From kinetic to macroscopic models through local Nash equilibria

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Summary

1. Motivation
2. Nash equilibria vs kinetic equilibria
3. Hydrodynamics driven by local Nash equilibria
4. Wealth distribution
5. Conclusion

1. Motivation

Particles vs rational agents

Social or biological agents can be

mechanical particles subject to forces: kinetic theory

rational agents trying to optimize a goal: game theory

Our goal: try to reconcile these viewpoints

show that kinetic theory can deal with rational agents

incorporate time-dynamics in game theory

Applications:

Pedestrians with C. Appert-Rolland . . . & G. Theraulaz, JSP 2013 & KRM 2013, based on D. Helbing, . . . & G. Theraulaz, PNAS 2011

Social herding behavior with J-G. Liu & C. Ringhofer, JNLS 2014

Economics with J-G. Liu & C. Ringhofer, JSP 2014 and arXiv1403.7800

2. Nash equilibria vs kinetic equilibria

P. D., J-G. Liu, C. Ringhofer, J. Nonlinear Sci. 24 (2014), pp. 93-115

N players $j = 1, \dots, N$

Each player can play a strategy Y_j in strategy space $\mathcal{Y}$

The cost function of player j playing strategy Y_j in the presence of the other players playing strategy $\hat{Y}_j = (Y_1, \dots, Y_{j-1}, Y_{j+1}, \dots, Y_N)$ is $\phi_j(Y_j, \hat{Y}_j)$

Players try to minimize their cost function by acting on their strategy Y_j, not touching the others' strategies $\hat{Y}_j$

Nash equilibrium

Strategy $Y = (Y_1, \dots, Y_N)$ such that no player can improve on its cost function by acting on his own strategy variable

Y Nash equilibrium $\Longleftrightarrow$

$$\phi_j(Y) = \min_{Z_j} \phi_j(Z_j, \hat{Y}_j), \quad \forall j = 1, \dots, N$$

Best reply strategy

Describe behavior of the agents in time

Agents march towards the local optimum by acting on their own strategy variable assuming the other agents will not change theirs

$$\dot{Y}_j(t) = -\nabla_{Y_j} \phi_j(Y_j, \hat{Y}_j), \qquad \forall j = 1, \ldots, N$$

Add noise to account for mixed strategies

$$dY_j(t) = -\nabla_{Y_j} \phi_j(Y_j, \hat{Y}_j)\, dt + \sqrt{2d}\, dB_t^j, \qquad \forall j = 1, \ldots, N$$

Continuum of players

Anonymous game with a continuum of player

- Players with the same strategy cannot be distinguished
- Agents described by strategy probability distribution $dF(y)$

Non-atomic:

- $dF(y) = f(y)\,dy$ is absolutely continuous
- Cost function is $\phi(y; f)$

General framework of

- Non-Cooperative, Non-Atomic, Anonymous game with a Continuum of Players (NCNAACP)

  Aumann, Mas Colell, Schmeidler, Shapiro & Shapley
- Mean-field games Lasry & Lions, Cardaliaguet

The probability distribution f_{NE} is a Nash Equilibrium (NE) iff

$$\exists K \quad \text{s. t.} \quad \begin{cases} \phi(y; f_{NE}) = K, & \forall y \in \text{Supp}\, f_{NE}, \\ \phi(y; f_{NE}) \geq K, & \forall y \end{cases}$$

This is equivalent to the following "mean-field equation"

$$\int \phi(y; f_{NE})\, f_{NE}\, dy = \inf_f \int \phi(y; f_{NE})\, f\, dy$$

Distribution of players $f(y,t)$ satisfies kinetic eq.

$$\partial_t f - \nabla_y \cdot \big(\nabla_y \phi_f \, f\big) - d\Delta_y f = 0, \qquad \phi_f = \phi(\cdot; f)$$

Define: "collision operator" Q:

$$Q(f) = \nabla_y \cdot \big(\nabla_y \phi_f \, f\big) + d\Delta_y f$$

Kinetic Equilibria (KE) are solutions of $Q(f) = 0$

For a given potential $\phi(y)$, define Gibbs measure M_ϕ

$$M_\phi(y) = \frac{1}{Z_\phi} \exp\big(-\frac{\phi(y)}{d}\big), \quad \int M_\phi(y)\, dy = 1$$

