Discrete functional inequalities & finite volume approximation of parabolic models

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Example 1 : porous media equation

- The porous media equation in self-similar variables

$$\partial_t u = \text{div}(xu + \nabla r(u)) \text{ on } \mathbb{R}^d \times (0,T).$$

$r(s) = s^\gamma$ with $\gamma \geq 1$ ($\gamma > 1$: degenerate parabolic equation).

- Convergence to steady states. **J. A. Carrillo, G. Toscani (2000)**
  Convergence of the solution to the Barenblatt-Pattle distribution

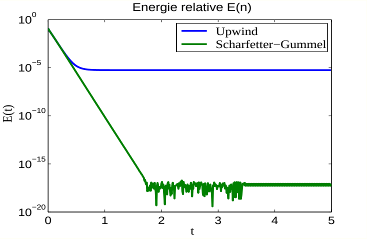
$$u_\infty(x) = \left(C_1 - \frac{\gamma - 1}{2\gamma} |x|^2 \right)_+^{\frac{1}{\gamma-1}}.$$

Notations

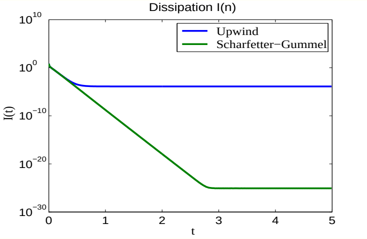
- enthalpy function $h(s) = \int_1^s \frac{r'(\tau)}{\tau} d\tau,$
- generalized inverse of h : $g(s) = \begin{cases} h^{-1}(s) & \text{if} \quad h(0^+) < s < \infty, \\ 0 & \text{if} \quad s \leq h(0^+). \end{cases}$

Proofs are based on an entropy estimate with the control of the entropy dissipation.

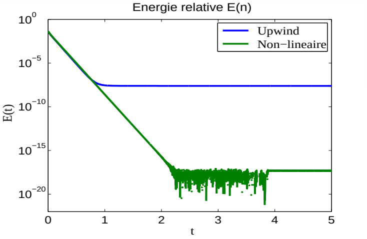
Example 1 : porous media equation



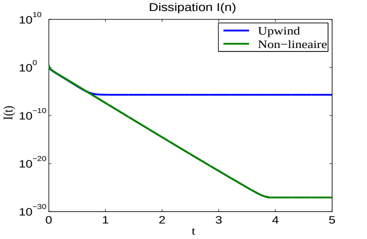
(a) Relative energy, $r(s) = s$



(b) Dissipation, $r(s) = s$



(c) Relative energy, $r(s) = s^2$



(d) Dissipation, $r(s) = s^2$

Example 2 : Boltzmann equation

Consider the following equation

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f \;=\; Q(f) \;:=\; \int_{S^2\times\mathbb{R}^3} \mathcal{B}(\sigma,|u|)\left(f' f'_* - f_* f\right) d\sigma du.$$

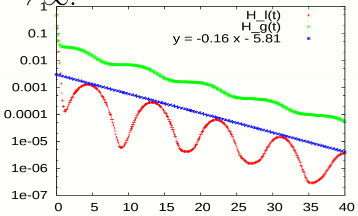
We introduce a truncated operator $Q^R(f)$ breaking some invariants (**FF & C. Mouhot** [1]).

- mass, invariance by translation and entropy production are preserved
- steady state are given by constant functions (wrong steady states !)

In the space homogeneous case, we adapt the method developped by **G. Toscani & C. Villani** [2] and get

$$\|f^R(t) - C_R\|_1 \approx \exp(-K_R t), \quad \text{when} \quad t \to \infty.$$

> It is a negative results but it gives information about the validity of the time intervall for which the numerical approximation is correct.



1. Analysis of spectral methods, Trans. AMS (2010)
2. Sharp entropy dissipation bounds and explicit rate of trend to equilibrium, CMP (1999).

Aim of this work

We want to

- construct numerical schemes which preserve entropy/entropy production property ;
- prove the long time asymptotic is correct with a correct convergence rate.

