Convex bodies and representation theory

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1 Introduction and Objectives of the workshop

The theory of toric varieties connects the combinatorics of convex integral polytopes with the geometry of toric varieties. In the case of a toric variety X, its associated polytope Δ coincides with its moment polytope (in the sense of symplectic geometry) and fully encodes the geometry of X, but this is not true in the general case. In ground-breaking work which was originated by Okounkov [17, 18] Kaveh-Khovanskii [8] and Lazarsfeld-Mustata [14] construct polytopes $\Delta(X, \nu)$ (called *Newton-Okounkov bodies* or *Okounkov bodies*) with $\dim_{\mathbb{R}} \Delta(X, \nu) = \dim_{\mathbb{C}} X$ associated to a projective variety $X \subseteq \mathbb{P}(V)$ and a valuation ν on its homogeneous coordinate ring, *even without the presence of any group action*. In fact, this construction in fact works in an even more general setting, associating convex bodies to *linear systems* on a projective variety. These Okounkov bodies carry interesting geometric information about X: for instance, one can prove a generalization of the Bernstein-Kushnirenko theorem to arbitrary varieties which relates intersection numbers of divisors with the volumes of the corresponding Okounkov bodies. This theory is still quite new (the foundational papers [14] and [8] are from 2008-2009) and the subject is both promising and still wide open. The fundamental question is: *What geometric data of X do the combinatorics of these Okounkov bodies encode, and how?*

There has been a burst of research activity surrounding Okounkov bodies since their introduction. It is already clear that these convex bodies are related to a wide range of research areas: for instance, Kaveh shows [7] that the *Littelmann-Berenstein-Zelevinsky string polytopes* from representation theory, which generalize the well-known *Gel'fand-Cetlin polytopes*, are examples of $\Delta(X, \nu)$. Also, recent work of Anderson [1] and Kiritchenko-Smirnov-Timorin [12] suggest many connections and possible applications to Schubert calculus. Furthermore, Harada and Kaveh [5] use a toric degeneration associated to $\Delta(X, \nu)$ to construct, in a very general situation, an integrable system on a projective variety; this opens the door to many applications in e.g. symplectic topology. We also mention that, at an MFO Mini-workshop on Okounkov bodies in August 2011, Victor Batyrev suggested that toric degenerations associated to Okounkov bodies may provide new methods for constructing *mirror pairs*, thus connecting this area also to mirror symmetry.

The main goal of this half-workshop was to bring together (1) researchers active in the area of Newton-Okounkov bodies and (2) mathematicians working in the closely related area of representation theory, particularly with a focus on combinatorial and convex-geometric techniques. By doing so, we were able to foster an active conversation in both directions; people working on Newton-Okounkov bodies gained new perspectives and ideas for future applications, and the researchers working in representation theory were introduced to the relatively new theory of Newton-Okounkov bodies.

As an additional bonus, we had an exciting synergy between our half-workshop and the concurrent half-workshop on "Positivity and linear series". We held a joint session with them on the Wedsnesday morning of our week-long workshop.

2 Schedule of the talks

We had the following schedule of talks.

Monday

9:00-10:00	Askold Khovanskii, *Convex bodies and representation theory, Part 1*
10:00-10:30	Coffee Break
10:30-11:30	Kiumars Kaveh, *Convex bodies and representation theory, Part 2*
11:30–13:00	Lunch
14:00-15:00	Susan Tolman, *Cohomology of quotients of Hamiltonian loop group actions*
15:00-15:30	Coffee Break
15:30-16:30	Nicholas Perrin, *Quantum K-theory of homogeneous spaces*

Tuesday

7:00–9:00	Breakfast
9:00-10:00	Valentina Kiritchenko, *Okounkov polytopes of Bott-Samelson varieties*
10:00-10:30	Coffee Break
10:30-11:30	Vladlen Timorin, *On the theory of coconvex bodies*
11:30–13:30	Lunch
13:30-14:30	Klaus Altmann, *Okounkov bodies and versal deformations of toric singularities*
14:30-15:00	Coffee Break
15:00-16:00	Boris Kazarnovskii, *Exponential sums: Kusnirenko–Bernstein theorem and convex polyhedra in complex space (old and recent results)*

