COHOMOLOGICAL REALIZATIONS OF MOTIVES

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1 Overview of the Field

The subject of motives is arguably the most fascinating and far reaching goals of Grothendieck's legacy. Grothendieck's unifying vision had to do with what all cohomology theories have in common. Perhaps contrary to popular misconception, Grothendieck did propose an explicit candidate construction of a category of motives, as documented by Manin. The problem is that to move forward with his construction, one needed the full strength of number of conjectures (Grothendiek's standard conjectures (see [Kl])). The standard conjectures, as well as the related celebrated Hodge ([Lew1]) and Tate conjectures are still unresolved. But this has not stopped the progress towards constructing suitable candidate motivic cohomology theories meeting the original goal of Grothendieck's vision. Indeed the input of D. Quillen's higher K-theory K_m in algebraic geometry (generalizing Grothendieck's K_0) led to a natural candidate for motivic cohomology (absolute motivic cohomology). Although K_0 has a cycle theoretic description in terms of a cycle group (classical Chow groups), it was through the work of S. Bloch [Blo] that a cycle theoretic description of K_m in terms of higher Chow groups led to an explosion of activity on the subject. A key milestone in the development of motivic cohomology is Voevodsky's revolutionary construction of motivic cohomology, which incorporates the powerful homological machinery necessary to meet the original goal of Grothendieck. The most spectacular recent success of the idea of motives was Voevodsky's proof of the Milnor conjecture for which he received the Fields medal in 2002. More generally, Rost-Voevodsky have announced a proof of the Bloch-Kato conjecture, which was later written up by C. Weibel [We]. In both proofs, Voevodsky's triangulated category of "mixed" motives is an essential tool.

It seems that the (once almost completely conjectural) picture of motives is becoming more tangible. Until recently, even the existence of a good category of mixed motives was conjectural whereas now we have various constructions of such a category due to Voevodsky, Nori (unpublished), Levine, Hanamura, and others. That is, we are beginning a phase where things are concrete enough that motives can be used as a tool for answering questions rather than simply forming a language in which to express some of the most difficult unsolved conjectures in algebraic geometry and related areas.

As with many other great mathematical ideas, the reasons for studying motives morphed over time. Our best attempts to "calculate" motivic cohomology into something more earthly, viz., via cohomological realizations, is the subject of this workshop. Historically, the subject of regulators (a regulator being a "realization" of motivic cohomology), which encompasses cohomological realizations, and from the number theoretic side, began with the works of Dirichlet, Dedekind, and on the geometric side with the works of Abel and Jacobi. The geometric aspects would be later fortified by the monumental works of Weil and Griffiths. At a later stage, it was the more recent works of S. Bloch and A. Beilinson that elevated the subject of regulators to a whole new level. These other important applications of motives to problems in algebraic geometry have had a much broader impact reaching into algebraic number theory and representation theory via the Langlands' program. Furthermore, recent results exhibit a connection between motives, periods and physics, as for example seen from the collaborative works of S. Bloch and D. Kreimer.

The timeliness of this workshop fits in beautifully with a number of recent developments particularly on the subject of regulators. The Beilinson Hodge theoretic realizations of motivic cohomology, have been explicitly worked out in detail by M. Kerr, J. D. Lewis, S. Mueller-Stach [KLM], and together with recent developments by K. Kato and S. Usui on logarithmic Hodge structures, the techniques are in place for degeneration arguments as well. R. de Jeu and J. D. Lewis have found an interpretation of the recent Bloch-Kato theorem to a version of the Hodge conjecture for Milnor K-theory [dJ-L]. More recent works on Hodge realizations of Voevodsky's motivic cohomology (as well as real variants) are due to P. L. Filho and P. des Santos, and J. D. Lewis. Then there is the recent work by M. Walker, et al, on morphic cohomological realizations. One also has the p-adic syntomic regulator realizations, with some recent progress by A. Besser and M. Asakura. While it would be unrealistic to include the works of all the major contributors in the subject, no less significant are the recent works due to C. Deninger, W. Raskind, R. Sreekantan, C. Schoen, J.-L. Colliot-Thélène, P. Griffiths and M. Green, G. Pearlstein and P. Brosnan, et al, that are having a major impact on this subject.

