

# Classifying subfactors and fusion categories

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## 1 Introduction

This workshop brought together mathematicians studying subfactors, fusion categories, and Hopf algebras. The majority of the participants were from Canada and the US, but we also had many international participants, including mathematicians from Argentina, Australia, Colombia, Denmark, France, Great Britain, Israel, Japan, and the Netherlands. Seven women and 33 men participated. Seven graduate students and four postdocs attended.

This is a very exciting time for these fields, as it's only recently been appreciated how many of the techniques and ideas from each field can be fruitfully applied in the others. Many new classification results have appeared in the last few years, and the change of perspective offered by thinking about old questions using the techniques of another field have suggested many new approaches, and new problems.

In order to facilitate collaborations between researchers in different areas, we scheduled a smaller number of research talks than in a typical conference, and set up focused discussion groups in the afternoons. After each morning of talks, we all came together to propose ideas for working groups for the afternoon, and then divided up the conference amongst a few smaller groups. Sometimes this groups worked as intensive tutorials led by an expert in one field, and other times the groups worked together on new problems. Some of the most exciting sections involved discussions of the material of a morning talk, with participants realizing new applications of the ideas in their own subject.

We really enjoyed organizing the conference in this format, and we highly recommend it to other organizers! It requires a certain amount of planning, direction from the organizers on the day, and care to ensure that everyone has an interesting group to join, but we think that the outcomes justify this extra work.

The remainder of this report consists of the following subsections:

- (1) Exciting new developments reported in the lectures
- (2) Advances made by small group discussions
- (3) Open problems

## 2 New developments reported in lectures

### 2.1 The Asaeda-Haagerup subfactor

One of the most exciting results discussed at the conference was the current work of Grossman-Izumi-Snyder on the Asaeda-Haagerup subfactor. Until this work, the Asaeda-Haagerup subfactor was among the least well understood subfactors of the currently known examples.

It was first constructed by Asaeda and Haagerup [AH99], by a very direct construction that does not seem very enlightening! The possibility of its existence was first noticed by Haagerup [Haa94], in the first project to exhaustively describe possible principal graphs for small index subfactors. Thus the Asaeda-Haagerup subfactor, along with the “extended Haagerup” subfactor, was considered quite exotic!

This new result of Grossman-Izumi-Snyder shows how the Asaeda-Haagerup subfactor is naturally associated with the family of  $3^G$  subfactors. These subfactors are natural generalizations of the Haagerup subfactor at index  $\frac{5+\sqrt{13}}{2}$ . A  $3^G$  subfactor has principal graph a spoke graph with all arms having length 3. The even bimodules consists of a group  $G$  of invertible bimodules together with one other  $G$ -orbit of simple bimodule. Such fusion categories were constructed by Izumi in [Izu01], and later, more were constructed by Evans-Gannon [EG11].

These  $3^G$  subfactors are not completely understood. While it seems likely that there will be an infinite family of  $3^G$  subfactors, they have only been constructed in specific cases for  $|G| \leq 19$ . Each construction requires solving a system of polynomial equations, which scales poorly as the size of the group increases.

In [GS12b], Grossman-Snyder found the quantum subgroups of the Haagerup  $3^{\mathbb{Z}/3}$  subfactor, i.e., all Frobenius algebra objects in the even half. They were able to determine the Brauer-Picard groupoid, whose objects are the Frobenius algebras, and whose morphisms are invertible bimodules in the category. In a subsequent article [GS12a], they studied the Brauer-Picard groupoid of the Asaeda-Haagerup subfactor, but were left with some open cases.

In this recent work, Grossman-Izumi-Snyder complete the open cases to show that the even half of the Asaeda-Haagerup subfactor is Morita equivalent to a generalized Haagerup category, obtained by a de-equivariantization of the even half of a  $3^{\mathbb{Z}/4 \times \mathbb{Z}/2}$  subfactor. Thus the Asaeda-Haagerup subfactor fits into the family of  $3^G$  subfactors together with (de-)equivariantizations and Morita equivalences.

Further, their work on the  $3^{\mathbb{Z}/4 \times \mathbb{Z}/2}$  subfactors allows them to completely describe the centre of the Asaeda-Haagerup subfactor, including its  $S$  and  $T$  matrices.