The drift-diffusion model for semiconductors

- Drift-diffusion model. Consider the density of electrons $N_e(t,x)$ and of ions $N_i(t,x)$,

$$\begin{cases} \partial_t N_e - \mathrm{div}(\nabla r(N_e) - N_e \nabla V) = 0, & \text{on } \Omega \times (0,T), \\ \partial_t N_i - \mathrm{div}(\nabla r(N_i) + N_i \nabla V) = 0, & \text{on } \Omega \times (0,T), \\ \triangle V = N_e - N_i, & \text{on } \Omega \times (0,T). \end{cases}$$

- Convergence to steady states solutions. **A. Jüngel (1995)**
  Convergence of the transient solution to the thermal equilibrium state

$$\begin{cases} N_e^\infty = g(\alpha_{N_e} + V^\infty), \quad N_i^\infty = g(\alpha_{N_i} - V^\infty) \\ \Delta V^\infty = g(\alpha_{N_e} + V^\infty) - g(\alpha_{N_i} - V^\infty). \end{cases}$$

Notations

- enthalpy function $h(s) = \displaystyle\int_1^s \frac{r'(\tau)}{\tau} d\tau$,
- generalized inverse of h : $g(s) = \begin{cases} h^{-1}(s) & \text{if} \quad h(0^+) < s < \infty, \\ 0 & \text{if} \quad s \le h(0^+). \end{cases}$

The cross-diffusion & PKS models

1. Cross-diffusion model.
   - Consider the density $r(t,x)$ and $b(t,x)$,

$$\partial_t r - \text{div}\,((1-b)\nabla r \,+\, r\,\nabla b \,+\, r(1-r-b)\nabla V) = 0,$$
$$\partial_t b - \text{div}\,((1-r)\nabla b \,+\, b\,\nabla r \,+\, b(1-r-b)\nabla W) = 0,$$

   - **M.Burger, M. Di Francesco, J.-F. Pietschmann, B. Schlake (2010)** Convergence of the transient solution to an equilibrium state, gradient flow structure, energy estimates
2. The Patlak or Keller-Segel model for chemotaxis.
   - Consider the density of bacteries $N(t,x)$ and of chemical substance $C(t,x)$,

$$\partial_t N - \text{div}(\nabla r(N) - N\,\nabla C) \;=\; S_N(N,C) \quad \text{on} \quad \Omega \times (0,T),$$
$$\varepsilon\partial_t C - \text{div}\nabla C = S_C(N,C) \quad \text{on} \quad \Omega \times (0,T),$$

   - Blow-up or steady states : $(\varepsilon = 0)$ **J. Dolbeault & B. Perthame (2004)** Blow-up or global existence of solutions
   - Entropy dissipation : **V. Calvez & J.A. Carrillo (2008)** Energy estimates.

Long time behavior of PDEs

Carrillo, Jüngel, Markowich, Toscani, Unterreiter (2001).

$$\partial_t u = \mathrm{div}\left(u\nabla V(x) + \nabla r(u)\right), \quad x \in \Omega, \quad t > 0.$$

$\exists h$ such that $r'(s) = h'(s)\, s$, H primitive of h, strictly convex.

$$\partial_t u = \mathrm{div}\left(u\nabla\left(V(x) + h(u)\right)\right)$$

$$\Longrightarrow \frac{dE(t)}{dt} = -\mathcal{I}(t) \leq 0,$$

$u^{eq} \in L^1(\Omega)$ is an equilibrium if and only if u^{eq} minimizes E in $\mathcal{C} := \{u \in L^1_+(\Omega),\ \int_\Omega u = \int_\Omega u_0\}$.

$$\frac{dE(t)}{dt} = 0 \Rightarrow u^{eq}\left|\nabla\left(V + h(u^{eq})\right)\right|^2 = 0$$
$$\Rightarrow u^{eq} = 0 \text{ or } V + h(u^{eq}) = C^{te}.$$

- $\rightarrow$ $\exists!\, u^{eq}$ if V and h are smooth enough,
- $\rightarrow$ exponential decay rate of the relative entropy and its dissipation.

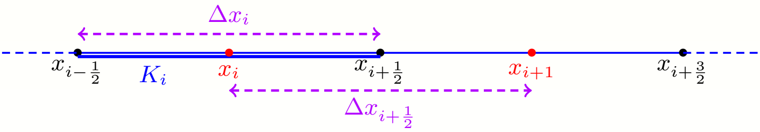
General framework of Finite Volume of 1D

$$\partial_t u - \text{div}(\nabla r(u) + \nabla V\, u) = 0 \text{ pour } (x,t) \in \Omega \times (0,T).$$

Goal (explained it in 1D). We want to approximate the flux

$$\mathcal{F}_{i+\frac{1}{2}} = [\partial_x r(u) + \partial_x V\, u]\,|_{x=x_{i+1/2}}$$

