Existence and singularity in the α-patch model

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2D (incompressible) Euler equation

- 2D Euler equation (in vorticity form):

$$\omega_t + u \cdot \nabla\omega = 0,$$

$$u(x) = \nabla^\perp(-\Delta)^{-1}\omega = C\int_{\mathbb{R}^2} \frac{(x-y)^\perp}{|x-y|^2}\omega(y)dy.$$

- Global well-posedness for weak solution with $\omega_0 \in L^1 \cap L^\infty$: Yudovich ('63).
- For smooth ω_0, a classical solution exists for all time, with $\|\nabla\omega(\cdot,t)\|_\infty$ at most growing double-exponentially.

Patch solution for 2D Euler

- If $\omega_0 = 1_{\Omega_0}$, the solution will evolve as a "patch" in a sense that $\omega(t) = 1_{\Omega(t)}$ for some $\Omega(t)$.
- Global regularity of patches is first proved by Chemin ('93), then a shorter proof is given by Bertozzi-Constantin ('93).
- If Ω_0 has smooth boundary, then $\Omega(t)$ will stay smooth for all times.
- Two patches can at most approach each other double-exponentially.

SQG (Surface Quasi-Geostrophic) Equation and α-equation

- Inviscid SQG equation in $\mathbb{R}^2$: $\omega_t + u \cdot \nabla \omega = 0$,

$$u = \nabla^{\perp}(-\Delta)^{-1/2}\omega = C \int_{\mathbb{R}^2} \frac{(x-y)^{\perp}}{|x-y|^3} \omega(y) dy.$$

- Introduced by Constantin-Majda-Tabak ('94), it resembles the 3D Euler equations in many aspects.
- Resnick ('95) proved global existence of weak solutions, as well as local-in-time existence of classical solutions.
- It is unknown whether there is a global classical solution for any smooth initial data.
- α-equation (with $0 < \alpha < 1/2$):

$$u = \nabla^{\perp}(-\Delta)^{-1+\alpha}\omega = C_\alpha \int_{\mathbb{R}^2} \frac{(x-y)^{\perp}}{|x-y|^{1+2\alpha}} \omega(y) dy.$$

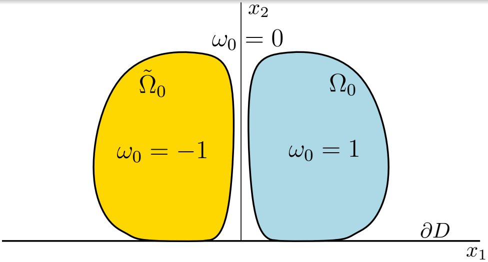
Global existence of classical solutions also unknown!

SQG patch and α-patch in $\mathbb{R}^2$

- For SQG patch and α-patch, local existence in Sobolev space (H^3 or higher) is proved by Gancedo ('07).

  Here we say Ω has H^3 boundary if $\partial\Omega = \{(x_1(\gamma), x_2(\gamma)) : \gamma \in \mathbb{T}\}$, where $x_1(\cdot), x_2(\cdot) \in H^3(\mathbb{T})$.
- It is unknown whether such patch solution exists globally in time.
- Numerical evidence by Córdoba-Fontelos-Mancho-Rodrigo ('05) suggests that for both SQG and α-equation, the patch may develop a finite-time singularity – two patches may touch each other in finite time and form sharp corners.

Problem setting in the upper half plane



- Consider the α-patch problem in the upper half plane D, with boundary condition $u \cdot n = 0$.
- The initial patch is $\omega_0 = 1_{\Omega_0} - 1_{\tilde{\Omega}_0}$, where Ω_0 and $\tilde{\Omega}_0$ is symmetric through the x_2-axis. (so $\omega(\cdot, t)$ is odd through x_2-axis for all time.)
- Such problem setting is motivated by Kiselev-Šverák ('14), where they prove that smooth solution to 2D Euler equation in a bounded domain (with $u \cdot n = 0$) can have double-exponential gradient growth.

Finite time singularity for α-patch problem in half plane

Theorem (Kiselev-Ryzhik-Yao-Zlatos)

Let $0 < \alpha < 1/400$.

- *(Existence) For any patch initial data where patches are initially separated, not touching themselves and have H^3 boundary, the α-patch problem in the half plane D has a unique H^3 solution for a short time.*
- *(Finite-time singularity) We can find some Ω_0 with H^3 boundary, such that the H^3 patch solution with initial data $\omega_0 1_{\Omega_0} - 1_{\tilde{\Omega}_0}$ cannot be extended for all time.*

That is, (at least) one of the following three things happen for such patch solution:

1. The boundary stopped being H^3 at some finite time;
2. Some patch touches itself in finite time;
3. Two patches touch each other in finite time.