### 2.2 Computing centers of quadratic categories

A quadratic fusion category is a fusion category with a non-invertible simple object  $X$  such that every simple object is either invertible or is isomorphic to  $X$  tensored with an invertible object. Most known examples of fusion categories that do not come from finite groups or quantum groups are Morita equivalent to quadratic fusion categories. The quantum doubles, or Drinfeld centers, of quadratic fusion categories provide interesting examples of modular tensor categories.

These quantum doubles can be explicitly described, and invariants such as  $S$  and  $T$

matrices computed, using two ideas of Izumi. The first idea is that many quadratic fusion categories can be represented as categories of endomorphisms of (von Neumann algebra closures of) Cuntz algebras. The second idea is that the Cuntz algebra representations of these fusion categories allow for an explicit description of their tube algebras.

The tube algebra of a unitary fusion category  $\mathcal{C}$  is, as a vector space, the direct sum of the intertwiner spaces  $\text{Hom}(XY, YZ)$  where  $X$ ,  $Y$ , and  $Z$  range over representatives of the simple objects of  $\mathcal{C}$ . This vector space is then endowed with an associative multiplication and an involution. The minimal central projections of the tube algebra are in bijection with the simple objects of the quantum double, and the half-braidings and  $S$  and  $T$  matrices can also be described in terms of the tube algebra.

For a quadratic fusion category represented as endomorphisms of a Cuntz algebra, one can often choose a basis for the tube algebra where the intertwiners are words in the Cuntz algebra (together with labels for the intertwiner space), and the tube algebra multiplication is given by a combination of multiplications in the Cuntz algebra and endomorphisms of the Cuntz algebra. The structure constants for the tube algebra can thus be written down explicitly in terms of a certain basis. However, this description of the tube algebra does not reveal its simple summands or indicate a formula for finding matrix units.

To actually find matrix units for the tube algebra, one first looks at components of the tube algebra of the form  $\text{Hom}(g-, -g)$ , where  $g$  is invertible. Such components are easy to understand using the structure of the group of invertible objects. Then to understand components of the form  $\text{Hom}(X-, -X)$ , with  $X$  noninvertible, one first looks at the intertwiner spaces  $\text{Hom}(g-, -X)$  to find the projections of  $\text{Hom}(X-, -X)$  which have nontrivial intertwiners to projections in  $\text{Hom}(g-, -g)$  for invertible objects  $g$ . The remaining parts of  $\text{Hom}(X-, -X)$  are typically harder to describe and usually require some computer linear algebra. Once one has matrix units for the tube algebra, the half-braidings and modular data can be computed and analyzed.

Izumi worked out a number of examples in his seminal papers [Izu00, Izu01], including near-group categories and generalized Haagerup categories associated to groups of odd order. Further examples of these two types were computed and studied by Evans and Gannon [EG11, EG14], who also analyzed the modular data of these quadratic categories and found striking patterns and relations to modular data of finite groups. Their work persuasively argues that these quadratic fusion categories are not exotic but rather fit into larger families. More recently, Grossman-Izumi-Snyder announced that the even parts of the Asaeda-Haagerup subfactor are Morita equivalent to three quadratic categories, including a de-equivariantization of a generalized Haagerup category for a group of even order, which allowed for the computation of the quantum double by similar methods.

## 2.3 Rank finiteness for modular tensor categories

A long standing open conjecture in the study of modular tensor categories is rank-finiteness, namely that there are only finitely many equivalence classes of MTCs in each rank (the rank is simply the number of simple objects). A proof of this conjecture was announced shortly before the conference, by Bruillard-Ng-Rowell-Wang [BNRW13], and Eric Rowell gave the first public lecture on their argument at the conference. This lecture was then followed by an afternoon discussion section which went through some of the more technical details.

Unfortunately the bounds provided by this proof seem to be very large, and hard to make effective, but it is still extremely exciting to have a solution to this puzzle!

The method of proof relies heavily on modularity, but nevertheless a group at the conference had ideas for extending the result to spherical braided fusion categories.