Kinetic Equilibria

Write Q as

$$Q(f) = d\,\nabla_y \cdot \Big(M_{\phi_f} \nabla_y \big(\frac{f}{M_{\phi_f}} \big) \Big)$$

Implies:

$$\int Q(f)\,\frac{f}{M_{\phi_f}}\,dy = -d \int \Big| \nabla_y \big(\frac{f}{M_{\phi_f}} \big) \Big|^2 M_{\phi_f}\,dy$$

Theorem: f_{KE} Kinetic Equilibrium (and normalized, i.e. $\int f_{KE} = 1$) iff f_{KE} is a solution of the fixed point eq.

$$f = M_{\phi_f}$$

or equivalently $f_{KE} = M_{\phi_{KE}}$ with ϕ_{KE} a solution of the fixed point eq.

$$\phi = \phi_{M_\phi}$$

Let a NCNAACP - game be defined by the cost function

$$\mu_f = \phi_f + d \log f$$

Theorem: Suppose ϕ_f continuous; $\forall f$. Then, f_{KE} Kinetic Equilibrium associated to $Q(f)$ iff it is Nash Equilibrium of this game

Proof: "$\Rightarrow$": ϕ_f is locally finite $\forall f$. So,

$$M_{\phi_f}(y) = Z_{\phi_f}^{-1} \exp(-\phi_f(y)/d) > 0, \quad \forall y,$$

and,

$$\mu_{M_{\phi_f}} = -d \log Z_{\phi_f} = \text{Constant}, \quad \forall y.$$

So, if $f = M_{\phi_f}$, i.e. if $f = f_{KE}$ Kinetic Equilibrium then, it is a Nash Equilibrium for the game with cost function μ_f

Proof (cont): "$\Leftarrow$": Suppose $f = f_{NE}$ Nash Equilibrium. Then $f > 0$, $\forall y$. Otherwise $\exists y$ s.t. $f(y) = 0$ and $\mu_f(y) = -\infty \geq K$. Then $K = -\infty$ and $f \equiv 0$: contradiction with $\int f = 1$. Therefore, $\mu_f = K$, $\forall y$, which implies $f = M_{\phi_f}$, implying that f is a Kinetic Equilibrium.

Special case: potential games (Monderer & Shapley)

Suppose $\exists$ a functional $\mathcal{U}(f)$ s.t.
$$\phi_f = \frac{\delta \mathcal{U}}{\delta f}$$

Define free energy:

$$\mathcal{F}(f) = \mathcal{U}(f) + d \int f \, \log f \, dy.$$

Then, Cost function μ_f is a "Chemical potential":

$$\mu_f = \frac{\delta \mathcal{F}}{\delta f}$$

In general: $Q(f) = \nabla_y \cdot \left(f \, \nabla_y \, \mu_f\right)$

If potential game, leads to gradient flow:

$$\partial_t f = \nabla_y \cdot \left(\nabla_y \left(\frac{\delta \mathcal{F}}{\delta f}\right) f \right)$$

Free-energy dissipation:

$$\frac{d}{dt}\mathcal{F}(f) = -\mathcal{D}(f) < 0, \quad \mathcal{D}(f) = \int f \left| \nabla_y \left(\frac{\delta \mathcal{F}}{\delta f}\right) \right|^2 dy$$

We have the equivalence (i) $\Leftrightarrow$ (ii):

(i) f critical point of $\mathcal{F}$ subject to the constraint $\int f \, dy = 1$

(ii) f Nash equilibrium

Ground state, metastable equilibria, phase transition, hysteresis

3.Hydrodynamics driven by local Nash equilibria

P. D., J-G. Liu, C. Ringhofer, J. Nonlinear Sci. 24 (2014), pp. 93-115

Add configuration (aka "type") variable X_j (e.g. space)

Motion depends on both type X_j and strategy Y_j

$$\dot{X}_j = V(X_j, Y_j), \qquad \forall j = 1, \ldots, N$$

Cost function depends also on types $X = (X_j)_{j=1,\ldots,N}$

$$dY_j(t) = -\nabla_{Y_j} \phi_j(Y_j, \hat{Y}_j, X)\, dt + \sqrt{2d}\, dB_t^j, \qquad \forall j = 1, \ldots, N$$

