An entropic approach to Navier-Stokes equation

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Euler

Motion of a perfect incompressible fluid

- position: $x \in \mathcal{X} := (\mathbb{R}/\mathbb{Z})^n$, time: $t \in [0,1]$
- force per unit of mass: $-\nabla U(t,x) \in \mathbb{R}^n$

Unknown quantities:

- density field: $\rho(t,x) \in [0,\infty)$
- velocity field: $v(t,x) \in \mathbb{R}^n$
- pressure field: $p(t,x) \in \mathbb{R}$

Euler

Incompressibility

$\nabla \cdot v = 0$

Conservation of mass

$\partial_t \rho + \nabla \cdot (\rho v) = 0$

Newton's second law

$\rho \left(\partial_t v + v \cdot \nabla v + \nabla U\right) + \nabla p = 0$

- $D_t := \partial_t + v \cdot \nabla$
  $\frac{d}{dt} f(t, x_t) = \partial_t f(t, x_t) + \nabla f(t, x_t) \cdot \dot{x}(t) = (D_t f)(t, x_t)$
- $\partial_t v + v \cdot \nabla v = D_t v =$ acceleration
- the pressure force field ∇p is necessary for the fluid to be incompressible

Arnold

- Lagrange representation:
  flow map $g(t,x) \in \mathcal{X}$: $g(0,x) = x, \quad \partial_t g(t,x) = v(t, g(t,x))$
- $\rho(t, g(t,x)) = \rho(t = 0, x) := \rho_0(x)$
- $\rho_0(x) \left\{ \partial_t^2 g(t,x) + \nabla U(t, g(t,x)) \right\} + \nabla p(t, g(t,x)) = 0$

Arnold

Smoothness

$g(t, \cdot) \in G := \{\text{diffeomorphisms}\}$

Incompressibility

$g(t, \cdot) \in G_{\mathrm{vol}} := \{\text{volume preserving diffeomorphisms}\}$

- Lie group of G: space of all vector fields
- Lie group of G_{vol}: space of all *divergence-free* vector fields

Arnold

- $g(t,\cdot) \in G_{\mathrm{vol}}$
- switch from a Cauchy problem to a fixed endpoint problem
  with $\begin{cases} g(0,\cdot) = \mathrm{Id}, \\ g(1,\cdot) = g_1 \end{cases}$ prescribed
- no external force: $\nabla U = 0$ (to make things easy)

Arnold's insight

$t \mapsto g(t,\cdot)$ is a constant speed geodesic in G_{vol}

i. e. it solves

Arnold's geodesic problem

$$\int_0^1 \frac{1}{2} \|\partial_t g(t,\cdot)\|^2_{L^2_{\mathbb{R}^n}(\rho_0)}\, dt \to \min; \quad g \in G_{\mathrm{vol}}^{[0,1]} : \begin{cases} g(0,\cdot) = \mathrm{Id} \\ g(1,\cdot) = g_1 \end{cases}$$

- $\nabla p \in G_{\mathrm{vol}}^{\perp} \subset G$ is the force that keeps the motion in G_{vol}

Brenier

relaxation of Arnold's problem

- path space: $\Omega \subset \mathcal{X}^{[0,1]}$
- probability path measure on Ω: $P \in \mathcal{P}(\Omega)$
- $E_P := \int_\Omega dP$
- canonical process: $X_t(\omega) = \omega_t \in \mathcal{X}, \quad \omega = (\omega_t)_{0 \le t \le 1} \in \Omega$
- $P_t := (X_t)_\# P \in \mathcal{P}(\mathcal{X})$
- $P_{0t} := (X_0, X_t)_\# P \in \mathcal{P}(\mathcal{X}^2)$

Brenier

- $\mathcal{P}_{\mathrm{vol}}(\mathcal{X}^2) := \{\pi \in \mathcal{P}(\mathcal{X}^2); X_\# \pi = Y_\# \pi = \mathrm{vol}\}$
- $\mathcal{P}_{\mathrm{vol}}(\Omega) := \{P \in \mathcal{P}(\Omega) : P_t = \mathrm{vol},\ \forall t\}$

- for any $g \in G_{\mathrm{vol}}$, $\quad \pi^g(dxdy) := \mathrm{vol}(dx)\delta_{g(x)}(dy) \in \mathcal{P}_{\mathrm{vol}}(\mathcal{X}^2)$

(the subscript vol means that volume is preserved)

Suppose $P \in \mathcal{P}_{\mathrm{vol}}(\Omega)$. Then,

- $P_{0t} \in \mathcal{P}_{\mathrm{vol}}(\mathcal{X}^2)$
- $t \mapsto P_{0t}$ is a relaxation of $t \mapsto g(t, \cdot)$

Brenier

Brenier's problem

$$E_P \int_0^1 \rho_0(X_0)\Big(|\dot{X}_t|^2/2 - U(t,X_t)\Big)\,dt \to \min; \quad P \in \mathcal{P}_{\mathrm{vol}}(\Omega),\ P_{01} = \pi$$

$\pi \in \mathcal{P}_{\mathrm{vol}}(\mathcal{X}^2)$: final configuration

Result

Any $P \in \mathcal{P}_{\mathrm{vol}}(\Omega)$ such that

$$\rho_0(X_0)\Big(\ddot{X}_t + \nabla U(t,X_t)\Big) + \nabla p(t,X_t) = 0, \quad P\text{-a.e.}$$

for some pressure field p, solves Brenier's problem

- $(P_{0t})_{0\le t\le 1} \in \mathcal{P}_{\mathrm{vol}}(\mathcal{X}^2)^{[0,1]}$ is the generalized solution of the Euler equation with an endpoint boundary condition

Navier-Stokes

- Simplifying assumptions
  - spatial homogeneity: $\rho \equiv 1$
  - no external force: $U \equiv 0$