- generalize the Scharfetter-Gummel scheme for diffusion $(r(s) = \epsilon s)$
  $$\mathcal{F}_{i+\frac{1}{2}} = \frac{1}{\Delta x_{i+\frac{1}{2}}}\Big(B(-\Delta x_{i+\frac{1}{2}} dV_{i+\frac{1}{2}})U_i - B(\Delta x_{i+\frac{1}{2}} dV_{i+\frac{1}{2}})U_{i+1}\Big),$$
  where $B(x) = \dfrac{x}{e^{x/\varepsilon} - 1}$, $x \neq 0$, $B(0) = 1$ is the Bernoulli function.
- keep order of accuracy even in the degenerate case
- preserve the asymptotic behavior the the solution.



$$\Delta x_i \frac{U_i^{n+1} - U_i^n}{\Delta t} + \mathcal{F}^*_{i+\frac{1}{2}} - \mathcal{F}^*_{i-\frac{1}{2}} = 0.$$

Construction of the numerical flux

$$\partial_t u - \partial_x \left(u\,\partial_x V(x) + \partial_x r(u)\right) = 0, \quad x \in \Omega, \quad t > 0$$

$$u\partial_x V + \partial_x r(u) = \underbrace{\partial_x \left(V + h(u)\right)}_{\text{“velocity” A}} u.$$

$$\partial_t u + \partial_x (A(u, \nabla u)u) = 0, \quad x \in \Omega, \quad t > 0$$

$$-\partial_x \left(V + h(u)\right)|_{x_{i+\frac{1}{2}}} \approx A_{i+\frac{1}{2}} = -dV_{i+\frac{1}{2}} - dh(U)_{i+\frac{1}{2}},$$

with

$$dV_{i+\frac{1}{2}} = \frac{V(x_{i+1}) - V(x_i)}{\Delta x_{i+\frac{1}{2}}}, \quad dh(U)_{i+\frac{1}{2}} = \frac{h(U_{i+1}) - h(U_i)}{\Delta x_{i+\frac{1}{2}}}$$

Classical upwind flux for $\partial_t u + \partial_x (Au) = 0$

$$\Rightarrow \boxed{\mathcal{F}_{i+\frac{1}{2}} = A^+_{i+\frac{1}{2}} U_i - A^-_{i+\frac{1}{2}} U_{i+1}}$$

Properties

Semi-discrete scheme : $\frac{d}{dt}U_i + \frac{1}{\Delta x_i}\left(\mathcal{F}_{i+\frac{1}{2}} - \mathcal{F}_{i-\frac{1}{2}}\right) = 0.$

- Results :
  - preservation of positivity,
  - preservation of equilibrium : $dh(U)_{i+\frac{1}{2}} + dV_{i+\frac{1}{2}} = 0 \Longrightarrow \mathcal{F}_{i+\frac{1}{2}} = 0,$
  - entropy estimate : $\forall\, 0 < t_1 \leq t_2 < +\infty,$

$$0 \leq \mathcal{E}_\Delta(t_2) + \int_{t_1}^{t_2} \mathcal{I}_\Delta(t)\, dt \leq \mathcal{E}_\Delta(t_1).$$

  - uniform accuracy even for degenerate diffusion.

Fully discrete scheme : $\frac{1}{\Delta t}(U_i^{n+1} - U_i^n) + \frac{1}{\Delta x_i}\left(\mathcal{F}^n_{i+\frac{1}{2}} - \mathcal{F}^n_{i-\frac{1}{2}}\right) = 0.$

- Results :
  - preservation of equilibrium,
  - preservation of positivity under a CFL condition

$$\Delta t \max_i |V(x_{i+1}) - V(x_i) - h(U^n_{i+1}) + h(U^n_i)| \leq \frac{1}{2} \min_i \Delta x_i^2.$$

Other problem : nonlinear transport

$$\begin{cases} \partial_t u = \text{div}\left(f(u)\nabla V(x) + \nabla r(u)\right), & x \in \Omega, \quad t \geq 0, \\ u(x,0) = u_0(x) \geq 0, & x \in \Omega, \end{cases}$$

where $u : \Omega \subset \mathbb{R}^d \to \mathbb{R}_+$ is a density, $f(u) \geq 0$, $r \in \mathcal{C}^1(\mathbb{R}_+)$ such that $r(0) = 0$, $r'(u) \geq 0$.