Probability distribution depends on type x and strategy y:

$$f = f(x, y, t)$$

Satisfies space-dependent Kinetic Eq.:

$$\partial_t f + \nabla_x \cdot (V(x, y)\, f) - \nabla_y \cdot \big(\nabla_y \phi_f\, f\big) - d\Delta_y f = 0$$

with

$$\phi_f = \phi_{f(t)}(x, y)$$

Goal of this work:

Provide continuum model for moments of f wrt strategy y such as agent density $\rho_f(x, t)$ or mean strategy $\bar{\Upsilon}_f(x, t)$

$$\rho_f(x, t) = \int f(x, y, t)\, dy, \quad \rho \bar{\Upsilon}_f(x, t) = \int f(x, y, t)\, y\, dy$$

Mean-field game approach directly provides continuum eq.

Without Kinetic Eq. step

Relies on an optimal control approach within a finite horizon time $[0,T]$ and terminal data

$$-\partial_t \Upsilon - \nu \Delta \Upsilon + H(x, \rho, D\Upsilon) = 0, \quad \text{in} \quad \mathbb{R}^d \times (0,T),$$

$$\partial_t \rho - \nu \Delta \rho - \text{div}(D_p H(x, \rho, D\Upsilon) \rho) = 0, \quad \text{in} \quad \mathbb{R}^d \times (0,T),$$

$$\rho(x,0) = \rho_0(x), \quad \Upsilon(x,T) = G(x, \rho(T))$$

In this model

$H \sim$ cost function

$G =$ cost function for reaching target at terminal time T

ρ satisfies convection-diffusion in field determined by H

Υ acts as a control variable and satisfies backwards eq.

Main hypothesis of our work

Scale separation hypothesis

- Variation of strategy y much faster than that of type x
- Fast equilibration of strategy leads to slow evolution of type

Passage to slow time scale

Let ε ratio of time scales. Then

$$\varepsilon\Big(\partial_t f^\varepsilon + \nabla_x \cdot (V(x,y)\, f^\varepsilon)\Big) = \nabla_y \cdot \Big(\nabla_y \phi^\varepsilon_{f^\varepsilon}\, f^\varepsilon\Big) + d\Delta_y f^\varepsilon$$

Scale separation $\Rightarrow$ decoupling of slow and fast scales

$$\phi^\varepsilon_f = \phi_{\rho(x,t),\nu_{x,t}}(x,y) + \mathcal{O}(\varepsilon^2)$$

$$\rho(x,t) = \int f(x,y,t)\, dy, \quad \nu_{x,t}(y) = \frac{f(x,y,t)}{\rho(x,t)}$$

Leading order cost function ϕ only depends on the local density $\rho(x,t)$ and (functionnally) on the conditional probability $\nu_{x,t}$ conditionned on position and time being (x,t).

ϕ only depends on local quantities at position x

All non-local effects are contained in the $\mathcal{O}(\varepsilon^2)$

Macroscopic limit $\varepsilon \longrightarrow 0$

Kinetic Eq. with scale separation written as:

$$\partial_t f^\varepsilon + \nabla_x \cdot (V(x,y)\, f^\varepsilon) = \frac{1}{\varepsilon} Q(f^\varepsilon)$$

$$Q(f) = \nabla_y \cdot \big(f\, \nabla_y \phi_{\rho(x,t),\nu_{x,t}} + d\nabla_y f\big)$$

$$\rho(x,t) = \int f(x,y,t)\; dy, \quad \nu_{x,t}(y) = \frac{f(x,y,t)}{\rho(x,t)}$$

Using degree 1 homogeneity of Q, we write

$$Q(f) = \rho \mathcal{Q}_\rho(\nu), \qquad \mathcal{Q}_\rho(\nu) = \nabla_y \cdot \big(\nu\, \nabla_y \phi_{\rho,\nu} + d\nabla_y \nu\big)$$

Local Kinetic Equilibria: f s.t. $Q(f) = 0$

are of the form $f(x,y,t) = \rho(x,t)\nu_{KE,\rho(x,t)}(y)$ where $\nu_{KE,\rho}(y)$ is a solution of $\mathcal{Q}_\rho(\nu) = 0$, i.e.