Wednesday

Joint session:	*(with the parallel half-workshop "Positivity of linear series and vector bundles")*
9:00-9:30	Alex Kuronya, *Local positivity in convex geometric terms*
9:30-9:45	Break
9:45-10:15	Xin Zhou, *Asymptotic Schur decomposition of Veronese syzygy functors*
10:15-10:45	Coffee Break
10:45-11:15	Tomek Szemberg, *Minkowski decomposition of Okounkov bodies on surfaces*

Thursday

9:00-10:00	Jonathan Weitsman, *Integrable systems and Berenstein-Zelevinsky polytopes*
10:00-10:30	Coffee Break
10:30-11:30	Michel Brion, *On linearization of line bundles*
11:30–13:30	Lunch
13:30-14:30	Chris Manon, *Okounkov bodies and Kaveh-Harada construction for character varieties*
14:30-15:00	Coffee Break
15:00-16:00	Jaehyouk Lee, *Gosset polytopes and Del Pezzo surfaces*

Friday

9:00-10:00	Evgeny Smirnov, *Schubert calculus and Gelfand-Zetlin polytopes*
10:00-10:30	Coffee Break
10:30-11:30	Informal discussions

3 Presentation Highlights

The following are concise synopses of lectures given during the workshop.

Askold Khovanskii: *Convex bodies and representation theory, Part 1*

Khovanskii's was the first lecture of the workshop and served as Part 1 of an introductory series of 2 lectures on the background and context of the main themes of the workshop. In particular, Khovanskii focussed on Newton polyhedra theory. As a motivating question he presented the following. Suppose given a Laurent polynomial $p = \sum_m c_m x^m$ where $m = (m_1, \ldots, m_n)$ is an integer exponent vector, $x = (x_1, \ldots, x_n)$ are the variables, and the coefficients c_m are complex numbers. Let $\Delta(p)$ be the convex hull of $\{m : c_m \neq 0\}$; this is the so-called *Newton polytope* of p. Given a collection $\Delta(p_1), \ldots, \Delta(p_k)$ of such Newton polyhedra, consider the set of common solutions $X = \{p_1 = \cdots = p_k = 0\}$ in $(\mathbb{C}^*)^k$. The motivating question is: assuming the p_i are sufficiently generic, what invariants of X do the $\Delta(p_i)$ encode? Starting with this question and the 'first' answer along these lines (the Kushnirenko theorem, which deals with the case where all the Newton polytopes are equal, $\Delta(p_i) = \Delta \forall i$ and gives an answer in terms of the Euclidean volume of Δ, Khovanskii gave a broad historical overview of this subject. Topics touched upon included toric varieties, geometric genus, mixed Hodge numbers, f-vectors of simple polytopes, volume polynomials of polytopes, and Hilbert's 16th problem. At the end of the talk, Khovanskii motivated Part 2 of this series of introductory lectures by formulating the non-abelian analogue of the Kushrinenko theorem (i.e. replacing the abelian group $(\mathbb{C}^*)^n$ by a general reductive algebraic group G).

Kiumars Kaveh: *Convex bodies and representation theory, Part 2*

Building on the previous lecture ("Part 1"), Kaveh discussed the generalization to non-abelian groups of many of the themes discussed by Khovanskii. He chose as his starting point a result of Kazarnovskii, which can be interpreted as the non-abelian version of the Kushnirenko theorem. Let G be a reductive algebraic group. Let $\pi : G \to GL(N)$ be a representation and $f_1, \ldots, f_k$ sufficiently generic linear combinations of the matrix entries of π. The Brion-Kazarnovskii theorem then gives a formula for the cardinality of the set of common zeros $\{f_1 = \cdots = f_k = 0\} \subset G$ in terms of an integral over the so-called *weight polytope* Δ_{wt} of π, which is computed in terms of the set of irreducible representations V_λ appearing in the representation π. The main motivating question for Kaveh's talk was: Can we build a convex polytope $\tilde{\Delta}$ such that the integral over the weight polytope Δ_{wt} can be interpreted (more simply) as the Euclidean volume of $\tilde{\Delta}$? Can we also account for the multiplicities of the representations which occur in π? For some special cases, a clever answer was given by Okounkov in the 1990s. Motivated by Okounkov's results, Kaveh and Khovanskii recently gave a general construction of *Newton-Okounkov bodies*, which in this context are maximal-dimensional polytopes which account (asympotically) for not only the weights of the representation but also the multiplicities, and a basis for each irreducible representation. Kaveh gave a broad overview of these and related results.

