remark: This would work for Legendrians in $J^1 M$ for $M \neq R^n$. For example, if $M = S^1$ and the fiber is S_1, a birth followed by a death can produce a a non-trivial Legendrian knot. This works for $M = S^2$, etc, as well. A follow-up (vague) question was whether or not this can put into a more general theory.

Problem 44 *(Hutchings) Can we apply symplectic geometry to solve the Schonflies conjecture? Can we deform any S^3 to be pseudo-convex?*

Problem 45 *(Bourgeois)How do overtwisted disks presented by Eliashberg for high-dimensional contact manifolds compare to others in the literature? More specifically, take the contact manifold (M, ξ) with an open book given by a Dehn twist on $ST^* S^n$. Is this overtwisted in Eliashberg's sense?*

Remark: One difficulty in answering this question is that Eliashberg's theorem is not constructive, so the disk is hard to "see."

Problem 46 *(Wehrheim) What do Lagrangians look like in moment polytopes?*

Remark: Apparently this relates to work by Denis Auroux and his student.

Problem 47 *(Eliashberg) Are holomorphic curves the only way to detect symplectic/contact rigidity results? More concretely, consider the Arnold Conjecture in $T^* M$. The number of intersections of a Hamiltonian deformation of the zero-section O with O is bounded from below by the stable morse number of O. This follows from generating families. Can the bound be improved? Can holomorphic curves prove this bound?*

Remark: In some cases, the stable Morse number bound might follow from the bifurcation analysis proofs of Floer invariance in [45] or [29].

Problem 48 *(Eliashberg) Prove flexibility results for symplectic 6-manifolds and 4-manifolds of general type.*

Problem 49 *(Wehrheim) Prove an h-cobordism like theorem for Floer homology.*

Remark: A key step would be to replicate Milnor's cancelling disk trick.

3 Abstracts for talks

This section lists the titles and abstracts of the talks, in alphabetical order of the speakers' last names.

Speaker: **Bruce Bartlett**
Title: *Three-dimensional bordism representations via generators and relations*
Abstract: The three-dimensional oriented bordism bicategory has closed 1-manifolds as objects, 2-dimensional cobordisms as 1-morphisms, and diffeomorphism classes of 3-dimensional cobordisms as 2-morphisms. We use higher Morse theory to find a simple generators-and-relations presentation of it. Dropping a certain relation leads to a "signature" central extension of the oriented bordism bicategory. The presentation allows for an elementary proof that a representation of this bicategory (i.e. a "123 TQFT") corresponds in a 2-1 fashion to a modular category, which must be anomaly-free in the oriented case. J/w Chris Douglas, Chris Schommer-Pries, Jamie Vicary.

Speaker: **Stefan Behrens**
Title: *Singular Fibrations on 4-Manifolds*
Abstract: The last 15 years of 4-manifold theory have seen a revival of the study of smooth maps to surfaces. While this subject had already enjoyed popularity in the third quarter of the 20th century, the current developments were motivated by work of Donaldson and Gompf on symplectic 4-manifolds and Lefschetz pencils as well as Taubes's work on the Seiberg-Witten invariants of near-symplectic 4-manifolds. In this talk I will begin with a brief historical overview and then go on to describe the basic structure of generic maps from

4-manifolds to surfaces and 1-parameter families thereof. I will point out relations to 3- and 4-dimensional Morse theory and the theory of (broken) Lefschetz fibrations. Finally, I will describe how this "surface valued Morse theory" leads to pictorial descriptions of 4-manifolds in terms of curve configurations on surfaces. If time permits, I will discuss some (potential) applications and open problems.

Speaker: **Ryan Budney**
Title: *Triangulating 4-manifolds and a table of knots in homotopy 4-spheres.*
Abstract: I will update the group on an ongoing project which creates a census of triangulated smooth 4-manifolds. The ultimate goal of the project is to see how computationally useful triangulations of 4-manifolds can be, as compared to with 3-manifold theory. The table of knot exteriors in homotopy 4-spheres is nearing completion, this has included the discovery of a new 2-knot type.

