Application of the Implicit Particle Filter to a Model of Nearshore Circulation

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Abstract

- We apply the implicit particle filter to a model of nearshore circulation
- This is a model with $\approx$ 30,000 state variables.
- We assimilate gridded observations of the two horizontal velocity components
- In the implicit particle filter the trajectory of each particle is informed by observations.
- In its simplest form, the implicit particle filter reduces to the method of optimal importance sampling.
- The system runs efficiently on a single workstation

A Shallow Water Model of Nearshore Circulation

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Representer-based variational data assimilation in a nonlinear model of nearshore circulation

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Why did I choose this model?

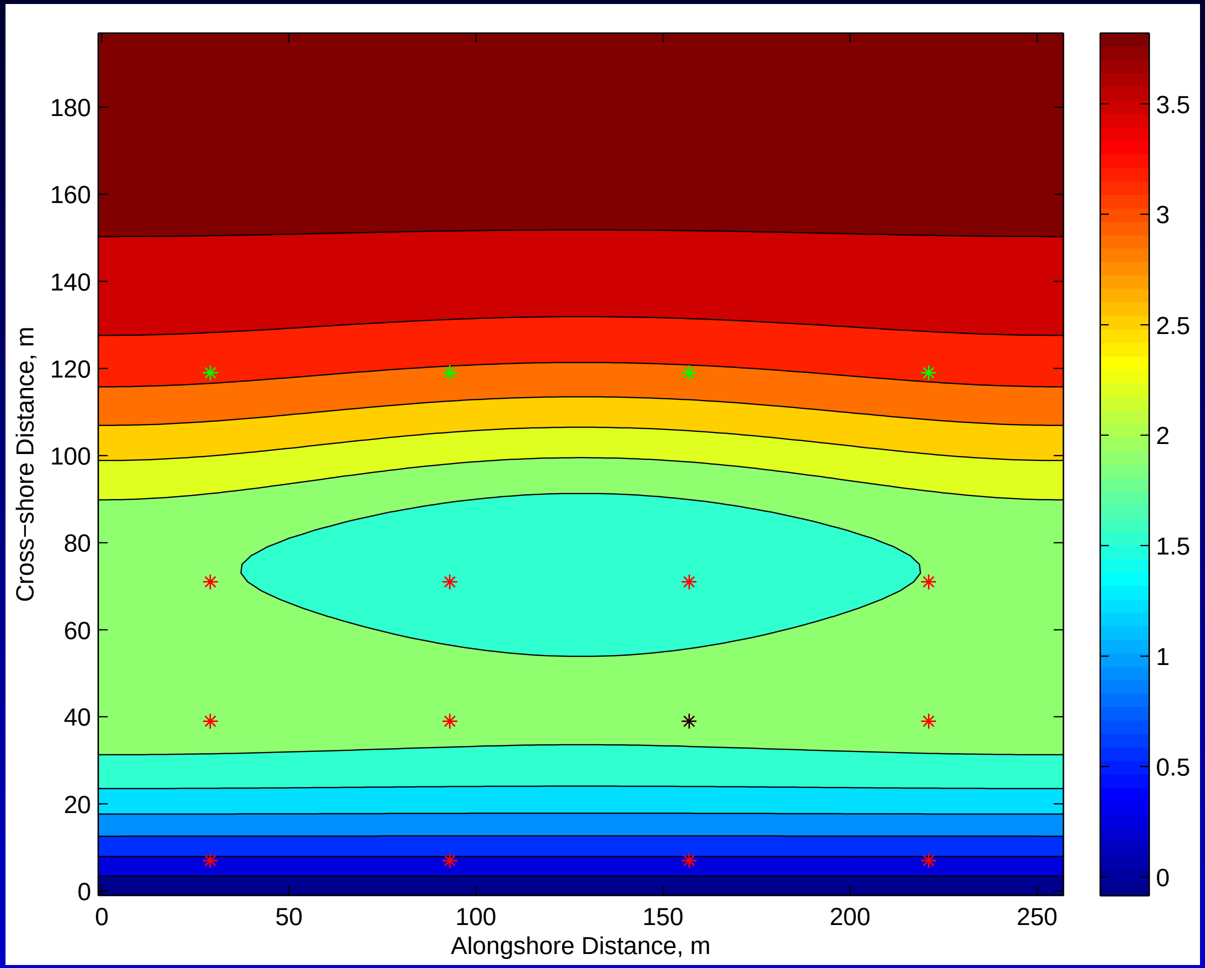
- It’s a highly nonlinear model with large state dimension
- Kurapov et al. described problems with application of 4DVAR to this problem that bear investigating