Assumptions : $\exists h$ such that $r'(s) = h'(s)\, f(s)$, H primitive of h, strictly convex.

$$\partial_t u = \text{div}\left(f(u)\nabla\left(V(x) + h(u)\right)\right)$$

$$\Longrightarrow \frac{dE(t)}{dt} = -\mathcal{I}(t) \leq 0,$$

with

- entropy : $E(t) = \displaystyle\int_\Omega \left(V\, u + H(u)\right)\, dx$,
- dissipation : $\mathcal{I}(t) = \displaystyle\int_\Omega f(u)\left|\nabla\left(V + h(u)\right)\right|^2 dx.$

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- Convergence to steady states solutions. **A. Jüngel (1995)**
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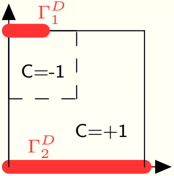
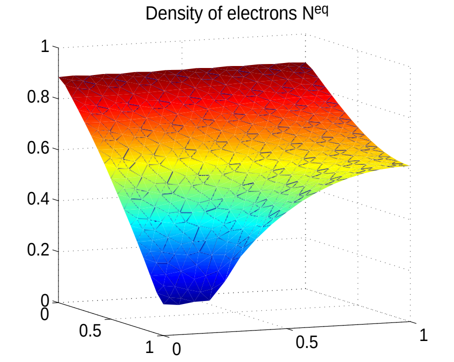
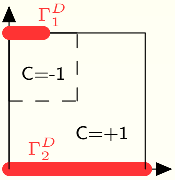
Drift-diffusion system



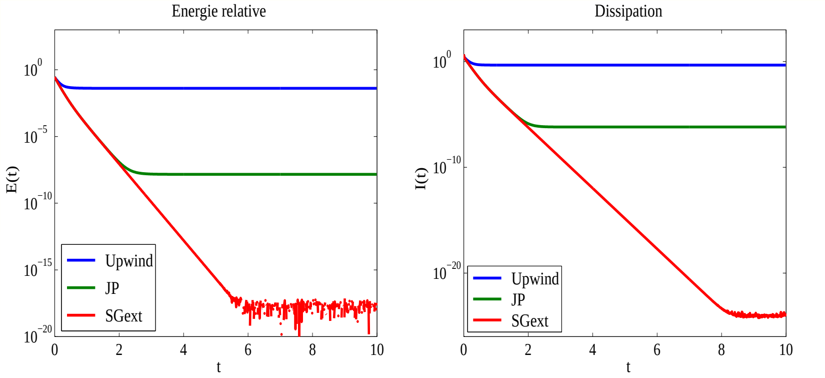
$\Delta t = 5.10^{-4}$, $T = 0.05$,
admissible mesh with 896 triangles,
$r(s) = s^{5/3}$,
$N_e^D = 0.1$, $N_i^D = 0.9$ on Γ_1^D,
$N_e^D = 0.9$, $N_i^D = 0.1$ on Γ_2^D.



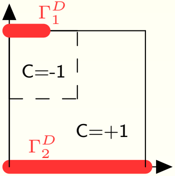
Drift-diffusion system



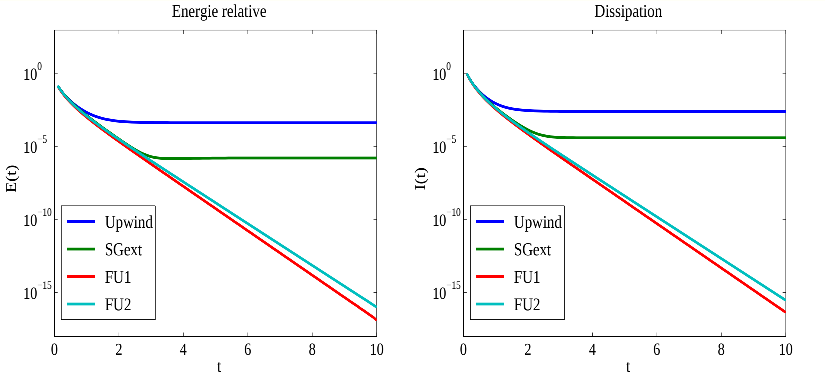
$\Delta t = 10^{-2}$, $T = 10$,
admissible mesh of 896 triangles,
$r(s) = s^{5/3}$,
$N_e^D = 0.1$, $N_i^D = 0.9$ on Γ_1^D,
$N_e^D = 0.9$, $N_i^D = 0.1$ on Γ_2^D.



Drift-diffusion system



$\Delta t = 10^{-2}$, $T = 10$,
admissible mesh with 896 triangles,
$r(s) = s^{5/3}$,
$N_e^D = \mathbf{0}$, $N_i^D = 1$ on Γ_1^D,
$N_e^D = 1$, $N_i^D = \mathbf{0}$ on Γ_2^D.