Speaker: **Yasha Eliashberg**
Title: *All manifolds are contact except those which are obviously not.*

Speaker: **M. Brad Henry**
Title: *A combinatorial differential graded algebra for Legendrian knots from generating families*
Abstract: We outline recent work that assigns a differential graded algebra (DGA) to a Legendrian knot in the standard contact structure on $\mathbb{R}^3$. The definition of the DGA is motivated by considering Morse-theoretic data from a generating family. A generating family f_x for a Legendrian knot is a 1-parameter family of functions whose Cerf diagram is a projection of the knot. Generating family homology is a useful invariant of Legendrian knots extracted from the generating family. The new DGA is defined combinatorially using the Cerf diagram and handleslide data from f_x. Although defined combinatorially, the differential of the DGA is geometrically motivated by a conjectured extension of generating family homology using gradient flow trees. We will discuss this motivation and how it informs the combinatorial definition of the DGA, and relate the new DGA to the Chekanov-Eliashberg DGA. This work is joint with Dan Rutherford (University of Arkansas).

Speaker: **Jesse Johnson**
Title: *Minsky Models and Morse two-functions on three-manifolds*
Abstract: Morse two-functions have recently become popular in three-dimensional topology via the closely related of a (Rubinstein-Scharlemann) graphic. In this talk, I will describe how a Morse two-function on a three-dimensional manifold encodes topological and geometric information about the three-manifold via a combinatorial structure called a Minsky model.

Speaker: **Daniel Rutherford**
Title: *Cellular computation of Legendrian contact homology and generating families*
Abstract: This is joint work with Mike Sullivan. We consider a Legendrian surface, L, in $\mathbb{R}^5$ (or more generally in the 1-jet space of a surface). Such a Legendrian, L, can be conveniently presented via its front projection which is a surface in $\mathbb{R}^3$ that is immersed except for certain standard singularities.

We associate a differential graded algebra (DGA) to L by starting with a cellular decomposition of the base projection (to $\mathbb{R}^2$) of L that contains the projection of the singular set of L in its 1-skeleton. A collection of generators is associated to each cell, and the differential is determined in a formulaic manner by the nature of the singular set above the boundary of a cell. Our motivation is to give a cellular computation of the Legendrian contact homology DGA of L. In this setting, the construction of Legendrian contact homology was carried out by Etnyre-Ekholm-Sullvan with the differential defined by counting holomorphic disks in $\mathbb{C}^2$ with boundary on the Lagrangian projection of L. In work in progress, we hope to establish equivalence of our DGA with LCH using work of Ekholm on gradient flow trees.

As an application we discuss connections between the cellular DGA and generating families. Here, augmentations arise from Morse complexes and bifurcation data appearing in 2-parameter families of functions.

Speaker: **Hyam Rubinstein**
Title: *Parametrised Morse theory for 3-manifolds*

Abstract: The first part of the talk will be a quick survey on the homotopy type of the group of Diffeomorphisms of a 3-manifold. Pioneering work was done by Hatcher and Ivanov in the late 70s and early 80s, computing this for Haken 3-manifolds and famously the Smale conjecture. Following Thurston, the natural question is whether Diff is homotopy equivalent to the group of isometries, for geometric 3-manifolds. Gabai showed this is true in the hyperbolic case. McCullough and others studied Seifert fibred spaces and many spherical classes of examples.

In the second part, I will talk about Heegaard splittings, a natural view of Morse theory for 3-manifolds. Casson-Gordon in the mid 1980s introduced a key idea of strong irreducibility and this was extended by Scharlemann-Thompson to telescoping. There have subsequently been many developments, including classification of splittings for all 7 non hyperbolic geometries of Thurston. Comparing splittings was introduced by Scharlemann and I, and distance of splittings by Hempel. If time permits the relationship with hyperbolic geometry will be sketched.

Speaker: **Josh Sabloff**
Title: *Families of Legendrian Submanifolds via Generating Families*
Abstract: I will introduce a framework to investigate families of Legendrian submanifolds using generating family homology through an application of the families theory to the analysis of a loop of Legendrian n-spheres in the standard contact space that is contractible in the smooth, but not Legendrian, category; this is joint work with Mike Sullivan. The computation of generating family homology necessary for the application comes from joint work with on Lagrangian cobordisms withFrederic Bourgeois and Lisa Traynor.

Speaker: **Martin Scharlemann**
Title: *The Schönflies Conjecture and its spin-offs*
Abstract: We briefly review the resolution of the Schnflies Conjecture in all dimensions other than four, discuss why the remaining conjecture is important, and the classic approach to its resolution. This approach has spawned much beautifully pictorial mathematics, without actually succeeding. An underlying theme is that, although the conjecture has not yet been settled, it interlocks with and has inspired much interesting topology in dimensions three and four.