It is tempting to dream of proving rank-finiteness at the level of fusion categories, as well!

## 2.4 A new sequence of subfactors from the $E_{N+2}$ quantum subgroup of $SU(N)$

Jones' index rigidity theorem classified the possible indices of  $II_1$  subfactors to the range

$$\{4 \cos^2(\pi/n) | n \geq 3\} \cup [4, \infty)$$

In the same article, he constructed an example with each allowed index. The subfactors arising from the discrete series in his article come from  $SU_q(2)$  at a root of unity. Soon afterward, Ocneanu announced the complete classification of subfactors with index less than 4, resulting in an ADE classification. In more detail, the principal graph of a subfactor with index less than 4 must be one of  $A_n$ ,  $D_{\text{even}}$ ,  $E_6$ , and  $E_8$ . Interestingly,  $D_{\text{odd}}$  and  $E_7$  do not occur. There is a unique subfactor with each allowed type  $A$  or  $D$  principal graph, and a pair of complex conjugate subfactors for both  $E_6$  and  $E_8$ .

This result can be understood as a classification of all quantum subgroups of  $SU(2)$ . A type I quantum subgroup of a fusion category  $\mathcal{C}$  can be thought of in two ways: a commutative Frobenius algebra object  $A \in \mathcal{C}$ , or equivalently the category of  $A$ -bimodules in  $\mathcal{C}$ . It is well known that finite-depth subfactors are in 1-to-1 correspondence with Frobenius algebra objects in unitary fusion categories, so quantum subgroups correspond to certain subfactors. For  $SU(2)$  at roots of unity, the non-trivial quantum subgroups exactly correspond to the  $D_{2n}$  and  $E_6$  and  $E_8$  subfactors.

In [Ocn02], Ocneanu gave the complete list of quantum subgroups of  $SU(3)$  and  $SU(4)$ . He also proved some general facts about quantum subgroups. There is always a  $D$ -series which arises from a  $\mathbb{Z}/n\mathbb{Z}$ -equivariantization corresponding to the dimension 1 representations. He also claimed that there are only finitely many exceptional quantum subgroups for any fixed  $SU(N)$ .

In his talk in the Wednesday morning session, Ostrik gave an expository talk in which he outlined the proof that there are only finitely many exceptional quantum subgroups for  $SU(2)$ . Of course, this is known from the classification of subfactors of index less than 4, but this proof could hopefully generalize to the higher  $SU(N)$ . The details have not yet been completely worked out, and it would be highly important and worthwhile for someone to do this. One of the most interesting and exciting small group discussions focused on extending this proof to  $SU(3)$  (see §3.2).

In recent work, Zhengwei Liu has constructed the  $E_{N+2}$  quantum subgroup of  $SU(N)$  for all  $N$ . Motivated by the recent classification article of Liu-Morrison-Penneys [LMP14]

which constructs subfactors with principal graphs

$$\mathcal{S} = \left( \begin{array}{c} \text{---} \bullet \begin{array}{c} \nearrow \bullet \\ \searrow \bullet \end{array} \bullet \text{---} \\ \text{---} \bullet \begin{array}{c} \nearrow \bullet \\ \searrow \bullet \end{array} \bullet \text{---} \end{array} \right), \left( \begin{array}{c} \text{---} \bullet \begin{array}{c} \nearrow \bullet \\ \searrow \bullet \end{array} \bullet \text{---} \\ \text{---} \bullet \begin{array}{c} \nearrow \bullet \\ \searrow \bullet \end{array} \bullet \text{---} \end{array} \right) \text{ or}$$

$$\mathcal{S}' = \left( \begin{array}{c} \text{---} \bullet \begin{array}{c} \nearrow \bullet \\ \searrow \bullet \end{array} \bullet \text{---} \\ \text{---} \bullet \begin{array}{c} \nearrow \bullet \\ \searrow \bullet \end{array} \bullet \text{---} \end{array} \right), \left( \begin{array}{c} \text{---} \bullet \begin{array}{c} \nearrow \bullet \\ \searrow \bullet \end{array} \bullet \text{---} \\ \text{---} \bullet \begin{array}{c} \nearrow \bullet \\ \searrow \bullet \end{array} \bullet \text{---} \end{array} \right).$$

using a ‘twisted’ version of a braiding, Liu gives a uniform construction for an infinite family of related subfactors.