2 Logistics, and the Overall Program

The choice of title of this workshop served as a broad enough umbrella to include a number of areas that interact with the subject of motives, such as regulators, Hodge theory [De] or number theory, some aspects of Physics connect to Calabi-Yau varieties, or that connected to Feynman integrals, motivic cohomology, as so forth. And yet we limited our scope so as to involve meaningful exchange amount the various groups involved here, given the limitation in time.

A number of guiding principles ensured the successfulness of this workshop, namely:

• We limited the number of talks per day (four), except for the half day (two). This allowed for adequate time for those to meet in break-a-way rooms for collaborations. Indeed, some of those collaborations were newly established, being a result of the talks (more on this later). Ironically, this did not result in denying anyone the opportunity to give a talk. For the most part, the senior people were more than happy to let our mid-career colleagues (some of whom are at the top of their fields!) present their results.

• Although there is a dearth of females working in this subject (or for that matter mathematics!), three of the female participants did indeed plan and give very interesting talks; one of which may further solidify some collaboration between J. Lewis and B. Kahn (see $\bullet_e$ below).

• By the nature of the organizing committee, this conference engaged a number of top rate colleagues of different nationalities (outside of N. America, those include hispanic (Spain/Mexico), Japan, China, India, Russia, Europe, and so forth). There was also a significant component from Alberta, due to some prejudices in hiring at the UofA. Another important point is that BIRS is within rather easy reach of UofA, making it very easy for candidates there to accept invitations.

• Every attempt was made to ensure that the participation at this conference would be maximal. Indeed, the organizers aggressively sought promising researchers to fill the vacant slots on our participant list.

• We generally like to open and close the conference with very good speakers. Bruno Kahn was the opening speaker, and we ended with Stefan Gille. Stefan Gille, being local (Alberta) turned out to be a natural choice, as many others had to leave earlier on Friday to catch a plane. Still, those many others wanted to attend his talk, albeit resulting in a reduced audience. Apart from all of this, attendance at each talk was excellent, which indicated a desire from various research groups of colleagues to learn what each other was doing. We also avoided concentrations of particular topics on any one day, allowing people to fully focus on a particular topic if they so desired, but also ensuring good cross-fertilization between topics.

3 Recent Developments and Open Problems

3.1 Recent developments.

To the layperson, a regulator can be thought of as a generalization of the logarithm. In its current incarnation, the precise definition is that it is a map from the K-theory of an algebraic variety to a cohomology theory. There are for example Betti realizations (related to the Hodge conjecture), l-adic, Hodge (such as in Beilinson-Deligne absolute Hodge cohomology), and so forth. This is well documented in [Ja2]. All such realizations can equivalently called regulators but with *a caveat*: our current understanding of motivic cohomology, as well as a correct definition of the category of mixed motives, is still lacking, and is included in our workshop. Indeed, besides Voevodsky, Nori has also presented a version of motivic cohomology. In both instances, how does one compute extension classes? As abstract as this sounds, the repercussions would be revolutionary (e.g., related to the kernel and image of the classical Abel-Jacobi map, and so forth...).

$\bullet_0$ (Number theory) The naive concept of a regulator being a generalization of the logarithm, is in fact typical of new developments in recent years. For example, Deninger's interpretation of Mahler measures in terms of real regulators involving mixed motives and expressed in terms of special values of polylogarithms, is a decisive development in number theory.

$\bullet_1$ (Physics) The interpretation of Feynman integrals in terms in regulators of cycles (periods) leading to multiple zeta values and polylogarithms, along the lines of Bloch, Kreimer, and more recently using the regulator formulas of Kerr-Lewis-Mueller-Stach [KLM], and Kerr-Lewis [K-L] and Kerr-Doran [D-K], has significantly had an impact in this area of Mathematical Physics. There is also the works by Griffiths-Green-Kerr [GGK], Doran, Pearlstein-Brosnan, Robbles, Usui-Kato on period domains in Hodge theory, and their impact on Calabi-Yau Geometry and Mirror symmetry (these domains serve as spaces capturing the images of variational regulators (normal functions) at the boundary.