Susan Tolman: *Cohomology of quotients of Hamiltonian loop group actions*

Tolman reported on her joint work with Raoul Bott and Jonathan Weitsman on the computation of the cohomology of quotients of Hamiltonian loop group actions. It is well-known that the theory of Hamiltonian group actions and symplectic quotients is intimately linked with representation theory through Borel-Weil theory and the "quantization commutes with reduction" theorem. One powerful technique in equivariant symplectic geometry is the Kirwan surjectivity theorem, which roughly states that given a Hamiltonian G-space (for G a compact Lie group) (M, ω), there is a natural surjection of cohomology rings $H^*(M) \to H^*(M//G)$ where $M//G$ denotes the symplectic quotient of M by the G-action. This theorem allows one to compute explicitly the cohomology rings of symplectic quotients. Tolman explained her joint work with Bott and Weitsman, which generalizes this Kirwan surjectivity to the case of Hamiltonian loop group quotients, i.e. to the situation when the (infinite-dimensional) loop group LG acts on a symplectic Banach manifold

$(\mathcal{M}, \omega)$, under some mild technical conditions. This is also related to the computation of the cohomology of quotients of quasi-Hamiltonian G-spaces.

Nicholas Perrin: *Quantum K-theory of homogeneous spaces*

Nicolas Perrin talked about his joint work with A. Buch, P.-E. Chaput and L. Mihalcea. Let X be a generalized flag variety with Picard groups of rank one. Given a degree d, they consider the Gromov-Witten variety of rational curves of degree d in X that meet three general points. In their joint work, they prove that the product in the small quantum K-theory ring of X is finite and has some positivity properties. One of the main issues is to prove rational connectedness of of the Gromov-Witten variety for all large degreed d.

Valentina Kiritchenko: *Okounkov polytopes and Bott-Samelson varieties*

Kiritchenko reported on her recent work which defines an elementary convex-geometric operation on polytopes which mimics the famous Demazure operators in representation theory and Schubert calculus. These operators are used to construct inductively polytopes that capture Demazure characters of representations of reductive groups. In particular, Gelfand-Zetlin polytopes and twisted cubes of Grossberg-Karshon are obtained in a uniform way. Kiritchenko gave an introduction to her operators, with many examples; in particular, she explained how her operators may be applied to a study of the Okounkov bodies of Bott-Samelson varieties.

Vladlen Timorin: *On the theory of coconvex bodies*

Timorin reported on recent joint work with Askold Khovanskii on the theory of coconvex bodies. If the complement of a closed convex set in a closed convex cone is bounded, then this complement minus the apex of the cone is called a coconvex set. Coconvex sets appear in singularity theory (they are closely related to Newton diagrams) and in commutative algebra. Such invariants of coconvex sets as volumes, mixed volumes, number of integer points, etc., play an important role. Timorin and Khovanskii's joint work aims at extending various results from the theory of convex bodies to the coconvex setting. These include the Aleksandrov-Fenchel inequality and the Ehrhart duality. Timorin gave a well-presented lecture introducing the subject and explaining the basic philosophy behind the theory, as well as the "main theorem" which shows how to prove theorems in co-convex geometry by interpreting co-convex bodies as virtual convex polytopes in the sense of Pukhlikov and Khovanskii.

Klaus Altmann: *Okounkov bodies and versal deformations of toric singularities*

Klaus Altman talked about his old and new works about the notion of versal deformation of toric singularities (i.e. those singularities occurring in toric varieties). The theory of toric varieties assigns to combinatorial objects (such as cones, fans or lattice polytopes) algebraic varieties. Using this construction, cones supported by lattice polytopes correspond to the germs of toric Gorenstein singularities. In a nice earlier paper, Altmann discusses their deformations and in the case in which the singularities are isolated, he gives a complete description of the versal deformation. The basic approach to understanding deformations of toric varieties is to split certain cross cuts of the defining cone into a Minkowski sum of specific polyhedra. In the case of toric Gorenstein varieties, one has to deal with the distinguished cross cut provided by the defining polytope. Its Minkowski summands are parametrized by a convex cone C which determines an affine toric variety itself. The pair consisting of C and the "universal" Minkowski summand C constitutes the material from which the versal deformation is finally built. Altman discussed ideas about possible extensions to a more general setup involving Newton-Okounkov bodies.