Liu constructs these examples using a partial braiding, as is the case for quantum subgroups corresponding to commutative algebra objects. Moreover, he gives a complete classification of singly-generated Yang-Baxter planar algebras. A Yang-Baxter planar algebra is generated by 2-boxes such that one triangle can be simplified in terms of the other triangle and lower order terms.

## 2.5 Connections to conformal field theory

Subfactors and fusion categories are intimately connected to conformal field theory. It is conjectured that the infinite family of subfactors that Liu constructed also comes from an infinite family of conformal inclusions. Also, it is conjectured (by Kawahigashi) that every modular tensor category arises from conformal field theory. This is an important open question in the field.

Kawahigashi, together with Longo and Bischoff proved a conjecture due to Kong-Runkel. Starting with a non-local extension of a chiral CFT  $\mathcal{A}$ , one can construct a full CFT based on  $\mathcal{A}$  via boundary CFT, which gives a natural subfactor. One can also describe the non-local extension in terms of a non-commutative Frobenius algebra object, and perform the ‘full center’ construction to obtain another natural subfactor. Bischoff-Kawahigashi-Longo proved these subfactors are canonically isomorphic. Thus they showed that there is a bijective correspondence between Morita-equivalence classes of non-local extensions of  $\mathcal{A}$  and equivalence classes of full CFT’s based on  $\mathcal{A}$ .

## 2.6 Quantum Computation

Quantum computation is an exciting field of research, as it offers the tantalizing possibility of redefining ‘hard’ and ‘not hard’ problems in computer science. For example, Shor’s algorithm [Sho99] gives a way to factor integers in polynomial time using a quantum computer.

However, quantum computers are a long way from being able to do these calculations in practice: despite decades of research, the largest verified quantum computer ever built had only 14 qubits.

Quantum ‘qubits’ are hard to maintain for long enough to do even simple computations. While Shor, Steane, and Kitaev have independently discovered fault tolerance schemes for quantum computation, the decoherence rate in present implementations is still too high.

A proposed workaround to the fragility of quantum systems, due to Freedman and Kitaev, is to use topology to make the system robust to decoherence. The group at Microsoft Station Q is studying topological phases of matter in connection to developing a topological quantum computer.

Topological phases of matter are quantum systems whose ground state space has dimension greater than 1, and whose eigenvalues have a spectral gap that survives in the limit as the number of particles increases. The assignment of ground states to surfaces in this case is a 2-dimensional topological quantum field theory. Since the ground state has dimension greater than 1, quasiparticles/anyons can be manipulated within this ground state. They are protected from jumping to other states via the spectral gap.

To perform quantum computations, we start with a certain finite gate set of unitary  $2^n \times 2^n$  matrices for varying  $n$ s. A quantum circuit is a composite of tensor powers of elements of our gate set. We perform a quantum computation by starting with a state, applying the quantum circuit, and measuring the the final state. We arrange the circuit such that the probability of observing some particular state is close to one when the answer ought to be ‘yes’, and otherwise this probability is close to zero.

Now the idea behind topological quantum computing is that a certain braid of anyons should correspond to a quantum gate. There are analogous notions of tensor products and composition of braids of anyons. The computation is then performed by braiding the anyons to create the corresponding quantum circuit, and measuring the output.

We are immediately led to the question of which quantum gates and circuits are possible to realize. A gate set is called universal for quantum computation if we can approximate any unitary matrix as a composite of tensor products of elements from our gate set (with the number of gates polynomial in the precision of the approximation).

In the topological computation approach, the anyons correspond to simple objects in a unitary modular category (UMC). The unitary modular category is called universal for quantum computation if given an object  $x$ , the image of the braid group in  $\text{End}(x^{\otimes N})$  is dense in  $PSU(2^N)$  for all  $N$ . It is known that the Ising theory (the  $A_3$  UMC) is not universal, whereas the Fibonacci theory (the  $A_4$  UMC) is universal [FLW02].