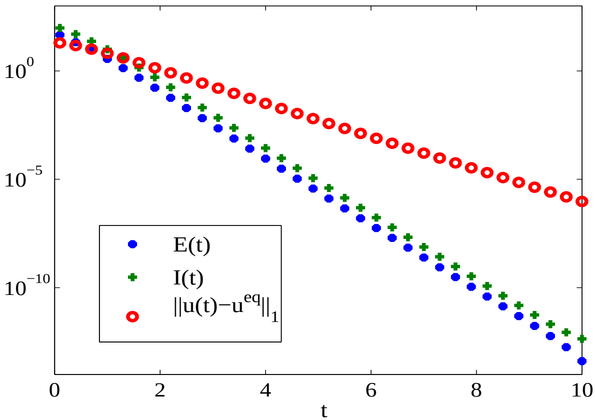


Fokker-Planck equation for Fermions

Consider a density u solution to

$$\partial_t u = \text{div}(xu(1-u) + \nabla u),$$

where $\Omega = (-8, 8)^3$. We apply our finite volume scheme with a cartesian mesh $40 \times 40 \times 40$, $\Delta t = 10^{-4}$, $T = 3$.



How to justify the long time asymptotic behavior of the numerical solution ?

We either consider the semi-discrete case or the fully implicit case (or some particular cases of explicit schemes). These schemes are constructed in such a way that

$$\frac{dE_h(t)}{dt} = -\mathcal{I}_h(t) \leq 0,$$

where E_h is the discrete energy and $\mathcal{I}_h(t)$ the discrete dissipation. The next step is to prove eventually that $C_\Omega E_h \leq \mathcal{I}_h(t)$ and then

$$\frac{dE_h(t)}{dt} \leq -C_\Omega E_h.$$

Example : heat equation with the discrete L^2 norm, for which we need some discrete Poincaré inequality.

Some functional inequalities

Let assume $N \geq 2$ and Ω be an open polyhedral bounded domain of $\mathbb{R}^N$.

- If $1 \leq p < N$, let $1 \leq s \leq m \leq p^* = \frac{pN}{N-p}$;
- If $p \geq N$, let $1 \leq s \leq m < +\infty$.

Then there exists a constant $C > 0$ depending on p, s, m, N and Ω such that

$$\|u\|_{L^m(\Omega)} \leq C \|u\|^{\theta}_{W^{1,p}(\Omega)} \|u\|^{1-\theta}_{L^s(\Omega)} \quad \forall u \in W^{1,p}(\Omega) \cap L^s(\Omega), \tag{1}$$

where θ depends on p,N, s, m.

Also, we can prove the Poincaré-Sobolev inequality

- If $1 \leq p < N$, let $1 \leq q \leq p^* = \frac{pN}{N-p}$;
- If $p \geq N$, let $1 \leq q < +\infty$.

$$\|u\|_{L^q(\Omega)} \leq C \|u\|_{W^{1,p}(\Omega)} \quad \forall u \in W^{1,p}(\Omega), \tag{2}$$

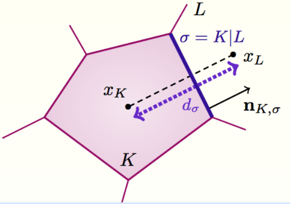
Assumption on the mesh

A family $\mathcal{E}$ of relatively open parts of hyperplanes in $\mathbb{R}^N$, which represent the faces of the control volumes, and a family of points $(x_K)_{K\in\mathfrak{M}}$ which satisfy the following properties :

- $\overline{\Omega} = \bigcup_{K\in\mathfrak{M}} \overline{K}$,
- for all $K \in \mathfrak{M}$, there exists $\mathcal{E}_K \subset \mathcal{E}$ such that $\partial K = \bigcup_{\sigma\in\mathcal{E}_K} \overline{\sigma}$,
- the family of points $(x_K)_{K\in\mathfrak{M}}$ is such that $x_K \in K$ and if $\sigma \in \mathcal{E}_K$, it is assumed that $x_L \neq x_L$.