A Shallow Water Model of Nearshore Circulation

- Shallow water, forcing by parameterized wave breaking

$$
\begin{aligned}
\frac{\partial \zeta}{\partial t} &+ \frac{\partial (Du)}{\partial x} + \frac{\partial (Dv)}{\partial y} = 0 \\
\frac{\partial (Du)}{\partial t} &+ \frac{\partial (Duu)}{\partial x} + \frac{\partial (Dvu)}{\partial y} = -gD\frac{\partial \zeta}{\partial x} + \\
&f_X - ru - a\nabla^2(H\nabla^2 u) \\
\frac{\partial (Dv)}{\partial t} &+ \frac{\partial (Duv)}{\partial x} + \frac{\partial (Dvv)}{\partial y} = -gD\frac{\partial \zeta}{\partial y} + \\
&f_Y - rv - a\nabla^2(H\nabla^2 v)
\end{aligned}
$$

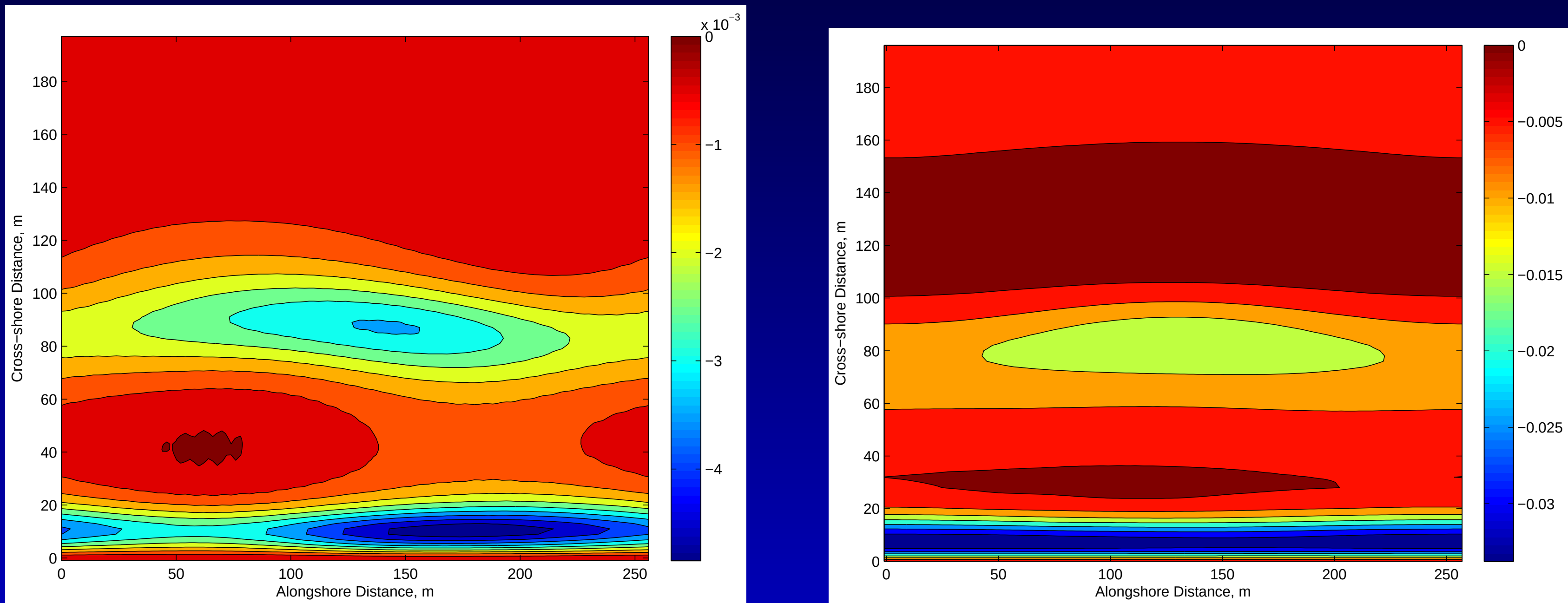
Model Domain



Periodic in x. Rigid walls at $y = 0, 200m$.
Stars show observation locations. Vector momentum assimilated at starred locations