Why half plane?

- For the half-plane D with boundary condition $u \cdot n = 0$, the relation between u and ω becomes

$$u(x) = \int_D \Big(\frac{(x-y)^\perp}{|x-y|^{2+2\alpha}} - \frac{(x-\bar{y})^\perp}{|x-\bar{y}|^{2+2\alpha}} \Big) \omega(y) dy \quad \text{for } x \in D,$$

where $\bar{y}$ is the reflection of y through x_1 axis.

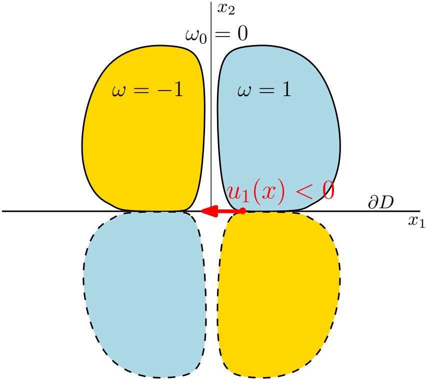
- This is identical to

$$u(x) = \int_{\mathbb{R}^2} \frac{(x-y)^\perp}{|x-y|^{2+2\alpha}} \omega(y) dy \quad \text{for } x \in D,$$

where in D^c, ω is defined by an odd extension through the x_1 axis.

Why half plane?

- Recall that a part of $\partial\Omega_0$ lies on ∂D. Due to $u \cdot n = 0$, they can only slide along ∂D.



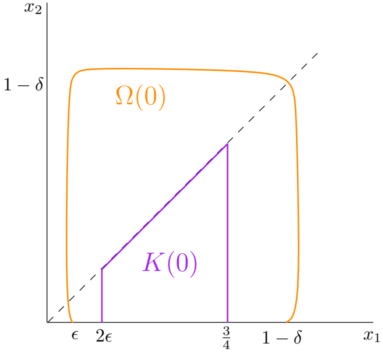
- Indeed, the velocity field of this part is always pointing towards the origin. (But this is not good enough to show it hits the origin in finite time!)

Comparing contour with an artificial barrier

Let $\Omega(t) = \{\omega(\cdot, t) = 1\}$, and $K(t)$ is an artificial barrier as below:

$$K(t) = \{(x_1, x_2) \in D : x_1 \geq x_2, X_1(t) \leq x_1 \leq \frac{3}{4}\}$$

where $\frac{d}{dt}X_1(t) = -\frac{1}{200}X_1(t)^{1-2\alpha}$, with $X_1(0) = 2\epsilon$.



- Note that $X_1(t)$ touches 0 at $T = 100(2\epsilon)^{2\alpha}$.

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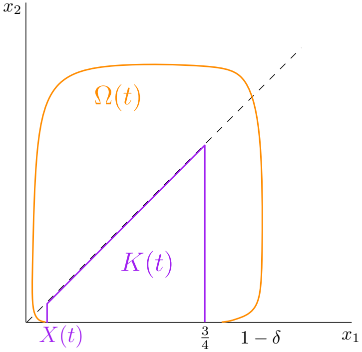

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- Our goal: $\Omega(t) \supset K(t)$ for all $0 < t < T$.

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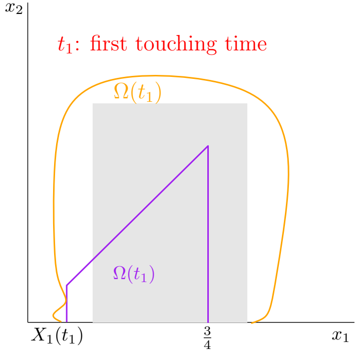
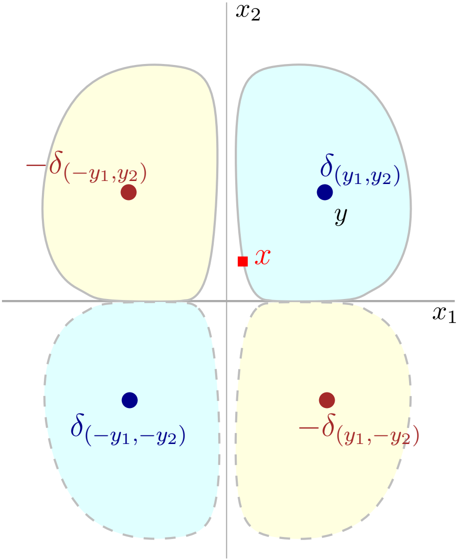


- Note that $X_1(t)$ touches 0 at $T = 100(2\epsilon)^{2\alpha}$.
- Our goal: $\Omega(t) \supset K(t)$ for all $0 < t < T$.
- At the first touching time $t_1 < T$, $\partial\Omega_1$ cannot touching the gray region.

