Positivity of Linear Series and Vector Bundles

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1 Positivity concepts for line and vector bundles

The main focus of the workshop was to study the existing positivity concepts of line bundles and vector bundles with an emphasis on the higher rank case.

There exists a highly developed and widely used theory of positivity for line bundles, which forms one of the fundamental building blocks of modern projective geometry. With the recent shift in emphasis towards birational geometry, there is a heightened interest for the behaviour of big line bundles — as opposed to ample ones, which describe the various birational models of the underlying variety. Both ample and big divisors have numerical, cohomological, and geometric characterizations, which makes for a quite satisfactory theory of positivity for linear series. Also the concept of nef divisors is well understood and commonly applied.

There have been several attempts to extend positivity concepts mentioned above to the case of higher rank vector bundles. In addition to ampleness, bigness and nefness there are various other concepts designed with vector bundles in mind, for instance Viehweg's weak semipositivity, Miyaoka's generic semipositivity and almost everywhere ampleness. These notions play a prominent role for example in the theory of moduli of higher-dimensional varieties, which is an active area of research with very significant recent progress due to Alexeev, Kollár, Kovács, and others.

It has recently come to light in the thesis of Kelly Jabbusch [2] that the two definitions of a big vector bundle put forward by Lazarsfeld and Viehweg, respectively, do not agree. This surprising but simple observation has opened the door to questions regarding the relationship between the various positivity concepts for vector bundles and their geometric consequences. The exploration of how the behaviour of positivity for line bundles extends or branches when considering higher-rank bundles was at the heart of the workshop activities. We explain them in more detail in the next section.

2 Various concepts of positivity for vector bundles

Let X be a projective variety and E a vector bundle on X. Let $\pi : \mathbb{P}(E) \to X$ be the projective bundle of one-dimensional quotients of E. Finally, let $\mathcal{O}_{\mathbb{P}(E)}(1)$ be the tautological quotient of π^*E. The following two notions are well established.

Definition 2.1. A vector bundle E is *ample* if $\mathcal{O}_{\mathbb{P}(E)}(1)$ is ample.
Similarly, we say that E is *nef*, if $\mathcal{O}_{\mathbb{P}(E)}(1)$ is so.

The next positivity notion for line bundles, the bigness does not generalize that easily. Finding the "correct" generalization for vector bundles is one of intriguing problems in this area. We recall several attempts, starting with the most obvious one.

Definition 2.2 (L-big, Lazarsfeld weakly big)**.** A vector bundle E is *L-big* if $\mathcal{O}_{\mathbb{P}(E)}(1)$ is big. In other words the function

$$m \to h^0(X, \operatorname{Sym}^m E)$$

has the maximal possible rate of growth.

A stronger property is the following.

Definition 2.3 (strongly big)**.** A vector bundle E is *strongly big* if it is L-big and

$$\pi(\operatorname{Bs}(\mathcal{O}_{\mathbb{P}(E)}(1))) \quad \text{is a proper subset of} \quad X,$$

where $\operatorname{Bs}(M)$ denotes the *base locus* of a line bundle M.

By definition strongly big implies L-big, but the converse implication is false in general.

Studying properties of cotangent bundles Miyaoka [6] introduced in passing the following notion.

Definition 2.4 (AEA, almost everywhere ample)**.** A vector bundle E is *almost everywhere ample* (AEA, for short), if there exists an ample line bundle A on X, a Zariski closed subset $T \subset \mathbb{P}(E)$, whose projection $\pi(T)$ is a proper subset of X, and a positive number $\varepsilon > 0$ such that

$$\mathcal{O}_{\mathbb{P}(E)}(1) \cdot C \geq \varepsilon \cdot \pi^*(A) \cdot C$$

for all curves $C \subset \mathbb{P}(E)$ that are not contained in T.

The next notion is due to Viehweg [8].

Definition 2.5 (V-big, Viehweg weakly positive)**.** A vector bundle E is *V-big* if for all ample line bundles A on X and all positive integers m the bundle

$$\operatorname{Sym}^{mk} E \otimes A^k$$

is generically spanned for $k \gg 0$.