Steady Momentum Forcing

Derived from parameterized wave breaking, Thornton & Guza, JGR 1983

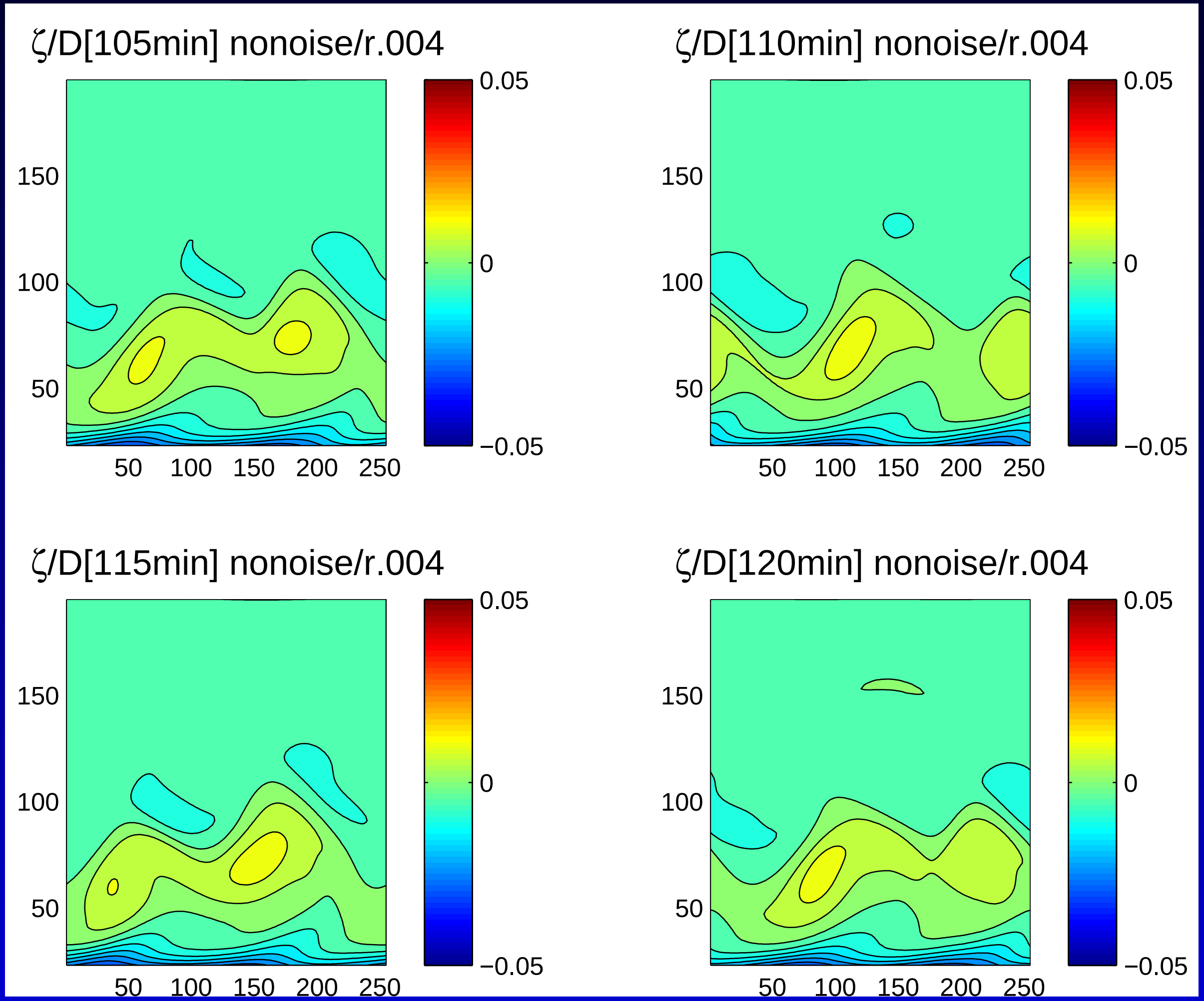