$\bullet_1'$ (Period domains) Although mentioned in $\bullet_1$, it is a separate subject by itself, and has connections to $\bullet_2$ below.

$\bullet_2$ (Classical Hodge conjecture) Further to the previous paragraph is the work of Griffiths-Green on singularities of normal functions (variational regulators), based on a key observation of Richard Thomas, has led to a new and interesting re-interpretation of the classical Hodge conjecture[1]. Much of this has also been raised at a new level by the works of Pearlstein-Brosnan-Schnell [BPS].

$\bullet_3$ (Beilinson-Hodge conjecture) The formulation of the amended Beilinson Hodge conjecture has evolved significantly over the years by S. Saito-Asakura [A-K], Lewis-de Jeu [dJ-L], Lewis-de Jeu-Patel (to appear, revisiting [SJK-L]), Arapura [A-K] and can be viewed as a top down approach to the classical Hodge conjecture, as it involves all the higher K-groups of complex varieties. It not only can connections to the Bloch-Kato theorem, but these generalized Hodge conjectures play a deep role in the conjectures about the zero loci of cycle induced normal functions, particularly being invariant under an absolute Galois action.

$\bullet_4$ (Revisting the classical Hodge conjecture - arithmetic considerations) The works of Voisin over the past 10 years have made it abundantly clear that any conceivable generalization of the Hodge conjecture beyond the category of complex algebraic varieties, is *false*. If one were dealing with the category of compact complex manifolds, then the key ingredient of being projective algebraic is the notion of polarization (Kodaira). Having said this, something more elementary (viz., on an earthly level) is at work here, and was observed and exploited independently by many, including Nori, Bloch, Beilinson, Deligne, Esnault, Griffiths, Green, Lewis [Lew2], M. Saito, S. Saito, M. Asakura [A], Schoen, Kerr [K-L], Raskind, Voisin, *et al.*; that being the notion of spreads. Consider any complex scheme X (of finite type, etc.). Working with the coefficients of the polynomials defining X, one can spread X to a smooth family $\rho : \mathcal{X} \to S$ over a number field k for which

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[1] If X is a projective algebraic manifold, then there is a decomposition of singular cohomology $H^\bullet(X, \mathbb{Q}) \otimes \mathbb{C} = \oplus_{p+q=\bullet} H^{p,q}(X)$ into pieces depending only on the the complex structure of X. For an integer $r \geq 0$, the Hodge conjecture states that $H^{2r}(X, \mathbb{Q}) \cap H^{r,r}(X)$ is generated by the fundamental classes of algebraic cycles of codimemention r in X. An arithmetic variant of this is the celebrated Tate conjecture.

$\mathcal{X} \times_{k(S)} \mathbb{C} = X$. If X is complete, then one can work with a (conjectural) regular model of $\mathcal{X}$ over the ring of integers in k, which would involve the world of Arakelov geometry. This obviously leads one in the world of arithmetic geometry. Griffiths-Green, Lewis, Asakura-M. Saito, Lewis-Kerr, independently exploited this idea in terms of developing a candidate "Bloch-Beilinson filtration" on the K-groups of a complex projective variety X, which measures the "complexity" of these groups. This spread idea already served a precursor to the notion of *arithmetic* variations of Hodge structures, which was independently studied by Griffiths-Green, and the Japanese school, including M. Saito, S. Saito and M. Asakura. The meeting of these two schools occurred at the 1998 NATO conference in Banff; a pivotal point in the creation of BIRS.