Estimating $u_1(x)$



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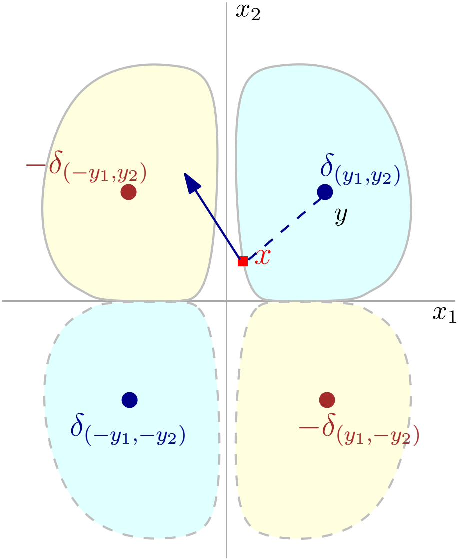
Estimating $u_1(x)$



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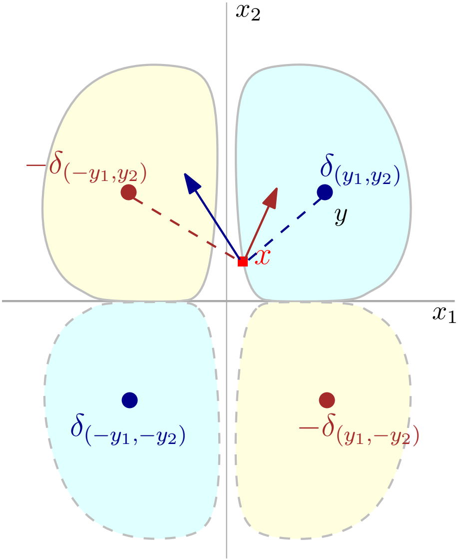

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$$u_{11} + u_{12} < 0,$$

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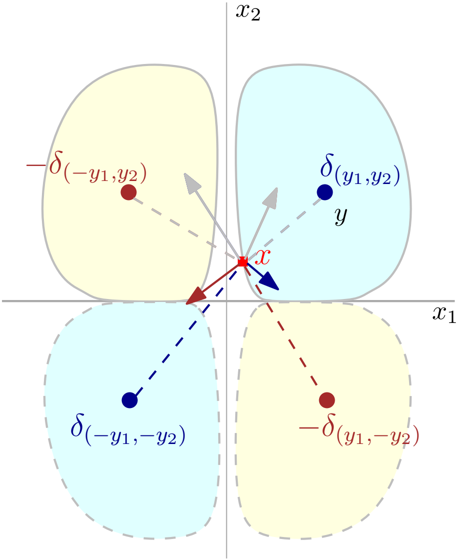

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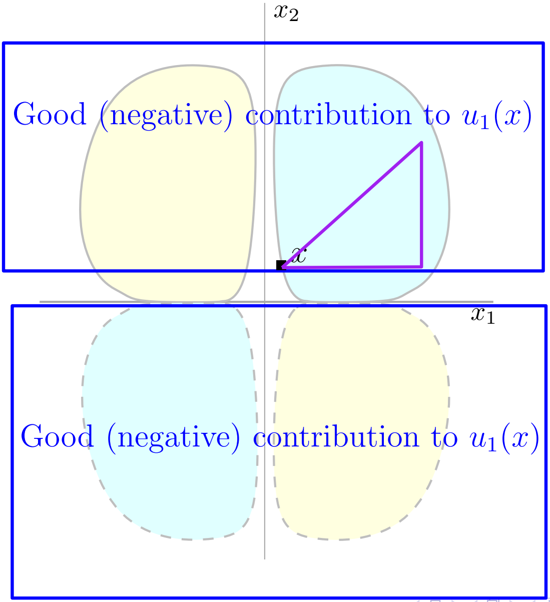
Why small α?