For all $\sigma \in \mathcal{E}$, we define

$$d_\sigma = \begin{cases} \mathrm{d}(x_K, x_L) & \text{for} \quad \sigma = K|L \\ \mathrm{d}(x_K, \sigma) & \text{for} \quad \sigma \in \mathcal{E}_{ext}, \end{cases}$$



We assume that the mesh satisfies the following regularity constraint : there exists $\xi > 0$ such that

$$\mathrm{d}(x_K, \sigma) \geq \xi\, d_\sigma, \quad \forall K \in \mathfrak{M},\ \forall \sigma \in \mathcal{E}_K.$$

General case without boundary conditions

We define the set $X(\mathfrak{M})$ of the finite volume approximation :

$$X(\mathfrak{M}) = \left\{ u \in L^1(\Omega) \ / \ \exists (U_i)_{i\in\mathfrak{M}} \text{ such that } u = \sum_{i\in\mathfrak{M}} U_i \mathbf{1}_i \right\}.$$

- We remind the continuous embedding of $BV(\Omega)$ into $L^{N/(N-1)}(\Omega)$ for a polyhedral domain Ω

$$\left(\int_\Omega |\, u \,|^{\frac{N}{N-1}} \, dx \right)^{\frac{N-1}{N}} \leq c(\Omega) \, \|\, u \,\|_{BV(\Omega)}, \quad \forall u \in BV(\Omega). \qquad (3)$$

- For $u \in X(\mathfrak{M})$, we have $\|\, u \,\|_{BV(\Omega)} = \|u\|_{L^1} + TV_\Omega(u)$

$$TV_\Omega(u) := \sum_{\substack{\sigma\in\mathcal{E}_{int} \\ \sigma_{i,j}}} \mathsf{m}(\sigma_{i,j}) \, |u_i - u_j| = |u|_{1,1,\mathfrak{M}} < +\infty.$$

The discrete space $X(\mathfrak{M})$ is included in $L^1 \cap BV(\Omega)$.

General case without boundary conditions

Let Ω be a bounded polyhedral subset of $\mathbb{R}^N$, $\mathfrak{M}$ a mesh of Ω.

1. For $p \in [1, +\infty)$, the discrete L^p norm is defined by

$$\| u \|_{L^p} = \left(\sum_{i \in \mathfrak{M}} \mathrm{m}(K_i) \, | U_i |^p \right)^{\frac{1}{p}}, \quad \forall u \in X(\mathfrak{M}).$$

2. In the general case, for $p \in [1, +\infty)$, the discrete $W^{1,p}$-seminorm is defined by :

$$| u |_{1,p,\mathfrak{M}} = \left(\sum_{\sigma = i|j \in \mathcal{E}_{int}} d_\sigma \mathrm{m}(\sigma) \frac{|U_i - U_j|^p}{d_\sigma d_\sigma^{p-1}} \right)^{\frac{1}{p}}, \quad \forall u \in X(\mathfrak{M})$$

and the discrete $W^{1,p}$-norm is defined by

$$\| u \|_{1,p,\mathfrak{M}} = \| u \|_{L^p} + | u |_{1,p,\mathfrak{M}}, \quad \forall u \in X(\mathfrak{M}). \tag{4}$$

General case without boundary conditions

Theorem (General discrete GNS inequality)

Let Ω be an open bounded polyhedral domain of $\mathbb{R}^N$, $N \geq 2$. Let $\mathfrak{M}$ be a mesh satisfying the previous definition.

- *If $1 \leq p < N$, let $1 \leq s \leq m \leq p^* = \dfrac{pN}{(N-p)}$,*
- *if $p \geq N$, let $1 \leq s \leq m < +\infty$.*

Then, there exists a constant $C > 0$ only depending on p, s, m, N and Ω such that :

$$\| u \|_{L^m} \leq \frac{C}{\xi^{(p-1)\theta/p}} \| u \|_{1,p,\mathfrak{M}}^{\theta} \| u \|_{L^s}^{1-\theta}, \ \forall u \in X(\mathfrak{M}), \tag{5}$$

where θ is given.

Proof of discrete GNS inequality

Part I. For $p = 1$, the results follows from the BV space property

$$\| v \|_{L^{N/(N-1)}} \leq c(\Omega) \left(| v |_{1,1,\mathfrak{M}} + \| v \|_{L^1} \right) \quad \forall v \in X(\mathfrak{M}).$$

Part II. Then, let $s \geq 1$ and $u \in X(\mathfrak{M})$, we choose $v \in X(\mathfrak{M})$ by $v_K = |u_K|^s$ for all $K \in \mathfrak{M}$. Hence, we apply successively

- one inequality

$$\mathrm{m}(\sigma) \left| |u_K|^s - |u_L|^s \right| \leq \mathrm{m}(\sigma) d(\sigma) s \left(|u_K|^{s-1} + |u_L|^{s-1} \right) \frac{|u_K - u_L|}{d_\sigma}.$$