In fact, the  $A_3$  UMC has Property (F): the images of the braid groups are finite for every object. This relates to an important conjecture of Rowell that Property (F) is equivalent to weak integrality, i.e., all objects have dimensions which are square roots of integers. Property (F) should be thought of as the polar opposite to universality.

## 3 Small group discussions

### 3.1 Cuntz algebras for beginners and intermediates

As mentioned above, Cuntz algebras play an important role in the construction of  $3^G$  subfactors, and a Cuntz algebra construction of a subfactor makes its center easier to compute. Gannon’s talk included some extremely hands-on examples of how to do computations with Cuntz algebras, and emphasized the point that there should be many, many more subfactors than we know about. He observed that every construction technique we know has allowed us to ‘shine a flashlight in one direction’ and see some new subfactors, but there are still many undiscovered ones waiting in the dark.

In light of this, there was enthusiasm among many participants to practice these Cuntz algebra construction techniques ourselves. This group began with a discussion, led by Izumi, about which subfactors one should expect Cuntz algebra techniques to succeed with; the

group then reproduced many calculations (which already appear in the literature) with the goal of understanding the role, in the subfactor, of each of the equations being solved.

### 3.2 Exceptional quantum subgroups of $SU(3)$

As stated before, Ostrik presented a proof that there are finitely many exceptional quantum subgroups of  $SU(2)$ . On one hand, this result due to Ocneanu is old. But the proof technique Ostrik presented was new to many experts in the field. During one of the afternoon sessions, about a dozen of us worked on extending the proof Ostrik presented that there are only finitely many exceptional quantum subgroups of  $SU(2)$  to the case of  $SU(3)$ .

The main hurdle to extending this proof was the fact that the fusion rules for  $SU(3)$  are more complicated than  $SU(2)$ . In more detail, as  $N$  grows, the fusion graph with respect to the standard representation forms an  $N$ -simplex, and writing down the explicit rules for fusion between any two objects becomes complicated.

Some participants were able to modify the technique to prove that ‘any non- $A_\infty$  subfactor with index less than 4.07 is at most 20-supertransitive’. This result follows already from classification techniques, and apparently cannot generalize to higher indices, but it is nevertheless exciting as it is the first direct bound on supertransitivity.

### 3.3 Weakly integral modular categories

During one of the break-out sessions at the workshop, it was suggested to study weakly integral modular categories, with the goal of classifying them in low rank. These are modular categories with integral categorical dimension, and include those categories whose objects have integer dimensions. An easy example is the Ising modular category, which has 3 simple objects with dimensions 1, 1 and  $\sqrt{2}$ . The group included Z. Wang, C. Galindo, E. Rowell, J. Plavnik and S.-H. Ng. At the end of the Banff workshop they had made some nice progress, including an idea of how to classify modular categories of dimension  $4p$  where  $p$  is prime. The working group had several more occasions to meet in 2014 (adding P. Bruillard to the group) and have now classified all weakly integral modular categories of rank 6 and 7, as well as all modular categories of dimension  $4m$  where  $m$  is odd and square-free. Integral modular categories of rank at most 7 are all pointed, with the first non-pointed example in rank 8 being the familiar representation category of the double of the symmetric group  $S_3$ . Building upon this work, the cases of ranks 8 and 9 have now been completed under the assumption that some object has non-integral dimension (so-called “strictly weakly integral”), by a slightly different group of researchers. The current goal is to show that all strictly weakly integral modular categories of rank at most 11 can be obtained via direct products of  $\mathbb{Z}_2$ -de-equivariantization of Tambara-Yamagami categories and pointed categories. There is an example in rank 12 that is not of this form, which motivates this goal.

### 3.4 Computing $S$ and $T$ matrices

Terry Gannon gave a talk on calculating the  $S$  and  $T$  matrices for the center of a fusion category, leveraging the representation theory of the modular group. After this talk, Terry Gannon, Scott Morrison, and David Evans began work on the center of the extended Haagerup

subfactor. This built on an article posted to the arXiv during the Banff conference by Scott Morrison and Kevin Walker [MW14], which used a combinatorial technique to prove that the center of the extended Haagerup subfactor has 22 simple objects.