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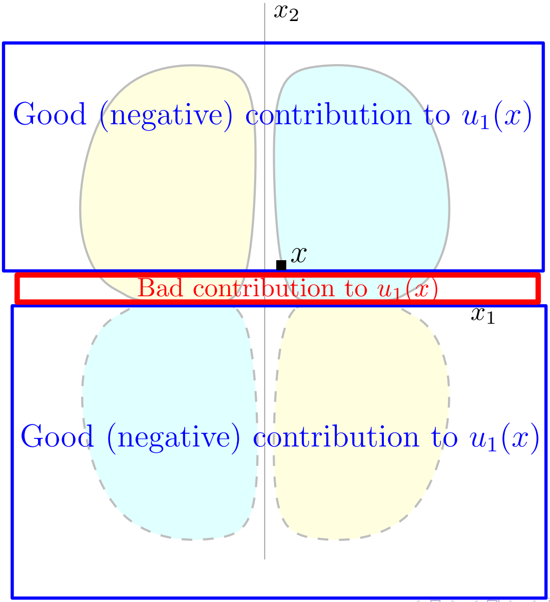

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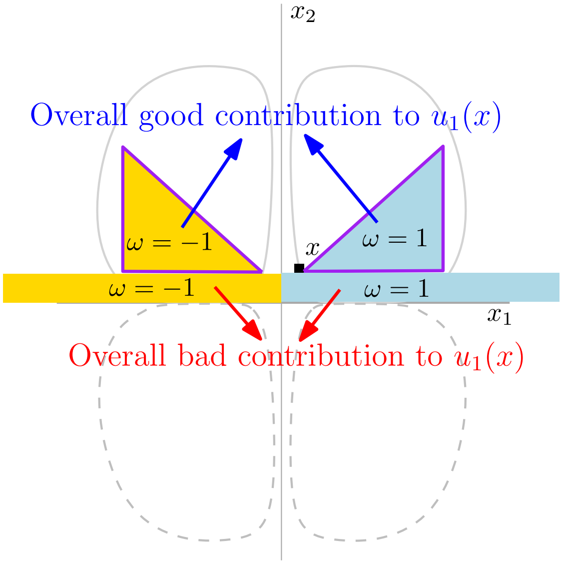
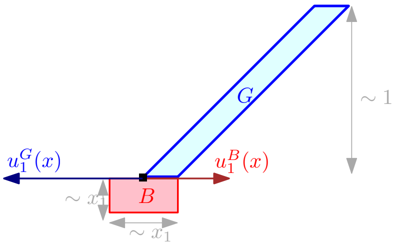


Why small α?

- To control $u_1(x)$, we eventually reduce to the competition to two terms: $u_1^G(x)$ v.s. $u_1^B(x)$.

$$u_1^G(x) \sim \int_G |x|^{-(1+2\alpha)} dx \sim C_G(\alpha)\, x_1^{1-2\alpha}$$

$$u_1^B(x) \sim \int_B |x|^{-(1+2\alpha)} dx \sim C_B(\alpha)\, x_1^{1-2\alpha}.$$



- As $\alpha \to 0$, $u_1^G(x) \gg u_1^B(x)$. As $\alpha \to \frac{1}{2}$, $u_1^G(x) \ll u_1^B(x)$.
- So our proof only works when α is small.

Why patches?

- Indeed, the above proof works for smooth initial data in D too!
- However, when the domain is the half plane D, there is no local existence for Sobolev space or $C^{1,\gamma}$ space!
- Reason: due to the presence of boundary, even ω is smooth in D, $|\nabla u(x_1, x_2)|$ can be of order $x_2^{-2\alpha}$ near ∂D.
- But for patch solution (for small α), we do have local existence in Sobolev space for a short time, so this finally gives local existence + finite-time singularity!

2D Euler patches in half plane

- To compare with the finite-time singularity of α-patch in D, we need to know whether Euler patch in the half plane is also globally regular.
- Problem: our patch is initially touching ∂D. If not, can use results for $\mathbb{R}^2$.
- Depauw ('98) gives local existence of $C^{1,\gamma}$ patch solution if patches are initially touching ∂D.
- If patches are initially $C^{1,\gamma}$ (that might be touching D), Dutrifoy ('03) shows the solution remain globally $C^{1,s}$ regular with $s < \gamma$.
- We prove that for 2D Euler equation in half plane D, $C^{1,\gamma}$ patches in D are globally $C^{1,\gamma}$ regular.

Thank you for your attention!