In the late 1970's Deligne introduced the notion of absolute Hodge classes on Betti cohomology (as well as l-adic). With regard to projective algebraic manifolds $X = X/\mathbb{C}$, any element $\sigma \in \mathrm{Gal}(\mathbb{C}/\mathbb{Q})$ acts on the arithmetic de Rham cohomology of X in a functorial way (hence by the work of Serre, on classical de Rham cohomology), but $\sigma : X \to X^\sigma$ is not continuous, unless σ is either the identity or complex conjugation. Thus σ does not define a map on the level of $\mathbb{Q}$-Betti cohomology, albeit it does preserve algebraic cycles. The subspace of Hodge classes $= H^{2r}(X, \mathbb{Q}) \cap H^{r,r}(X)$ preserved under $\mathrm{Gal}(\mathbb{C}/\mathbb{Q})$ is called the space of absolute Hodge class. Deligne proved that X is an Abelian variety, then $H^{2r}(X, \mathbb{Q}) \cap H^{r,r}(X)$ is absolute Hodge. One conjectures that for any projective algebraic manifold X, all Hodge classess are absolute Hodge, that being implied by the classical Hodge conjecture. Using the idea of $\mathbb{Q}$-spreads of a proiective algebraic manifold X, together with an interpretation of Deligne's ideas in this context (involving a Noether-Lefschetz locus), and Deligne's global invariant cycle theorem, *Voisin showed that if every Hodge class on a projective algebraic manifold is absolute Hodge, then the classical Hodge conjecture can be reduced to those X of the form $X = X_0 \times \mathbb{C}$, where X_0 is defined over a number field.*

$\bullet_5$ (Beilinson/Bloch regulator to Deligne cohomology) In 1995 [Go], Goncharov provided an explicit description of the regulator of a projective algebraic manifold X to real Deligne cohomology, and more generally to integral Deligne cohomology as well. That there are often mistakes in Goncharov's work is no surprise to anyone in this field. For instance, as first observed by Lewis, his regulator to integral Deligne cohomology is clearly flawed. An amended version appeared in [KLM] involving polylogarithmic currents, where we tacitly acknowledged that his real regulator is correct. Indeed in Matt's thesis [K], it is proven that it would be correct if one adopted a cubical representation of the higher K theory of X. But the Goncharov regulator is defined simplicially, and as first discovered by Matt, there are serious issues about his formula , with consequences involving a number of works. A paper soon to be submitted by Matt, Lewis and Patrick (Matt's student) [BKLP] provides the definitive formula, and truly qualifies as a new development. (See the present highlights below.)

$\bullet_6$ (Zero locus of admissible normal functions associated to variations of mixed Hodge structures) The work of Cattani-Deligne-Kaplan [CDK] on the algebraicity of the Noether-Lefschetz locus of Hodge classes on a family of projective algebraic manifolds (associated to a variation of Hodge structure) represented a significant milestone towards evidence in favour of the Hodge conjecture. The extension of these results to variations of *mixed* Hodge structures, is both a significant and highly non-trivial result, due to Pearstein-Brosnan-Schnell [BPS]. This plays a role in the open problem $\bullet_b$ below.

$\bullet_7$ (Essential dimension and related topics) The work of Nikita Karpenko, et al., on incompressibility, and its application to essential dimension, which also interacts with the theory of motives, has significantly evolved over the last few years.

3.2 Open problems

$\bullet_a$ The original Griffiths program on using normal functions as a line of attack on the classical Hodge conjecture, has been generalized to the higher K-groups, vis-á-vis the Beilinson-Hodge conjecture, and beyond that, using a concept of arithmetic normal functions (See [K-L], [L]). Is there an analog of the Griffith-Green program in $\bullet_2$ involving singularities of normal functions to this generalized situation? (The extension of Voisin's program in $\bullet_5$ is already done [L].)

$\bullet_b$ Take a cycle-induced arithmetic normal function, where the cycle arises from the K theory of a smooth projective X/k, $k \subseteq \mathbb{C}$. Is the zero-locus of that normal function defined over $\overline{k}$? This involves ongoing work of Lewis-Pearlstein. Some evidence in support of this appears in [L].

$\bullet_c$ Given a choice of universal family of polarized K3 surfaces, it is shown in [CDKL], based on Kodaira-Spencer deformation theory, that the "imaginary" of the K_1 regulator (also called transcendental regulator) for a general K3 surface of Picard rank ≤ 19, is non-zero. The holy grail is the situation where a given algebraic K3 surface has maximal Picard rank 20, and hence where deformation theory cannot be applied. We suspect it is non-zero. The problem then is to prove or disprove this.