Boris Kazarnovskii: *Exponential sums: Kusnirenko–Bernstein theorem and convex polyhedra in complex space (old and recent results)*

Boris Kazarnovskii talked about his older work and some new results on extending the celebrated Bernstein-Kushnirenko theorem to certain classes of analytic functions such as sums of exponential functions (instead of polynomials which are sums of monomials). The Bernstein-Kushnirenko theorem was one of the main motivating results in toric geometry behind the development of the theory of Newton-Okounkov bodies. More specifically, in his talk Kazarnovskii considered the common zero set of n exponential sums in C^n. Because systems of equations he deals with are not algebraic, the number of their solutions can be infinite.

Kazarnoskii explained how he studies the asymptotics of the number of solutions within a ball, whose radius he then allows to go to infinity. It turns out that the Newton polyhedra of such exponential sums are responsible for these asymptotics. Much more surprisingly, Kazarnoskii is able to apply methods of modern Algebraic Geometry in the study of this purely transcendental problem.

Alex Kuronya: *Local positivity in convex geometric terms*

Kuronya's talk was both an introduction to the use of theory of Newton-Okounkov bodies in the context of the study of linear systems, and a report on Kuronya's recent joint work with Victor Lozovanu. Kuronya first gave a quick overview of the definition of the (Newton-)Okounkov body $\Delta_{Y_\cdot}(D)$ associated to a smooth projective variety X over $\mathbb{C}$, a Cartier divisor D on X, and an admissible flag $Y_\cdot$ of subvarieties in X. He reviewed what is known about explicit computations of the Okounkov body $\Delta_{Y_\cdot}(D)$ in simple cases, e.g. when X has (complex) dimension 1 and 2, and when X is a toric variety and D and $Y_\cdot$ are T-invariant. He explained the main philosophy behind Okounkov-body theory within the context of the study of linear systems, namely, that Okounkov bodies provide a universal family of numerical invariants for D: if D and D' are such that for any admissible flag $Y_\cdot$ of subvarieties, $\Delta_{Y_\cdot}(D) = \Delta_{Y_\cdot}(D')$, then D and D' are numerically equivalent. So roughly speaking, the slogan is that Okounkov bodies encode the numerical invariants of D. As an instance of this philosophy, his recent joint work with Lozovanu shows that D is *nef* if and only if for all $x \in X$, there exists an admissible flag $Y_\cdot$ centered at x such that the origin is contained in $\Delta_{Y_\cdot}(D)$. In his talk Kuronya explained this result and gave a sketch of the proof.

Xin Zhou: *Asymptotic Schur decomposition of Veronese syzygy functors*

Xin Zhou discussed his recent results and on-going work join with Mihai Fulger. The syzygies of the d-th Veronese embedding of $\mathbb{P}(V)$ are functors of the complex vector space V. Xin Zhou obtains results about the asymptotic behavior of the Schur functor decomposition of these as d grows. Their result shows that, from a certain perspective, this decomposition is very rich whenever they are not zero. This is deduced from an asymptotic study of related plethysms.

Tomek Szemberg: *Minkowski decomposition of Okounkov bodies on surfaces*

Tomek Szemberg reported on decomposing Okounkov bodies on surfaces with rational polyhedral effective cone into Minkowski sums of some elementary "building bricks". This builds upon recent work of Patrycja Luszcz-Swidecka and David Schmitz (arXiv:1304.4246), where they prove that the Okounkov body of a big divisor with respect to a general flag on a smooth projective surface whose pseudo-effective cone is rational polyhedral decomposes as the Minkowski sum of finitely many simplices and line segments arising as Okounkov bodies of nef divisors.

Jonathan Weitsman: *Integrable systems and Berenstein-Zelevinsky polytopes*

In their influential work, Berenstein-Zelevinsky introduced polytopes whose number of integral points computes the tensor product multiplicities. In his talk, Jonathan Weitsman discussed his work in progress with Lisa Jeffrey and Paul Selick (also with his student Gouri Seal) on constructing integrable systems whose moment map images are the Berenstein-Zelevinsky polytopes. While it is expected that such integrable systems exist (e.g. form the work of Harada-Kaveh) he emphasized that even in the smallest examples such as $SU(3)$ it is not easy to explicitly construct one.

Michel Brion: *On linearization of line bundles*

Michel Brion talked about his recent work on the linearization of line bundles and the local structure of algebraic group actions in the setting of seminormal varieties equipped with an action of a connected linear algebraic group G. He shows that several classical results about normal G-varieties extend to that setting, if the Zariski topology is replaced with the étale topology.