$$\nu_{KE,\rho}(y) = Z^{-1}_{\phi_{\rho,\nu_{KE,\rho}}} \; \exp\Big(-\frac{\phi_{\rho,\nu_{KE,\rho}}}{d}\Big)$$

Continuity equation

We have $\int Q(f)\, dy = 0$

local conservation of the number of agents

Trading is so fast that the agents do not have time to move during one trading interaction

In Kinetic Theory: "1" is a "Collision Invariant"

Integrate Eq. wrt. y, take $\varepsilon \to 0$ limit and use equilibria

$$\partial_t \rho + \partial_x \big(\rho\, u(\rho)\big) = 0, \qquad u(\rho) = \int V(x,y)\, \nu_{\rho(x,t),KE}(y)\, dy$$

However, may $\exists$ more than 1 equilibria $\nu_{KE,\rho}$ for a given ρ

$\nu_{KE,\rho}$ may depend on other parameters

No general theory possible: requires a case by case study

4. Wealth distribution

P. D., J-G. Liu, C. Ringhofer, J. Stat. Phys., 154 (2014), pp. 751-780.

P. D., J-G. Liu, C. Ringhofer, Phil. Trans. Roy. Soc. A (to appear)

A model of conservative economy

Bouchaud & Mézard ; Cordier, Pareschi & Toscani ; Düring & Toscani

$$\partial_t f^\varepsilon + \partial_x \cdot (V(x,y)\, f^\varepsilon) = \frac{1}{\varepsilon} Q(f^\varepsilon)$$

$$Q(f) = \partial_y \left(f\, \partial_y \phi_{\nu_{x,t}} + d \partial_y (y^2\, f) \right)$$

$$\nu_{x,t}(y) = \frac{f(x,y,t)}{\rho(x,t)}, \quad \rho(x,t) = \int f(x,y,t)\, dy$$

Note: $y > 0$. Diffusion operator $\partial_y^2 (y^2\, f)$ associated to geometric Brownian motion (Bachelier)

Quadratic pairwise interaction potential (binary trading)

$$\phi_\nu(y) = \frac{\kappa}{2} \int (y-y')^2\, \nu(y')\, dy' = \frac{\kappa}{2} (y - \bar{\Upsilon}_\nu)^2, \quad \bar{\Upsilon}_\nu = \int \nu(y)\, y\, dy$$

$\bar{\Upsilon}_\nu =$ local mean wealth

Trading operator conserves wealth: $\int Q(f)\, y\, dy = 0$

Equilibria are parametrized by $\rho > 0$ and $\Upsilon > 0$:

$$f = \rho\, \nu_\Upsilon(y), \quad \nu_\Upsilon(y) = \frac{1}{Z_\Upsilon} \frac{1}{y^{\frac{\kappa}{2}+2}} \exp\big(- \frac{\kappa \Upsilon}{dy} \big)$$

Satisfy the equilibrium relation: $\bar{\Upsilon}_{\nu_\Upsilon} = \Upsilon$

Proof follows from a Poincaré inequality with Gamma distribution weight by Benaim & Rossignol

Are Nash equilibria for game associated to cost

$$\mu_\nu = (\kappa + 2d) \log y + \kappa \frac{\bar{\Upsilon}_\nu}{y} + d \log \nu$$

Have "fat" Pareto tails as $y \to \infty$

Macroscopic limit

Collision Invariant (CI)

Function $\psi(y)$ s.t. $\int Q(f)\,\psi\,dy = 0$, $\forall f$

The only CI are linear combination of 1 (mass) and y (wealth)

There are as many parameters (ρ, Υ) in the equilibrium as independent CI (1, y)

In the limit $\varepsilon \to 0$, leads to conservation eqs. for (ρ, Υ)

$$\partial_t \rho + \partial_x \big(\rho u_0(x; \Upsilon(x,t)) \big) = 0, \quad u_0(x;\Upsilon) = \int V(x,y)\, M_\Upsilon(y)\, dy$$

$$\partial_t (\rho \Upsilon) + \partial_x \big(\rho u_1(x; \Upsilon(x,t)) \big) = 0, \quad u_1(x;\Upsilon) = \int V(x,y)\, M_\Upsilon(y)\, y\, dy$$