### 3.5 Gannon’s new proof on the bound on the Frobenius-Schur exponent

Gannon pointed out a new way to bound the Frobenius-Schur exponent  $N$  of a modular category in terms of the rank  $r$ : look at the minimum dimension  $m$  of a faithful representation of  $SL(2, \mathbb{Z}_{p^k})$  (which is bounded below in terms of  $p^k$ ). If  $m > r$  then  $p^k$  does not divide  $N$ . This should improve the current best bound.

### 3.6 Various smaller groups

During the small group sessions, many mathematicians chose to collaborate in smaller groups. For instance, Brothier and Yamashita reported having a fruitful discussion on infinite depth subfactors, and Kawahigashi and Wang discussed modular tensor categories, anyons, and topological phases of matter.

#### 3.6.1 The generator conjecture for $3^G$ subfactor planar algebras

At this point, planar algebra techniques have not been as successful as Cuntz algebra techniques for analyzing quadratic categories. The first step to analyzing these  $3^G$  subfactor planar algebras is to find the formulas for the low weight rotational eigenvectors at depth 4 which generate the subfactor planar algebra. We know from [JP11, BP14, MP12] that the jellyfish algorithm can be used to construct such subfactors by finding them inside the graph planar algebra of the principal graph. However, one must know the formula for the generators in the graph planar algebra, and one must be able to compute the requisite 2-strand jellyfish relations. From the previous work [Jon12, Pet10, MP12, PP13], Penneys had conjectured a formula for the low weight rotational eigenvectors in the  $3^G$  subfactor planar algebra. Liu and Penneys proved this conjecture for the case  $|G|$  odd during the workshop.

#### 3.6.2 MTC with dimension $4p$

Shortly after the workshop, the group of Bruillard (not present at this conference), Galindo, Ng, Plavnik, Rowell, and Wang classified modular categories of dimension  $4p$  with  $p$  prime, generalizing  $p$  to any odd square-free number. They also extended the classification of weakly integral modular categories to rank 7.

## 4 Open Problems

- (1) Give a complete proof of Ocneanu’s theorem on finiteness of exception quantum subgroups of  $SU(N)$ .

- (2) Can the finite results for quantum subgroups of  $SU(N)$  be adapted to give supertransitivity bounds for subfactors? High supertransitivity subfactors appear to be very rare, yet our inability to control supertransitivity makes classifications very difficult.
- (3) (Kawahigahi) Does every unitary modular tensor category arise as the representation category of a conformal net?
- (4) Can one classify the quantum subgroups of  $U_q(\mathfrak{g})$  for simple Lie algebras  $\mathfrak{g}$  outside of type A? The exceptional group  $G_2$  in particular may be tractable [EP14].
- (5) Does weakly integral imply weakly group theoretical for fusion categories?
- (6) (Liu) What is the skein theory of subfactors from conformal inclusions, in particular, the ones in a family?
- (7) (Rowell) Which modular categories have trivial Galois group? This remains open, and seems hard.
- (8) (Rowell) Extend rank-finiteness to spherical braided fusion categories. (Plavnik, Galindo, Ng, Rowell, Wang and Bruillard have an approach to this.)
- (9) (Rowell) The Property F conjecture for braided fusion categories. This remains open.
- (10) What is the list of unitary fusion categories of rank 4? (The classification in the modular case is already known [RSW09].)
- (11) For how large of value of  $D$  can one classify all fusion rings with global dimension at most  $D$ ? ( $D \sim 12$  is certainly feasible, but it may be possible to go much further.)
- (12) The modular data for the Haagerup subfactor appears to be ‘grafted’ together out of simpler pieces [EG11]. Are there constructions making this precise?
- (13) The even part of the 4442 subfactor appears to be a ‘non-graded extension’ of  $RepA_4$ . (That is, it has the form  $\mathcal{C} \oplus \mathcal{M}$  where  $\mathcal{C} = RepA_4$  and  $\mathcal{M}$  is  $\mathcal{C}$  as a module over itself, but the tensor product structure on  $\mathcal{M}$  itself is more complicated.) Is there a construction making this precise?
- (14) Compute the center and the  $S$  and  $T$  matrices of the extended Haagerup subfactor.

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