Speaker: **Chris Schommer-Pries**
Title: *From the cobordism hypothesis to higher Morse theory*
Abstract: This talk will survey some recent developments in our understanding of extended topological field theories and their classification. This includes the cobordism hypothesis and related results. In the course of this talk we hope to make clear the role of higher Morse theory in this story.

Speaker: **Lisa Traynor**
Title: *An Introduction to Symplectic and Contact Topology and the Technique of Generating Families*
Abstract: I will give a brief introduction to some of the major objects in symplectic and contact topology: symplect and contact manifolds, Lagrangian and Legendrian submanifolds, and symplectic and contact diffeomorphisms. Then I will describe the technique of generating families: this is a way to encode a Lagrangian or Legendrian submanifold by a parameterized family of functions. Morse-theoretic constructions then lead to generating family (co)homology groups for a Legendrian submanifold and wrapped generating family (co)homology groups for a Lagrangian cobordism. I will also describe how from a Lagrangian cobordism with a generating family, one obtains a cobordism map that satisfies some of the typical properties of a TQFT.

Speaker: **Jamie Vicary**
Title: *Computations with topological defects*
Abstract: I will show how some fundamental computational processes, including encrypted communication and quantum teleportation, can be defined in terms of the higher representation theory of defects between 2d topological cobordisms, giving insight into fundamental questions in classical and quantum computation. No knowledge of computer science will be required to understand this talk.

Speaker: **Katrin Wehrheim**
Title: *How to extend 2+1 (symplectic but not quite) field theories to 2+1+1*

Abstract: In previous work with Chris Woodward, we gave constructions of 2+1 NQFT's (not quite field theories) via dimensionally reduced gauge theories and the symplectic 2-category. More precisely, these are functors from the category of connected 2+1 bordisms to Symp, composed with a natural functor from Symp to Cat. Using Morse 2-functions on 4-manifolds, I will explain that/how such theories naturally extend to 2+1+1 NQFT's under a single nontrivial axiom. And I hope to find help for translating this into the more algebraic TFT language during the workshop.

Speaker: **Jonathan Williams**
Title: *Weak Floer A-infinity algebras for smooth 4-manifolds*
Abstract: I will talk about how to apply constructions of Lipshitz and Akaho-Joyce to a certain class of maps from 4-manifolds to the 2-sphere to yield possibly new diffeomorphism invariants for general smooth, closed oriented 4-manifolds, and discuss future directions.

Speaker: **Alexander Zupan**
Title: *Knots with compressible thin levels*
Abstract:Thin positions theories have played a prominent role in 3-manifold topology over the last several decades, beginning with Gabai's definition of thin position for knots in the 3-sphere and proceeding up to Johnson's axiomatic thin position, which encompasses most existing adaptations. Modern notions of thin position are highly technical but exhibit the natural property that for a thin presentation of a knot or a 3-manifold, all thin surfaces are essential. This motivates the question, "For a knot in Gabai thin position, are all thin levels essential in the knot exterior?" We give a negative answer to this question, exhibiting an infinite family of knots whose thin positions have compressible thin levels. This is joint work Ryan Blair.

4 Feedback from some participants

This section records some of the post-conference feedback emailed to the conference organizers.

- Christopher Douglas: I just wanted to thank you for organizing the Banff workshop — I had a fabulous week. Really, it's been years since I enjoyed a math workshop or conference as much as I did this past week. It was engaging, informative, productive, inspiring, and fun to boot!

- Chris Schommer-Pries: Thanks for the great workshop. I definitely got several things from the conference. One of them was a better understanding of what the other approaches to 2-Morse theory do and do not do. For example I got a better feeling for the difference that occur when you consider being transverse to a foliation, such as in the tqft work.

  I think an important question that was raised is what is the analog of a framed generalised 2-Morse function? One requirement is that the space of these should be contractible in a suitable sense. This could have lots of important applications to tqfts and is also perhaps natural to consider in geometric contexts as well.

  One thing that was raised during the problem session was the question of whether there are results about 3-manifolds that can be proven or reproven using the presentations from tqfts. A similar reverse question would be to look at standard 3-manifold facts and then ask what do these mean when applied to a particular target n-category. For example I mentioned that one of the simplest 3-dimensional facts, that there is an immersed surface in R^3 connection the twice twisted circle to the untwisted circle (i.e. $\pi_1 SO(3) = Z/2$) leads, via extened tqfts, to interesting an important facts about tensor categories (representations of finite quantum groups/Hopf algebras). What happens when we take other facts about 3-manifolds? what do these topological results tell us about these algebraic categories? I hadn't yet considered taking important results in 3-manifolds and transporting them to this other situation.