Modern trading is trading with market rather than binary trading

Potential coefficients depend on market (i.e. $\nu_{x,t}$)

$$\phi_\nu(y) = \frac{1}{2} a_\nu y^2 + b_\nu y + c_\nu \sim \frac{a_\nu}{2}\big(y + \frac{b_\nu}{a_\nu}\big)^2 + c'_\nu$$

Define mean wealth $\bar{\Upsilon}_1(\nu)$ and variance $\bar{\Upsilon}_2(\nu) - \bar{\Upsilon}_1(\nu)^2$ by

$$\bar{\Upsilon}_1(\nu) = \int \nu \, y \, dy, \qquad \bar{\Upsilon}_2(\nu) = \int \nu \, y^2 \, dy$$

Choose: $$a_\nu = d \frac{\bar{\Upsilon}_2(\nu)}{\bar{\Upsilon}_2(\nu) - \bar{\Upsilon}_1(\nu)^2}, \quad b_\nu = -(1+\lambda) d \bar{\Upsilon}_1(\nu)$$

Trading frequency $a_\nu \nearrow$ when variance (market uncertainty) $\bar{\Upsilon}_2(\nu) - \bar{\Upsilon}_1(\nu)^2 \searrow$

Risk averse strategy

Note: $\int Q(f)\, y\, dy \neq 0$: no wealth conservation in trading

Same inverse gamma equilibria as before

$$\nu_\Upsilon(y) = \frac{1}{Z_\Upsilon} \frac{1}{y^{\lambda+3}} \exp\Big(- \frac{(1+\lambda)\Upsilon}{y} \Big)$$

ν_Υ satisfies: $\bar\Upsilon_1(\nu_\Upsilon) = \Upsilon$, $\bar\Upsilon_2(\nu_\Upsilon) = \big(1 + \frac{1}{\lambda}\big)\Upsilon^2$

Market uncertainty is $\lambda^{-1}\Upsilon^2$

How to find eq. for Υ ?

y is not a CI $\Rightarrow$ lacks a CI to close macroscopic system . . .

Answer: use Generalized Collision Invariant (GCI) concept

GCI $=$ CI which depends on (moments of) ν

Here GCI is: $\chi_{\bar\Upsilon_1(\nu)} = \frac{y^2}{2} - \bar\Upsilon_1(\nu) y$

Macroscopic limit

We have

$$\int Q(\nu^\varepsilon)\, \chi_{\bar{\Upsilon}_1(\nu^\varepsilon)}\, dy = 0$$

Then

$$\int \big(\partial_t(\rho^\varepsilon \nu^\varepsilon) + \partial_x \cdot (V(x,y)\, \rho^\varepsilon \nu^\varepsilon)\big)\, \chi_{\bar{\Upsilon}_1(\nu^\varepsilon)}\, dy = 0$$

And when $\varepsilon \to 0$

$$\int \big(\partial_t(\rho \nu_\Upsilon) + \partial_x \cdot (V(x,y)\, \rho \nu_\Upsilon)\big)\, \chi_\Upsilon\, dy = 0$$

Leads to a non-conservative eq. for evolution of Υ

Macroscopic system for local agent density ρ and mean wealth Υ is

$$\partial_t \rho + \partial_x \big(\rho u_0)\big) = 0,$$

$$\rho \partial_t \Upsilon + \frac{\lambda}{2\Upsilon} \partial_x(\rho u_2) - \lambda \partial_x(\rho u_1) - \frac{1-\lambda}{2} \Upsilon \partial_x(\rho u_0) = 0$$

$$u_k = u_k(x;\Upsilon) = \int V(x,y)\, \nu_\Upsilon(y)\, y^k \, dy$$

Remark: GCI concept first proposed in the context of herding model

D. & Motsch, Continuum limit of self-driven particles with orientation interaction, M3AS 18 Suppl. (2008) 1193-1215

5. Conclusion

Summary

Interplay between Kinetic Theory and Game Theory

- Best-reply strategy
- Nash equilibria are Kinetic equilibria of associated dynamics

Used this analogy to derive:

- large-scale evolution of system of agents
- subject to fast relaxation towards Nash equilibrium
- Hydrodynamic models of games

Application to wealth distribution

- Equilibria are inverse gamma distributions
- Parameters evolve through system of macroscopic equations
- Applied to non-conservative economy through GCI concept

Perspective

Development in other contexts of social dynamics

Comparisons with data in real-world applications

Rigorous proofs