Incompressibility

$\nabla \cdot v = 0$

Newton's second law

$\partial_t v + v \cdot \nabla v - \Delta v/2 + \nabla p = 0$

- $\Delta v/2$: viscosity force
- to obtain $\alpha \Delta v$, rescale time: $t \to at$

Boltzmann

Relative entropy

$H(P|R) := E_P \log(dP/dR) \in [0,\infty], \quad P, R \in \mathcal{P}(\Omega)$

- $R \in \mathcal{P}(\Omega)$ is the reference measure
  $R =$ Wiener measure (Brownian motion)
  $R_0 = \mathrm{vol}$

Entropy minimization problem

$$H(P|R) \to \min; \quad P \in \mathcal{P}_{\mathrm{vol}}(\Omega),\ P_{01} = \pi \qquad (H)$$

Nelson

$P \ll R$ implies

- $dX_t = \overrightarrow{D}_t^P(X)\, dt + dB_t, \quad P\text{-a.e.}$
- Nelson's *forward* stochastic derivative
$$\overrightarrow{D}_t^P(X) := \lim_{h \to 0^+} \frac{1}{h} E_P(X_{t+h} - X_t \mid X_{[0,t]})$$
- $H(P|R) = E_P \int_0^1 |\overrightarrow{D}_t^P(X)|^2/2\, dt$

Problem (H) is equivalent to

$$E_P \int_0^1 |\overrightarrow{D}_t^P(X)|^2/2\, dt \to \min; \quad P \in \mathcal{P}_{\mathrm{vol}}(\Omega),\ P \ll R, \quad P_{01} = \pi$$

compare with

Brenier's problem

$$E_P \int_0^1 |\dot{X}_t|^2/2\ dt \to \min; \quad P \in \mathcal{P}_{\mathrm{vol}}(\Omega),\ P_{01} = \pi$$

Lagrange

Back to entropy

$$H(P|R) \to \min; \quad P \in \mathcal{P}_{\mathrm{vol}}(\Omega),\ P_{01} = \pi \qquad (H)$$

The unique solution to (H) writes as

$$P = \exp\left(\theta(X_0, X_1) + \int_0^1 p(t, X_t)\, dt\right) R$$

Idea of proof:

- problem (H) is a *convex* problem
- existence: works for all final state π, as in Brenier's paper
- uniqueness: (H) is *strictly* convex
- constraint $P_{01} = \pi \quad \longrightarrow \quad$ Lagrange mult. $= \theta(X_0, X_1)$
- constraint $P \in \mathcal{P}_{\mathrm{vol}}(\Omega) \quad \longrightarrow \quad$ Lagrange mult. $= \int_0^1 p(t, X_t)\, dt$
- convex duality: $\quad t \log t - t \quad \longrightarrow \quad e^s$

Hamilton-Jacobi-Bellman

- y = final position
- $\varphi^y(t,z) := \log E_R\Big[\exp\Big(\theta(X_0,y) + \int_0^t p(s,X_s)\,ds\Big) \mid X_t = z\Big]$
- Hamilton-Jacobi-Bellman equation

$$\begin{cases} (-\partial_t + \Delta/2)\varphi^y + |\nabla\varphi^y|^2/2 + p = 0, & 0 < t \leq 1 \\ \varphi^y(0,\cdot) = \theta(\cdot,y), & t = 0. \end{cases}$$

Link with Navier-Stokes equation

$v^y := -\nabla\varphi^y$ solves

$$\begin{cases} \partial_t v^y + (v^y\cdot\nabla)v^y - \Delta v^y/2 + \nabla p = 0, & 0 < t \leq 1 \\ v^y(0,\cdot) = -\nabla_x\theta(\cdot,y), & t = 0. \end{cases}$$

- ∇(HJ) = Newton
- ∇(HJB) = stochastic Newton

Doob

- P solves (H) $\to$ (θ, p) $\to$ $v^y = -\nabla\varphi^y$ with
  $\partial_t v^y + (v^y \cdot \nabla)v^y - \Delta v^y/2 + \nabla p = 0$
- p doesn't depend on y
- $P^y := P(\cdot \mid X_1 = y)$ is a Doob h-process

Main result

- The *backward* velocity vector field of P^y is

$$\overleftarrow{D}_t^{P^y}(X) := \lim_{h\to 0^+} \frac{1}{h} E_P(X_{t-h} - X_t \mid X_{[t,1]}) = -v^y, \quad \text{i. e.}$$

$$\begin{cases} dX_t = v^y(t, X_t)\,dt + d\overleftarrow{B}_t^y, \\ X_1 = y, \end{cases} \qquad P^y\text{-a.e.}$$

- $P(\cdot) = \int_{\mathcal{X}} P^y(\cdot)\,\mathrm{vol}(dy) \in \mathcal{P}_{\mathrm{vol}}(\Omega)$ i.e. preserves volume

- $v^y = -\nabla\varphi^y$ implies not much turbulence
- same remark for Arnold and Brenier
- more turbulence under an additional constraint (later work)

Bernstein

Markov property fails

Except for very specific final states π, P is not Markov.
It is reciprocal.

- However, $(X_0, X_t)_{0\leq t\leq 1}$ and $(X_t, X_1)_{0\leq t\leq 1}$ are Markov under P.
  It is a general property of the reciprocal path measures.
- The fluid particles are tagged by their final position y.
- The velocity field of the y-particles is described by minus the *backward* drift field of the Markov measure P^y : $v^y = -\nabla\varphi^y$.

Beware

PDEs fit well with $(P_t)_{0\leq t\leq 1}$ when P is Markov, but might be odd when P fails to be Markov.

- Need for considering P_{t1} instead of P_t.

Thank you for your attention