- Hölder's inequality ;
- the assumption

$$\sum_{\sigma \in \mathcal{E}_K} m(\sigma) d_\sigma \leq \frac{1}{\xi} \sum_{\sigma \in \mathcal{E}_K} m(\sigma) d(x_K, \sigma) = \frac{1}{N\xi} m(K).$$

- Interpolation between L^p space which is still true in this framework

Proof of discrete GNS inequality

It first yields

$$\| u \|_{L^{sN/(N-1)}}^{s} \leq C \left(\frac{1}{\xi^{(p-1)/p}} \, | u |_{1,p,\mathfrak{M}} \, \| u \|_{L^q}^{s-1} + \| u \|_{L^p} \, \| u \|_{L^q}^{s-1} \right),$$

and finally playing with the parameter s and applying once more an interpolation inequality, we get the result.

Other results

Theorem (General discrete Sobolev-Poincaré inequality)

- *if* $1 \le p < N$, *for all* $1 \le q \le p^* := \dfrac{pN}{N-p}$,

$$\| u \|_{0,q,\mathfrak{M}} \le \frac{C}{\xi^{(p-1)/p}} \| u \|_{1,p,\mathfrak{M}}, \quad \forall u \in X(\mathfrak{M}), \tag{6}$$

- *if* $p \ge N$, *for all* $1 \le q < +\infty$,

$$\| u \|_{0,q,\mathfrak{M}} \le \frac{C}{\xi^{(p-1)/p}} \| u \|_{1,p,\mathfrak{M}}, \quad \forall u \in X(\mathfrak{M}). \tag{7}$$

From this, we can deduce a discrete Nash inequality :

Corollary (Discrete Nash inequality)

$$\| u \|_{0,2,\mathfrak{M}}^{1+\frac{2}{N}} \le \frac{C}{\sqrt{\xi}} \| u \|_{1,2,\mathfrak{M}} \| u \|_{0,1,\mathfrak{M}}^{\frac{2}{N}}, \quad \forall u \in X(\mathfrak{M}). \tag{8}$$

Other results

Theorem (Discrete Poincaré-Wirtinger inequality)

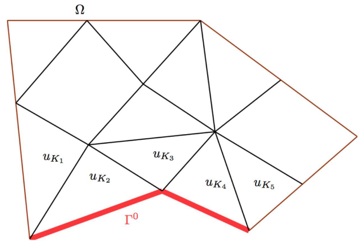
Let Ω be an open bounded connected polyhedral domain of $\mathbb{R}^N$. Let $\mathfrak{M}$ be a mesh satisfying our definition. Then for $1 \leq p < +\infty$ there exists a constant $C > 0$ only depending on Ω, N and p such that :

$$\| u - \overline{u} \|_{L^p} \leq \frac{C}{\xi^{(p-1)/p}} \, | u |_{1,p,\mathfrak{M}} \quad \forall u \in X(\mathfrak{M}). \tag{9}$$

We recall that $\bar{u} = \dfrac{1}{m(\Omega)} \displaystyle\int_\Omega u(x)dx = \dfrac{1}{m(\Omega)} \sum_{K \in \mathfrak{M}} m(K) u_K$, *for* $u \in X(\mathfrak{M})$.

General case with Dirichlet boundary conditions

For homogeneous Dirichlet boundary conditions, we need to take into account jumps on the boundary in the discrete $W^{1,p}$-seminorm. In the set of exterior faces $\mathcal{E}_{ext}$, we distinguish $\mathcal{E}^0_{ext}$ the set of boundary faces included in Γ^0.



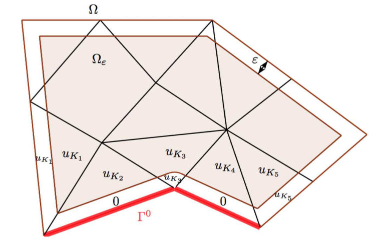
For $p \in [1, +\infty)$, we define the discrete $W^{1,p}$-seminorm (which depends on Γ^0) by

$$| u |_{1,p,\Gamma^0,\mathfrak{M}} = \left(\sum_{\sigma \in \mathcal{E}} \frac{\mathrm{m}(\sigma)}{d_\sigma^{p-1}} \, (D_\sigma u)^p \right)^{\frac{1}{p}}, \quad 1 \leq p < +\infty, \qquad (10)$$

where

$$D_\sigma u = \begin{cases} | U_i - U_j | & \text{if} \quad \sigma = i|j \in \mathcal{E}_{int}, \\ | U_i | & \text{if} \quad \sigma \in \mathcal{E}^0_{ext} \cap \mathcal{E}_i, \\ 0 & \text{if} \quad \sigma \in \mathcal{E}_{ext} \setminus \mathcal{E}^0_{ext}. \end{cases}$$