One can characterize V-bigness by means of asymptotic base loci. We recall first the following basic notion.

Definition 2.6. The base locus of a vector bundle E over X is the set

$$\operatorname{Bs}(E) := \left\{ x \in X \mid H^0(X, E) \to E(x) \text{ is not surjective} \right\}.$$

There are the following two related notions.

Definition 2.7. Let A be an ample line bundle on X and let $A^\vee$ denote its dual. The set

$$\mathbb{B}_+(E) := \bigcap_{m \geq 1} \operatorname{Bs}(\operatorname{Sym}^m E \otimes A^\vee)$$

is the *augmented base locus* of E.

The set

$$\mathbb{B}_-(E) := \bigcap_{m \geq 1} \operatorname{Bs}(\operatorname{Sym}^m E \otimes A)$$

is the *diminished base locus* of E.

Then we have the following result.

Proposition 2.8. *A vector bundle E is V-big if and only if $\mathbb{B}_-(E)$ is contained in a proper closed subset $Y \subset X$.*

We conclude this section with another definition which in fact motivated the introduction of augmented and diminished base loci for line bundles.

Definition 2.9. Let E be a vector bundle on X. The set

$$\mathbb{B}(E) := \bigcap_{m \geq 1} \operatorname{Bs}(\operatorname{Sym}^m E)$$

is the *stable base locus* of E.

3 Relations between positivity concepts for vector bundles

It is clear that one is interested in relations between various notions introduced in the previous section. We point out a couple below.

Problem 3.1. Is AEA vector bundle strongly big and vice versa?

At present we don't know the answer in general. A series of examined examples suggests that the answer in Problem 3.1 might be positive. Discussions during the workshop led to verifying this claim under additional assumption E being nef.

Proposition 3.2. *Let E be a nef vector bundle. Then the following conditions are equivalent:*

i) E is AEA;

ii) E is strongly big.

Moreover the strong bigness can be in general characterized by the means of the augmented base locus.

Proposition 3.3. *For a vector bundle E the following conditions are equivalent:*

i) E is strongly big;

ii) $\mathbb{B}_+(E) \neq X$.

In the view of this characterization and more generally in order to understand relations between the positivity of a vector bundle E and of a line bundle $\mathcal{O}_{\mathbb{P}(E)}(1)$, it is of interest to study relations between their base loci. We have established the following.

Proposition 3.4. *Let E be a vector bundle. Then the stable base loci of E agrees with the projection of the stable base locus of $\mathcal{O}_{\mathbb{P}(E)}(1)$, i.e. there is the equality*

$$\mathbb{B}(E) = \pi(\mathbb{B}(\mathcal{O}_{\mathbb{P}(E)}(1))).$$

It is natural to ask further

Problem 3.5. Is there also the equality

$$\mathbb{B}_+(E) = \pi(\mathbb{B}_+(\mathcal{O}_{\mathbb{P}(E)}(1))).$$

for augmented base loci?

It is important to know the answer to that problem, in particular since Vojta's preprint [9] defines the augmented base locus of E as the image of the augmented base locus of $\mathcal{O}_{\mathbb{P}(E)}(1)$. However Vojta works under the assumption E being semistable, so his results should be taken with caution in the general framework.

We were able to establish only a weaker but related fact.

Proposition 3.6.

$$\mathbb{B}_+(E) \supset \mathbb{B}_+(\det(E)).$$

Turning to the diminished base locus, we were able to establish the following property paralleling the well known result for line bundles.

Proposition 3.7. *Let E be a vector bundle. The following conditions are equivalent:*

i) *E is nef;*

ii) $\mathbb{B}_-(E) = \emptyset$.

This implies of course that $\mathbb{B}_-(E)$ and $\mathbb{B}_-(\mathcal{O}_{\mathbb{P}(E)}(1))$ are simultaneously empty or non-empty. However, an analogue of Problem 3.5 for diminished base loci remains open.

