Entropy method in some weighted fast diffusion equations and related functional inequalities

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Plan of the talk

Work in collaboration with J. Dolbeault and M. Muratori

- Entropy-entropy production method in a weighted fast diffusion equation.
- Related Caffarelli-Kohn-Nirenberg inequalities.
- Symmetry of the minimizers by a perturbation argument.

Weighted fast diffusion equations

- We consider the weighted fast diffusion equation :

$$\partial_t u = |x|^\gamma \nabla \left(u \nabla u^{m-1} \right) , \quad u(t=0,\cdot) = u_0$$

with $m < 1$ and $\gamma \in (0,2)$.

- Self-similar variables :

$$u(t,x) = R^{\gamma-d}\, v\left((2-\gamma)^{-1} \, \log R, \frac{x}{R} \right)$$

with $R = R(t)$ defined by

$$\frac{dR}{dt} = (2-\gamma)\, R^{(m-1)(\gamma-d)-1} , \quad R(0) = 1 \; .$$

The equation for v is of Fokker-Planck type and takes the form

$$\partial_t v = |x|^\gamma \, \nabla \cdot \left[v \, \nabla \left(v^{m-1} - |x|^{2-\gamma} \right) \right], \quad v(t=0,\cdot) = u_0.$$

Stationnary solutions and relative entropy

- Stationnary solutions :

$$\overline{v}(x) = \left(C + |x|^{2-\gamma}\right)^{\frac{1}{m-1}}$$

where $C = C_M$ is uniquely determined by the condition

$$\int_{\mathbb{R}^d} \overline{v}\ \frac{dx}{|x|^\gamma} = M := \int_{\mathbb{R}^d} u_0\ \frac{dx}{|x|^\gamma}\ .$$

- Relative entropy

$$\mathcal{F}[v] := \frac{1}{m-1} \int_{\mathbb{R}^d} \left(v^m - |x|^{2-\gamma}\, v\right)\ \frac{dx}{|x|^\gamma} + F_0(M)\ ,$$

$F_0(M)$ is chosen such that $\mathcal{F}[\overline{v}] = 0$.

Caffarelli-Kohn-Nirenberg inequalities

- Fisher information

$$\frac{d}{dt}\mathcal{F}[v(t,\cdot)] = -\frac{m}{1-m}\,\mathcal{I}[v(t,\cdot)],$$

where

$$\mathcal{I}[v] := \int_{\mathbb{R}^d} v\left|\nabla v^{m-1} - (2-\gamma)\,\frac{x}{|x|^\gamma}\right|^2 dx\ .$$

- According to $\gamma = 0$, a natural conjecture :

$$(1-m)\,(2-\gamma)^2\,\mathcal{F}[v] \le \mathcal{I}[v]\ , \tag{1}$$

so that the inequality is equivalent to

$$\left(\int_{\mathbb{R}^d}\frac{|w|^{2p}}{|x|^\gamma}\,dx\right)^{\frac{1}{2p}} \le \mathsf{C}^\gamma_{d,p}\left(\int_{\mathbb{R}^d}|\nabla w|^2\,dx\right)^{\frac{\theta}{2}}\left(\int_{\mathbb{R}^d}\frac{|w|^{p+1}}{|x|^\gamma}\,dx\right)^{\frac{1-\theta}{p+1}} \tag{2}$$

with $\overline{w} = \overline{v}^{m-\frac{1}{2}}$ realizing the equality in (2)

Range of parameters

- Link between the (1) and (2) :

$$\begin{cases} p = \dfrac{1}{2m-1} \\ w = v^{m-\frac{1}{2}} \end{cases}$$

- θ is a scaling parameter :

$$\frac{d-\gamma}{2p} = \frac{d-2}{2}\,\theta + \frac{d-\gamma}{p+1}\,(1-\theta), \text{ with } \theta \in (0,1]$$

- Range of validity :

$$1 < p \leq \frac{d-\gamma}{d-2} \Longleftrightarrow 1 > m \geq m_1(\gamma) := \frac{2d-\gamma-2}{2\,(d-\gamma)}.$$

Comments

- $\gamma = 0$: particular family of the well-known Gagliardo-Nirenberg inequalities, for which optimal functions have been completely characterized in [Del Pino-Dolbeault] :

$$\forall x \in \mathbb{R}^d,\ u_B(x+x_0) = \left(\frac{a}{b+|x|^2}\right)^{\frac{1}{p-1}},\ \text{with } x_0 \in \mathbb{R}^d \text{ and } a, b > 0.$$

- A crucial step in this characterization relies on the radial symmetry of the minimizers, and the uniqueness result by [Pucci-Serin]. Notice that symmetrization techniques also applies on the case where

$$0 < \gamma < 2, \quad p = \frac{d-\gamma}{d-2}.$$

- We follow here a perturbation approach for $\gamma \to 0$, with $1 < p < \frac{d-\gamma}{d-2}$, similar to the one used by [Dolbeault-Esteban-Loss-Tarantello].

Notations

From now on, we fix some $1 < p < \frac{d}{d-2}$, and consider the limit $\gamma \to 0$ in the range $(0, d-(d-2)p)$.