Alongshore (left) and Cross-shore (right) forcing, m^2/s^2

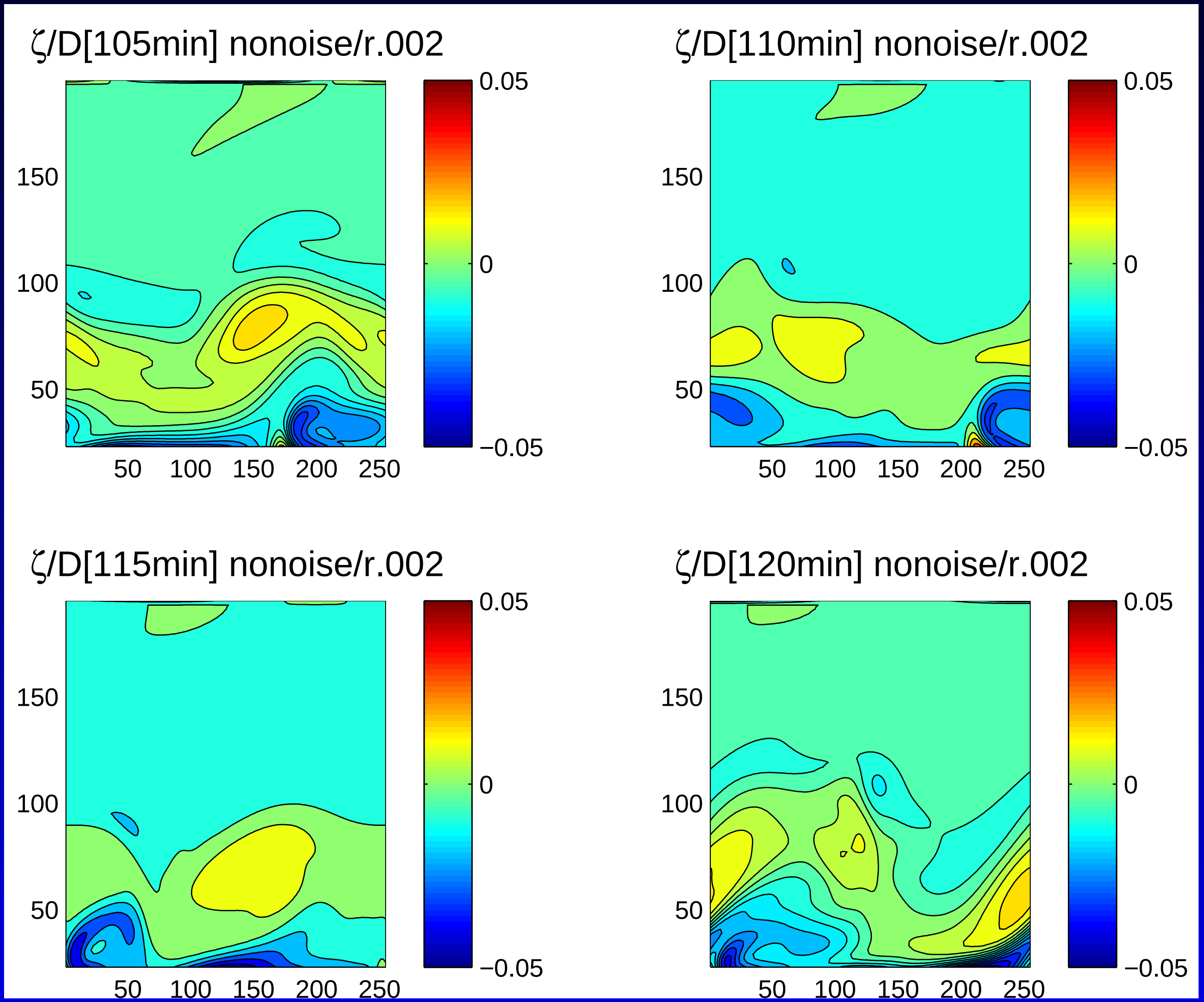
Why Did I Choose This Model?