$\bullet_d$ The classical Hodge conjecture is known to be false if one replaces $\mathbb{Q}$-coefficients by $\mathbb{Z}$-coefficients. The first counter-example was due to Atiyah-Hirzebruch [A-H], involving the existence of torsion non-algebraic integral classes. This was fascinating result at that time, given the fact that in codimension one, the classical Hodge conjecture holds with $\mathbb{Z}$-coefficients (Lefschetz (1,1) theorem). Turning the clock ahead 60 years, we now have a better understanding of this situation. Indeed the Bloch-Kato theorem implies that all torsion is supported in codimension one. Thus in higher codimension, it is conceptually easier to understand the phenomena of torsion non-algebraic integral classes. Later, it was discovered by Kollar that non-torsion, non-algebraic integral classes also exist. In recent years Voisin, J.-L. Colliot-Thélène, et al, have studied this phenomena in more detail, as to when to anticipate an integral version holding. The problem then is to consider an analogue of this for the Beilinson-Hodge conjecture. Some work in this direction appears in [dJ-L], as well as the appendix provided by Asakura.

$\bullet_e$ The Griffiths group is a very important invariant associated to K_0 of projective algebraic manifolds. It was a landmark result appearing in [Gr], using his program on normal functions. Indeed it may still be the most important accomplishment using normal functions. Philosophically speaking, a similar object is the notion of K_1 indecomposable classes, which can also be derived from a generalized program in $\bullet_a$. One fantasy of Lewis-Kahn is that there should be a direct connection between these two groups. One possible route goes back to Bloch, and A. Collino [Co], involving the degeneration of a K_0 class in a family of varieties to a K_1 class (and one can go even further to degenerating to a K_2 class). We were reminded of this idea in Jaya Iyer's talk, and Bruno Kahn quickly jumped on this idea as well. The central problem then is to connect a series of invariants, beginning with the Griffiths group at the K_0 level, to higher K theory.

4   Presentation Highlights

Our first speaker, Bruno Kahn, discussed the generalized Hodge and Tate conjectures for products of elliptic curves. It should be pointed out that a Grothendieck amended version of a generalized Hodge conjecture (and subsequent Tate analog) has been known for a long time. These new results were well presented and were certainly appealing to those in Hodge theory, as well as those experts on the Tate side (Rob de Jeu, W. Raskind, Mao Sheng, M. Asakura,,...). Susama Agarwala's talk on graphical motives, is very much connected to the mixed Tate motive regulator calculations appearing in $\bullet_1$. Ravindra's talk is very much connected to Nori connectivity (see [P]), the import of which is a step in the direction of a weak Lefschetz theorem for Chow groups (K_0 case). Such a theorem would be a consequence of the famous Bloch-Beilinson conjecture on the injectivity of the rational regulator on K_0 of smooth varieties defined over number fields. At the present time, this conjecture remains elusive. Matilde Lalin gave a very appealing lecture on recent developments on Mahler measures, and implicitly indicated the role of Deninger's observation that the Mahler measures are explained in terms of regulators of periods. Matt Kerr presented joint work with Lewis on the simplicial regulator to $\mathbb{Q}$-Deligne cohomology, and pointed out the errors by Goncharov indicated in $\bullet_5$. Jaya Iyer's interesting talk on the degeneration of the Gross-Schoen algebraic cycle, is what led Kahn and Lewis to revisit the ideas in $\bullet_e$. Chuck Doran's talk, which is in the realm of $\bullet_1$, also has some underpinnings with $\bullet_c$. M. Asakura's talk also has similar connections to $\bullet_c$. Pearlstein's talk, which was very well received, pertains directly to the subject in $\bullet'_1$. Roy Joshua's talk dealt with t-structures, which goes back to Verdier's introduction of triangulated categories, and the problem of placing derived categories in a category-theoretic context. A partial solution to this problem, due to Beilinson, Bernstein and Deligne in the 1980's, was

to impose a t-structure on the triangulated category D. Mao Sheng is an expert in both Hodge theory in characteristic zero, and its analogues in positive characteristic. This was clearly reflected in his talk. The contact organizer (Lewis) has spent time with Mao in China, discussing the wide range potential of arithmetic techniques in Lewis's recent works on regulators. Pablo Pelaez's work deals with filtrations on Chow groups, for which there are many such possibilities. The basic problem is to measure the "complexity" of Chow groups, and this can only be achieved via filtrations, as reflected by the title of his talk. Goncalo Tabuada's talk was an interesting albeit highly technical invitation to non-commutative motives of separable algebras over a field k.