Problem 3.8. Is there the equality

$$\mathbb{B}_-(E) = \pi(\mathbb{B}_-(\mathcal{O}_{\mathbb{P}(E)}(1))).$$

for diminished base loci?

Another problem which attracted our attention is to decide whether allowing $\varepsilon = 0$ in Definition 2.4 is equivalent to V-bigness.

Problem 3.9. Let E be a vector bundle on X. Let $T \subset \mathbb{P}(E)$ be a proper closed subset, whose projection $\pi(T)$ is a proper subset of X, and such that

$$\mathcal{O}_{\mathbb{P}(E)}(1) \cdot C \geq 0$$

for all curves $C \subset \mathbb{P}(E)$ that are not contained in T. Does this imply that E is a V-big vector bundle?

It is clear from the discussion above that there are so far more questions than answers and that the positivity of vector bundles will be an area of active research activities in the next future. It is also natural under these circumstances to study the above listed properties on specific families of varieties. Particular attention was devoted to toric vector bundles.

4 Toric vector bundles

Let X be a smooth complete toric variety. A *toric vector bundle* is a locally free $\mathcal{O}_X$-module E of finite rank equipped with a torus action that is compatible with the torus action on X. In other words there exists a torus action on the corresponding geometric vector bundle $V(E) := \mathrm{Spec}(\mathrm{Sym}\, E)$ such that the projection $\pi : V(E) \to X$ is torus equivariant and the torus acts linearly on the fibers of π.

If L is a line bundle on a toric variety, then various positivity properties of L can be read off directly from the convex geometry object, the rational polyhedron P_L associated to L.

In the case of a toric vector bundle E, there is naturally associated set of rational convex polytopes, called the *parliament of polytopes* for E. This is defined as follows. Let M denote the character lattice of the torus in X and Σ be the fan determining X. There is a finite set $\{v_i\}$ of uniquely determined minimal generators of the rays in Σ. We can view this set as the subset of $\mathrm{Hom}(M, \mathbb{Z})$. By Klyachko's Classification Theorem [3] there is a finite dimensional vector space E_0 associated to E equipped with decreasing filtrations

$$\ldots \supset E_0^{v_i}(j) \supset E_0^{v_i}(j+1) \supset \ldots$$

with $1 \leq i \leq n$ and $j \in \mathbb{Z}$. Then the polytopes P_e in the parliament are defined as

$$P_e := \{u \in M \otimes_{\mathbb{Z}} \mathbb{R} : \ \langle u, v_i \rangle \leq \max(j \in \mathbb{Z} : e \in E_0^{v_i}(j)) \text{ for all } 1 \leq i \leq n\}.$$

The role played by these polytopes in the study of positivity properties of a toric vector bundle E is expressed in the following result.

Proposition 4.1. *The lattice points in the parliament of polytopes for E correspond to a torus equivariant generating set of $H^0(X, E)$.*

Of course for a toric line bundle L everything reduces to the aforementioned polytope P_L.

Proposition 4.1 implies that base loci of toric vector bundles are torus invariant, which is very convenient for their study. As a sampling result relying on that (expected) observation we have the following.

Theorem 4.2. *A toric vector bundle is globally generated if and only if the associated characters correspond to lattice points in the parliament of polytopes and the vectors indexing these polytopes span E_0.*

A natural challenge coming out of the discussions preceding this section is the following.

Problem 4.3. Characterize on toric varieties various notions of positivity for vector bundles introduced in section 2 in terms of the parliaments of polytopes.

It is also interesting to compare positivity properties of line bundles and toric vector bundles. For example, contrary to the rank 1 case, there exist vector bundles which are ample but not globally generated or having an empty base locus (as defined in 2.6) and failing to be globally generated. There is a number of other properties which require more detailed study. In particular, the following problem, seems to us quite appealing.

Problem 4.4. Let E be a toric vector bundle on a smooth complete toric variety. Is then the section algebra

$$\bigoplus_{m\geq 0} H^0(X, \mathrm{Sym}^m(E))$$

finitely generated?