- The functional :

$$Q_{\gamma,p}(w) = \frac{1}{2}\int_{\mathbb{R}^d} |\nabla w(x)|^2 \, \mathrm{d}x + \frac{1}{p+1}\int_{\mathbb{R}^d} \frac{|w(x)|^{p+1}}{|x|^\gamma} \, \mathrm{d}x,$$

defined on the space $\mathcal{H}_{\gamma,p}(\mathbb{R}^d)$ with

$$\|w\|^2_{\mathcal{H}_{\gamma,p}} = \|\nabla w\|_2^2 + \|w\|^2_{p+1,\gamma} < \infty\,.$$

- For any $\gamma > 0$ (small), and $M > 0$, we consider

$$\inf_{\|w\|_{2p,\gamma}=M} Q_{\gamma,p}(w) \tag{3}$$

Main results

- Proposition. There exists some $\gamma^*(p) > 0$ s.t. $\forall \gamma < \gamma^*(p)$, the problem (3) admits a solution in $\mathcal{H}_{\gamma,p}(\mathbb{R}^d)$. In addition, it is possible to fix the mass M_γ in such a way that this solution w_γ solves

$$-\Delta w_\gamma + \frac{w_\gamma^p}{|x|^\gamma} = \frac{w_\gamma^{2p-1}}{|x|^\gamma} \tag{4}$$

- [DNM] Theorem 1. There exists some $\gamma_n \searrow 0$, such that (w_{γ_n}) converges strongly to u_B in the space $\mathcal{H}_{\gamma,p}(\mathbb{R}^d)$ and in $C^{1,\alpha}(\mathbb{R}^d)$.

  Remark : In u_B, (a, b) are fixed by the scaling, but the result says that $x_0 = 0$!

- [DNM] Theorem 2. There exists some $\gamma^*(p) > 0$ s.t. $\forall \gamma < \gamma^*(p)$, the minimizers of (2) are radially symmetric and all of the form

$$\forall x \in \mathbb{R}^d,\ \overline{w}(x) = \left(\frac{a}{b + |x|^{2-\gamma}} \right)^{\frac{1}{p-1}},\ \text{with } a, b > 0.$$

Proof of Theorem 1

- Concentration-compactness techniques leads to the convergence of $w_\gamma(\cdot + x_\gamma)$, for a sequence (x_γ) that can be proved to satisfy

$$\lim_{\gamma \to 0} |x_\gamma|^\gamma = 1.$$

- [Anzellotti-Baldo] : A selection principle based on asymptotic development by Γ-convergence. The idea : study the derivative w.r.t. the parameter γ, that is some logarithmic moments.

- First, we get the convergence of the sequence (u_γ) to the Barenblatt profile centered at some $\overline{y} \in \mathbb{R}^d$, which turns out the be

$$\overline{y} = \operatorname*{argmin}_{y \in \mathbb{R}^d} \int_{\mathbb{R}^d} \left[-\frac{1}{p+1} u_B^{p+1}(x-y) + \frac{1}{2p} u_B^{2p}(x-y) \right] \log|x| \, \mathrm{d}x \, .$$

- 0 is the only solution to the previous problem.

Proof of Theorem 2

- Introduce an angular derivative :

$$w_{A,\gamma} = \partial_{A_\gamma} w_\gamma(x) := (A_\gamma \nabla w_\gamma(x)) \cdot x, \quad A_\gamma \text{ being an antisymmetric matrix.}$$

Differentiating the Euler-Lagrange equation (4) gives the identities

$$\begin{aligned}
&\int_{\mathbb{R}^d} |\nabla w_{A,\gamma}|^2 + p \int_{\mathbb{R}^d} w_{A,\gamma}^2 \frac{w_\gamma^{p-1}}{|x|^\gamma} = (2p-1) \int_{\mathbb{R}^d} w_{A,\gamma}^2 \frac{w_\gamma^{2p-1}}{|x|^\gamma} \\
&\int_{\mathbb{R}^d} w_{A,\gamma} \frac{w_\gamma^{2p-1}}{|x|^\gamma} = 0
\end{aligned} \tag{5}$$

We then normalize and take the weak limit to get, for some $\overline{W}$,

$$\begin{aligned}
&\int_{\mathbb{R}^d} |\nabla \overline{W}|^2 + p \int_{\mathbb{R}^d} \overline{W}^2 u_B^{p-1} \leq (2p-1) \int_{\mathbb{R}^d} \overline{W}^2 u_B^{2p-1} \\
&\int_{\mathbb{R}^d} \overline{W} u_B^{2p-1} = 0
\end{aligned} \tag{6}$$

Proof of Theorem 2

- [BBDGV] : Introduce the functional

$$\mathcal{K}_B(v) = \int_{\mathbb{R}^d} |\nabla v|^2 + p \int_{\mathbb{R}^d} v^2\, u_B^{p-1} - (2p-1) \int_{\mathbb{R}^d} v^2\, u_B^{2(p-1)}$$

  Then, $\mathcal{K}_B(v) \geq 0$ for all $v \in \dot{H}^1(\mathbb{R}^d)$ such that

$$\int_{\mathbb{R}^d} v\, u_B^{2p-1}\, \mathrm{d}x = 0\,,$$

  and equality is attained if and only if there exists a vector $\underline{a} \in \mathbb{R}^d$ such that

$$v(x) = \nabla u_B(x) \cdot \underline{a} \quad \forall x \in \mathbb{R}^d\,.$$

- Consequence : $\overline{W} = \nabla u_B \cdot \underline{a}$.

Proof of Theorem 2

- The final identity : By differentiating again (4) w.r.t. any coordinate combined with (5) leads at the limit to

$$\forall 1 \leq i \leq d, \int_{\mathbb{R}^d} \frac{x_i}{|x|^2} \left[u_B^p(x) - u_B^{2p-1}(x) \right] \overline{W}(x) dx = 0,$$

and, after some computations,

$$\int_{R^d} \frac{1}{|x|^2} \left[-\frac{1}{p+1} u_B^{p+1}(x) + \frac{1}{2p} u_B^{2p}(x) \right] dx = 0,$$

which can be proved to be false.