Main idea : modify the mesh $\mathfrak{M}$



- if $K_i \subset \Omega_\varepsilon$, we set $K_\varepsilon = K$,
- if $K_i \cap (\Omega \setminus \Omega_\varepsilon) \neq \emptyset$, we split the control volume K_i into two control volumes $K_\varepsilon^1 = K \cap \Omega_\varepsilon$ and $K_\varepsilon^2 = K_i \cap (\Omega \setminus \Omega_\varepsilon)$.

Let us now define a function $u_\varepsilon \in X(\mathfrak{M}_\varepsilon)$, which is still a piecewise constant function but which takes into account some boundary values $u_\sigma = 0$ for $\sigma \subset \Gamma_0$:

$$u_\varepsilon = \sum_{i_\varepsilon \in \mathfrak{M}_\varepsilon} U_{i_\varepsilon} \mathbf{1}_{K_{i_\varepsilon}},$$

where

$$U_{i_\varepsilon} = \begin{cases} U_i & \text{if} \quad \mathrm{m}(\partial K_{i_\varepsilon} \cap \Gamma^0) = 0 \text{ and } K_{i_\varepsilon} \subset K_i, \\ 0 & \text{if} \quad \mathrm{m}(\partial K_{i_\varepsilon} \cap \Gamma^0) > 0. \end{cases}$$

Main idea : modify the mesh

This function verifies $u_\varepsilon = 0$ on Γ^0 and

$$\|u_\varepsilon\|_{L^{N/(N-1)}} \leq c(\Omega) TV_\Omega(u_\varepsilon). \tag{11}$$

In order to pass to the limit $\varepsilon \to 0$ in this last inequality, we analyze the limit of both sides of the inequality.

$$\lim_{\varepsilon \to 0} \| u_\varepsilon \|_{L^{N/(N-1)}(\Omega)} = \| u \|_{L^{N/N-1}} \tag{12}$$

When comparing $TV(u_\varepsilon)$ and $TV(u)$, it remains only quantities due to

- some edges included in $\partial\Omega_\varepsilon$ and tending to an edge included in Γ^0 when ε tends to 0
- edges such that $\mathrm{m}(\sigma_\varepsilon \cap \Gamma^0) = 0$ whose number does not depend on ε.

Discrete GNS inequality for Dirichlet boundary conditions

Theorem (Discrete Gagliardo-Nirenberg-Sobolev inequality)

Let Ω be an open convex bounded polyhedral domain of $\mathbb{R}^N$ and $\Gamma^0 \neq \emptyset$ be a part of the boundary. Let $\mathfrak{M}$ be an admissible mesh. Then for any $1 < p \leq N$ and $q \geq 1$, there exists a constant $C > 0$ only depending on p, q, N and Ω such that

$$\| u \|_{0,m,\mathfrak{M}} \leq \frac{C_1}{\xi^{(p-1)\theta/p}} \, | u |_{1,p,\Gamma^0,\mathfrak{M}}^{\theta} \, \| u \|_{0,q,\mathfrak{M}}^{1-\theta}, \quad \forall u \in X(\mathfrak{M}), \tag{13}$$

where θ and m satisfy

$$0 \leq \theta \leq \frac{p}{p + q(p-1)} \leq 1 \tag{14}$$

and

$$\frac{1}{m} = \frac{1-\theta}{q} + \frac{\theta}{p} - \frac{\theta}{N}. \tag{15}$$

Conclusion & Perspectives

This work gives a systematic way to derive discrete functional inequalities. We mention further works :

- M. Bessemoulin-Chatard & A. Jungel : Discrete log-Sobolev inequality : start from the discrete Sobolev inequality and then apply the Jensen inequality (see **A. Guionnet & B. Zegarlinski**[3])
- M. Bessemoulin-Chatard, C. Chainais-Hillairet & A. Jungel : Beckner inequalities
- K. Gartner, A. Glitzky & J. A. Griepentrog : Voronoi Finite Volume Methods for Semiconductor Problems.

3. Lectures on Log Sobolev inequalities

Good luck !

Today is : France - Germany