5 Newton-Okounkov bodies

Since the pioneering works of Lazarsfeld and Mustaţă on one hand and Kaveh and Khovanskii on the other, there are convex bodies $\Delta(D)$, the Newton-Okounkov bodies, associated to big linear series D on arbitrary normal projective varieties. These bodies are an interesting subject of study quite in their own right. There was a parallel workshop "Convex bodies and representation theory" held at BIRS at the same time. This circumstance has lead to interesting interchanges between both research groups. We shared a couple of lectured. Since the report from the other workshop will surely emphasize on the Newton-Okounkov bodies, we mention here just one path of thoughts particularly interesting from the positivity point of view.

Definition 5.1. A finite collection $\{D_1, \dots, D_r\}$ of pseudo-effective divisors on a smooth variety X is a *Minkowski basis* if

- For any pseudo-effective divisor D on X there exist non-negative numbers $a_1, \dots, a_r$ such that

$$D = \sum a_i D_i \quad \text{and} \quad \Delta(D) = \sum a_i \Delta(D_i);$$

- the Newton-Okounkov bodies $\Delta(D_i)$ are indecomposable in the sense of Minkowski sums.

Minkowski bases were first introduced by Łuszcz-Świdecka in [4] for del Pezzo surfaces. Shortly after Łuszcz-Świdecka and Schmitz [5] realized that using Zariski decompositions the construction carries over to arbitrary surfaces. The key point in this area is the question if Minkowski bases exist on a given variety and if they exist, if there is an explicit way to construct them. An effective algorithm for del Pezzo surfaces was described in [4]. It has been modified and successfully expanded to the case of toric varieties by Pokora, Schmitz and Urbinati [7].

During the workshop two new approaches were discussed. The first one concerns a decomposition of the pseudo-effective cone into *Minkowski chambers*. This decomposition is obtained as follows. If there is a Minkowski basis element D_i not contained in one of the rays spanning $\overline{\mathrm{Eff}}(X)$ then decompose $\overline{\mathrm{Eff}}(X)$ into subcones spanned by the sides of $\overline{\mathrm{Eff}}(X)$ and the ray spanned by D_i. Repeating this construction one arrives to the point where no Minkowski basis element lies in the interior of resulting subcones. Then, one can pass to a triangulation of each subcone into simplicial cones without having to add any new rays. The resulting subcones are Minkowski chambers of $\overline{\mathrm{Eff}}(X)$. It is natural to wonder how the decomposition into Minkowski chambers is related to the decomposition into Zariski chambers introduced in [1].

The second problem concerns the global Okounkov body. These objects are far more mysterious than the bodies associated to a single divisor. A general statement obtained in this direction is the following.

Theorem 5.2. *Let X be a smooth projective variety such that X admits a Minkowski basis $D_1, \dots, D_r$ whose corresponding Okounkov bodies, with respect to some fixed admissible flag $Y_\bullet$ are rational polyhedral. Then the global Okounkov body $\Delta(X)$ is rational polyhedral and it is spanned by the following set of vectors*

$$\bigcup \{(x, [D_i]) | x \text{ is a vertex of } \Delta(D_i)\} .$$

This result has a couple of nice consequences. For example

Corollary 5.3. *Let X be a toric variety. Then the global Okounkov body $\Delta(X)$ computed with respect to a torus invariant flag is rational polyhedral.*

It is worth to mention that all results listed above have been included in works which originated form the discussions during the workshop or at least were influenced by the workshop. The list of such papers is attached at the end of this report.

References

[1] Bauer, Th., Küronya, A., Szemberg, T.: Zariski chambers, volumes, and stable base loci, J. Reine Angew. Math. 576 (2004) 209-233

[2] Jabbusch, K.: Notions of positivity for vector bundles. Ph. D. thesis.

[3] Klyachko, A. A.: Equivariant bundles over toric varieties, Izv. Akad. Nauk SSSR Ser. Mat. 53 (1989), no. 5, 1001–1039, 1135 (Russian); English transl., Math. USSR-Izv. 35 (1990), no. 2, 337–375.