Phillipe Gille's talk, which is somewhat in the spirit of the research areas of Nikita and Stefan (below), dealt with certain homogeneous spaces X vover a field k for which $X(k) \neq \emptyset$.

The last two talks on Friday (given by our local colleagues in Edmonton), by Stefan Gille by Nikita Karpenko, attracted the interests of most participants, albeit a reduced audience due to travel itinerary issues. For this reason alone, we feel a need to mention their works in greater detail.

Nikita Karpenko's research deals with difficult and significant problems about algebraic cycles and motives of projective homogeneous varieties, their decompositions into simple pieces and their applications in the theory of algebraic groups. This was clearly evident in his well received talk on incompressibility of products of projective homogeneous varieties. A smooth projective variety X is called incompressible if every rational map from X into itself is dominant. The canonical dimension cdim X is a measure for the incompressibility of X. This positive integer is always smaller or equal than the dimension of X, and equal dim X if and only if X is incompressible. It is in general quite a task to show that a given variety is incompressible, or to compute the canonical dimension of it. However there exists a slightly weaker and more accessible notion, the so called p-incompressiblity, where p is a prime number, of a smooth and projective variety. This property is measured by the canonical p-dimension, which has been computed for many projective homogeneous varieties (for semisimple algebraic groups). All these results have strong implications on the cycles over such varieties, and applications to algebraic problems as for instance isotropy questions about algebras with involutions.

The last talk, by Stefan Gille, attracted the interests of K-theorists, on the Milnor-Witt groups of local rings. Milnor-Witt groups $K_n^{MW}(F)$, n an integer, of a field F show up as certain homotopy groups in Morel and Voevodsky's A^1-homotopy theory. Hopkins and Morel have found a nice presentation of these groups by generators and relations. This definition can be naively extended to local rings similar as the definition of Milnor K-Theory of fields extends to local rings. Morel has shown that for a field F of characteristic not 2 the n-th Milnor-Witt group of F is the pull-back of the nth Milnor K-group of F and the nth power of the fundamental ideal of the Witt ring of F over the nth Milnor K-group of F modulo 2. We have proven that the same holds for a regular local ring R which contains an infinite field of characteristic not 2. The proof uses a recent result of one of us which describes the nth fundamental power of the fundamental ideal in the Witt ring of such a ring. The proof of this presentation uses in turn a recent theorem of Panin and Pimenov on the existence of strictly isotropic vectors over such regular local rings. A corollary of our theorem is that the nth unramified Milnor-Witt group of such a regular local ring is equal the nth Milnor-Witt group of the ring for all integers n. This implies in particular that the unramified Milnor-Witt groups of smooth and proper schemes over an infinite field of characteristic not 2 are a birational invariant.

5 Scientific Progress Made

The contact organizer has initiated three projects, and invitations abroad (India, Japan, Mexico, N. America) as a direct result of this meeting. Judging by the interactions among other participants, one can well imagine a lot of new lines of research as a direct result of communication, and the interesting lectures that were presented. In particular several others initiated new connections/topics/invitations and discussions (there certainly was a lot of interaction between some of those seen more often at meetings in this field, and some that were perhaps more from other fields (Agarwala, Tabuada, for example). This is a great formula to produce good mathematics, and much of which can be expected to lead to fruition.

6 Outcome of the Meeting

The high intensity of this workshop, the broad range of participants approaching motives from various angles, and the good quality of the lectures, made this workshop a great experience, and a testament to the value of the BIRS facility; particularly with the backdrop of a bustling town and stunning scenery! Having said this, the real outcome of the meeting really lies in the future.

7 A Recommendation

The desire for mostly all participants to attend *all* talks was very apparent, and should be accommodated if feasible. Our main concern was the final day of the conference, where attendance was skewed due to travel itineraries. Perhaps an option is to stay overnight on the final day (viz., at a *reduced* cost, with or without meals) which could make a big difference.

8 Final Comment

The staff at BIRS were very professional, and certainly made our life agreeable at BIRS!

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