- Jamie Vicary: Let me say thanks again for an excellent workshop. I had a really great and productive time. Particular highlights: - Discovering the open questions lying at the boundary between my own work and other fields, e.g. symplectic/contact topology - Discovering that topologists are interested in what I'm doing!

- Katrin Wehrheim: Thanks for a wonderful conference! Banff is always productive for me, but this meeting has been exceptional. I learned a lot of TFT and was able to pretty much fully translate my results into an abstract result that mixes geometric "blob-complex-like" input with a cobordism hypothesis type result. And I was thrilled to find that the TFT specialists found this interesting, surprising, and yet believable - after several long discussions which allowed us to learn each others' language.

  I also have a very concrete new project and collaborator out of this week - we conjecturally constructed a new combinatorial 4-manifold invariant. I'd be prepared to say more if in ca 10 weeks I'll have heard from another specialist that it is believable and doesn't step on other people's toes.

- Hyam Rubinstein: I got a lot out of the conference as did my student David In particular I have two new ideas in mind One is to relate trisections of 4-manifolds to triangulations Dave Gay asked me about this and I also talked to Rob I am working on a project with Stephan Tillman on triangulations and Heegaard splittings of 3-manifolds and believe our methods apply in dimension 4 and most likely extend to higher dimensions as well Secondly after Ryan Budneys talk and also some informal conversations have some ideas for algorithms for 4-manifolds I have a project on normal 3-manifolds in triangulated 4-manifolds and have some new ideas for more interesting applications

- Josh Sabloff: First, thank you for organizing such an interesting and productive conference. It was great to interact with people I would not normally see, as well as those I donormally see.

  I got three main things from the conference:

  1. I got to talk with some of the TQFT folks about some algebraic structures that come up in Legendrian Contact Homology, and they had some helpful ideas about where else similar structures have appeared. I hope to use this in an ongoing project – in a sense, it gives me a target to aim for.

  2. Inspiration, once again from the TQFT people, about the structure into which generating family homology and the families framework fit.

  3. Frederic, Lisa, and I also made progress on an old project, and it was good to be in the same place at the same time to have some intensive discussions. This also goes for an old project with Joan.

- Paul Melvin: It was a great conference, superbly run! Sorry I had to leave early (and in particular, to miss the contact/symplectic day).

  For me, the most interesting outcome was a better appreciation of the unifying role of topological quantum field theories in low dimensional topology, as well as its application to other areas of mathematics. I can see many possibilities for increased impact of this perspective on my future research. I also benefited from discussions with Dave Auckly related to our joint work on stable isotopy in 4-manifolds; in particular we made progress on generalizing some of the cork-constructions that we have developed to a wider family of corks arising from symmetric ribbon knots.

  Thanks again to you and Dave for organizing a terrific conference.

- M. Brad Henry:

  1. Dan Rutherford and I began a new project to understand Legendrian invariants in the 1-jet space of the circle. We made very nice progress and left Banff with a clear intuition as to how to proceed.

  2. Dan Rutherford, Lisa Traynor, Paul Melvin, Josh Sabloff, and I discussed an on-going project related to constructing and distinguishing generating families.

  3. Stefan Behrens provided Paul Melvin with three references that may be of important use to the project from 2. above.

  4. This was my first exposure to symmetric monoidal 2-categories. Legendrian submanifolds may fit into this algebraic framework in a natural way.

  5. Lisa Traynor, Ziva Myer and I discussed Ziva's graduate research in Legendrian graph theory.

  6. Perhaps most importantly, I have a greater sense for the broad tapestry that all of the topics discussed are woven into.

- Marty Scharlemann: David Gay, Rob Kirby, Abby Thompson and I began discussing a prospective FRG proposal at the Banff workshop, and the setting was a definite plus for this activity. I had a good number of very helpful conversations, most memorably with Yasha Eliashberg about hopes and attempts at proving the Schonfliess Conjecture coming out of geometry. All in all a very nice conference - thanks for the invitation.

- Robion Kirby: This conference worked better than any conference I can remember. Well perhaps the one on 4-manifolds in Durham in 1982 just after both Freedman and Donaldson's great works was more exciting. But that is a very high standard. Banff had a lot of back and forth between the speakers and the audience which is always a good sign, so much better than a quiet zombie like audience that asks no questions. The organizers were either excellent choosers of participants, or plenty lucky!

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