[4] Łuszcz - Świdecka, P.: On Minkowski decomposition of Okounkov bodies on a Del Pezzo surface, Annales Universitatis Paedagogicae Cracoviensis, Studia Mathematica 10 (2011), 105–115

[5] Łuszcz - Świdecka, P., Schmitz, D.: Minkowski decomposition of Okounkov bodies on surfaces, Journal of Algebra 414 (2014), 159–174

[6] Miyaoka, Y.: Algebraic surfaces with positive indices. Proc. Symp. Katata/Jap. 1982, Prog. Math. **39** (1983), 281–301.

[7] Pokora, P., Schmitz, D., Urbinati, S.: Minkowski decomposition and generators of the moving cone for toric varieties, preprint arXiv:1310.8505

[8] Viehweg, E.: *Quasi-projective moduli for polarized manifolds*, Springer 1995

[9] Vojta, P.: Big semistable vector bundles. Preprint, Berkeley 2004

6 Presentation Highlights

During the workshop we heard altogether 24 talks. 5 of them were held jointly with the parallel program on ”Convex bodies and representation theory”, partly by the participants from the other workshop.

Some presentations given in the workshop were coupled talks, so that the lectures had the possibility to go into details. This was important for further discussions.

Here we present the list of talks roughly divided in subjects they revolved around. The contest of the talks is described in the preceding sections.

6.1 Toric vector bundles.

Sandra Di Rocco, Kelly Jabusch and Greg Smith were speaking on that subject. They went in particular through the construction of parliament of polytopes.

6.2 Convex geometric aspects of positivity

There were talks of Klaus Altmann, Alex Küronya, David Schmitz and Tomasz Szemberg devoted to this topic. In particular Minkowski decompositions on surfaces have been discussed in details.

6.3 Asymptotic constructions

Ernesto Mistretta, Joaquim Roé and Xin Zhou gave talks related to this subject. In particular various concepts of base loci for vector bundles have been discussed.

6.4 Moduli spaces and stability

The talks of Arend Bayer, Aaron Bertram, Daniel Greb and Jack Huizenga were revolving around these concepts. The interest was focused in particular on birational geometry aspects of moduli spaces of higher rank sheaves.

6.5 Singularities

They played prominent role in talks of Alberto Chieccio, Sándor Kovács, Karol Palka, Mihnea Popa, Stefano Urbinati. In particular there were many different notions of positivity for Weil divisors discussed in detail.

6.6 Revision of established techniques and specific examples

These have motivated talks by Michael Brion, Brian Harbourne, Yusuf Mustopa and Giuseppe Pareschi. In particular there was focus on Ulrich sheaves and splitting of the cotangent bundle on certain types of varieties.

7 Outcome of the Meeting

The discussions initialized during the workshop have already lead to a number of interesting preprints. Some ideas are waiting for further investigations. Here is the list of recent preprints which resulted more or less directly from the workshop.

1. Sandra Di Rocco, Kelly Jabbusch, Gregory G. Smith: Toric vector bundles and parliaments of polytope, arXiv:1409.3109

2. Mihnea Popa, Christian Schnell: On direct images of pluricanonical bundles, arXiv:1405.6125

3. Izzet Coskun, Jack Huizenga: The ample cone of moduli spaces of sheaves on the plane, arXiv:1409.5478

4. Thomas Bauer, Sándor J. Kovács, Alex Küronya, Ernesto Carlo Mistretta, Tomasz Szemberg, Stefano Urbinati: On positivity and base loci of vector bundles, arXiv:1406:5941

5. Karol Palka: Cuspidal curves, minimal models and Zaidenberg's finiteness conjecture, arXiv:1405.5346

6. Karol Palka: The Coolidge-Nagata conjecture, part I, arXiv:1405.5917

7. David Schmitz, Henrik Seppänen: On the polyhedrality of global Okounkov bodies, arXiv:1403.4517

8. David Schmitz, Henrik Seppänen: Global Okounkov bodies for Bott-Samelson varieties, arXiv:1409.1857