Chris Manon: *Okounkov bodies and Kaveh-Harada construction for character varieties*

Chris Manon talked about constructing families of Newton-Okounkov bodies for the free group character varieties and configuration spaces of any connected reductive group G. The character variety $X(\pi, G)$ associated to a finitely generated group π and a connected reductive group G is defined to be the moduli space of representations of π in G up to inner automorphisms. When π is taken to be the fundamental group of a smooth manifold M , $X(\pi, G)$ is the moduli space of flat, topological principal G bundles on M. His work is related to constructing toric degenerations and integrable systems for character varieties and is an important example of the general approach of Harada-Kaveh for constructing integrable systems on a large class of projective varieties (via toric degenerations).

Jaehyouk Lee: *Gosset polytopes and Del Pezzo surfaces*

Jaehyouk Lee from Ewha Women's University in Korea talked about his past work on the correspondences between the geometry of del Pezzo surfaces and the geometry of corresponding Gosset polytopes. In his talk he introduced main concepts involved such as the definition of a Gosset polytopes. He explained different results on how to read off information about the surface in particular divisor classes in the Picard group from the Gosset polytope. The corresponding leads, for example, to an understanding of Gieser transformations and Bertini transformations on the del Pezzo surface in terms of the symmetry of the Gosset polytope.

Evgeny Smirnov: *Schubert calculus and Gelfand-Zetlin polytopes*

Smirnov reported on joint work with Valentina Kiritchenko and Vladlen Timorin which proposes a new approach to the Schubert calculus on complete flag varieties using the volume polynomial associated with Gelfand-Zetlin polytopes. This approach allows us to compute the intersection products of Schubert cycles by intersecting faces of a polytope. One of their main tools is a construction of Pukhlikov and Khovanskii, which associates to a convex polytope P a graded commutative ring R_P (called the *polytope ring*). In the case of a smooth toric variety X and its associated polytope P, it is known that the polytope ring R_P is isomorphic to the cohomology ring $H^*(X, \mathbb{Z})$ of the toric variety, and that the product structure in $H^*(X, \mathbb{Z})$ is encoded by the intersections of faces of P. It was observed by Kaveh that the polytope ring of the Gel'fand-Zetlin polytope P_{GZ} is isomorphic to the cohomology ring of the flag variety $GL(n, \mathbb{C})/B$), which is additively generated by Schubert classes $[X_w]$. Motivated by this, Smirnov, Kiritchenko, and Timorin ask: is there an assignment to each $[X_w]$ a (linear combination of) faces $\mathcal{F}_w$ of P_{GZ} in such a way that multiplication of two such classes $[X_w] \cdot [X_v]$ in the cohomology ring corresponds to taking intersections $\mathcal{F}_w \cap \mathcal{F}_v$? The answer is yes, and involves so-called "reduced Kogan faces" of the Gel'fand-Zetlin polytope. In his talk, Smirnov carefully explained how this works, with many pictures, in the case of $n = 3$.

All lectures were well presented and invited further discussion and collaboration among the workshop participants. Some of this happened already at the workshop itself and we trust that the discussion will continue well after the workshop.

4 Developments and scientific progress

During the workshop, there were many promising informal discussions between the participants. We identify some of them here.

- Jonathan Weitsman and Chris Manon realized they are working on very similar problems/ideas regarding constructing integrable systems on varieties appearing in representation theory. In particular, Chris Manon mentioned to Jonathan the connection between their research and the recent results of Harada and Kaveh and they have begun ongoing discussions on this topic.
- Kiumars Kaveh, Chris Manon and Henrik Seppanen started a collaboration on geometric invariant theory and Newton-Okounkov bodies. Seppanen has already posted two preprints on the ArXiv related to these ideas.
- Kiumars Kaveh and J. B. Carrell collaborated on a paper in preparation on lifting the Springer action to equivariant cohomology.
- Kevin Purbhoo and Michel Brion had several discussions on topics related to Brion's talk at the workshop.
- Kiumars Kaveh and Askold Khovanskii collaborated on a paper in preparation on mixed multiplicities of ideals (related to local Newton-Okounkov bodies).

- Valentina Kiritchenko and Kiumars Kaveh discussed some future problems e.g. related to a recent result of Igor Makhlin in Moscow about projections of Gelfand-Zetlin polytopes and character formula for reductive groups.

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