- The linearized system is unstable, and the calculations blow up in a time comparable to the assimilation cycle
- Interesting behavior of 4DVAR
- Two distinct cases are considered:
  - High drag case, regular wavelike flow
  - Low drag case, aperiodic flow
- Assimilation fails for low drag case with assumption of steady forcing
- Must use *incorrect* assumption of unsteady forcing to get a solution to 4DVAR in the low drag case

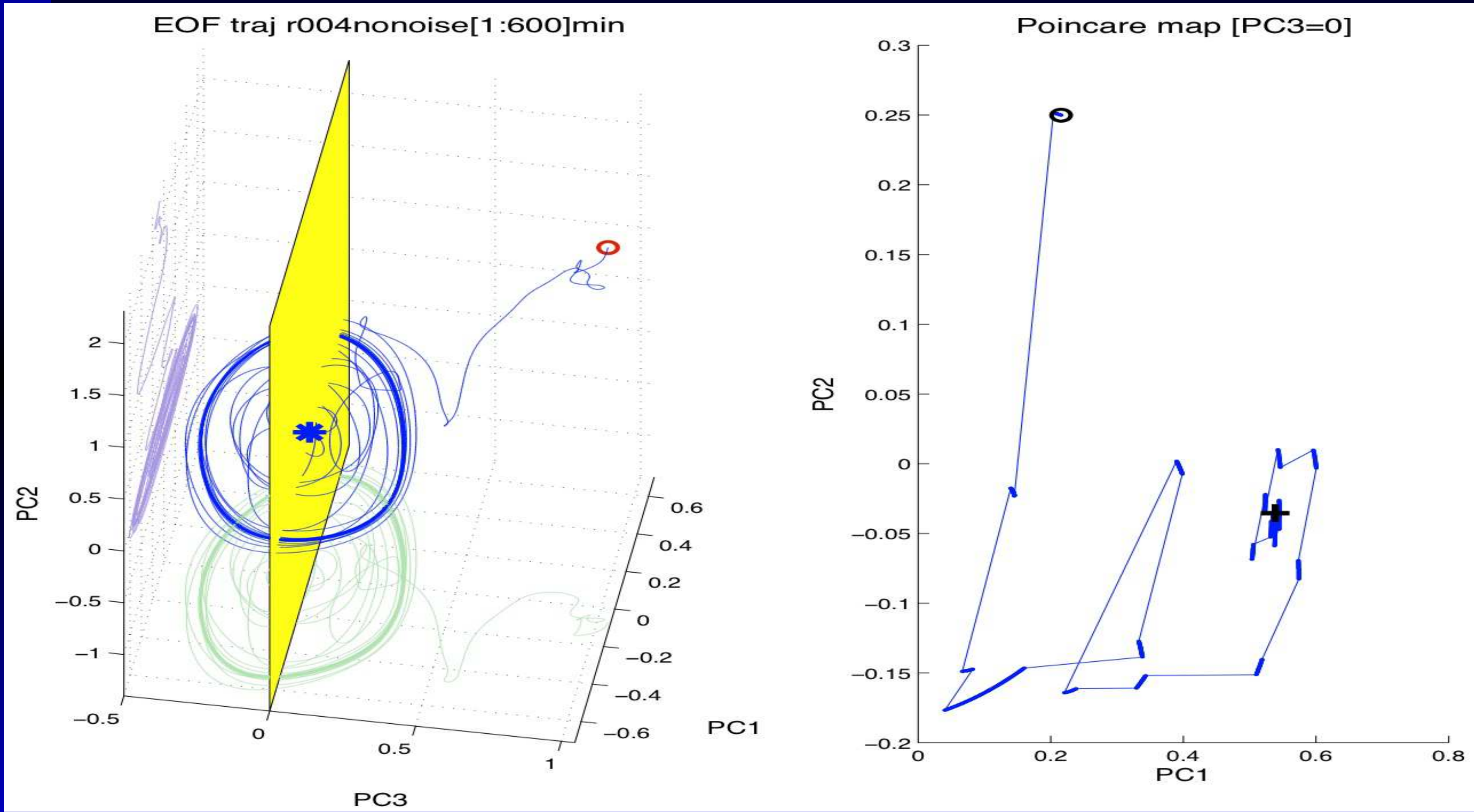
Equilibrated Wave Regime



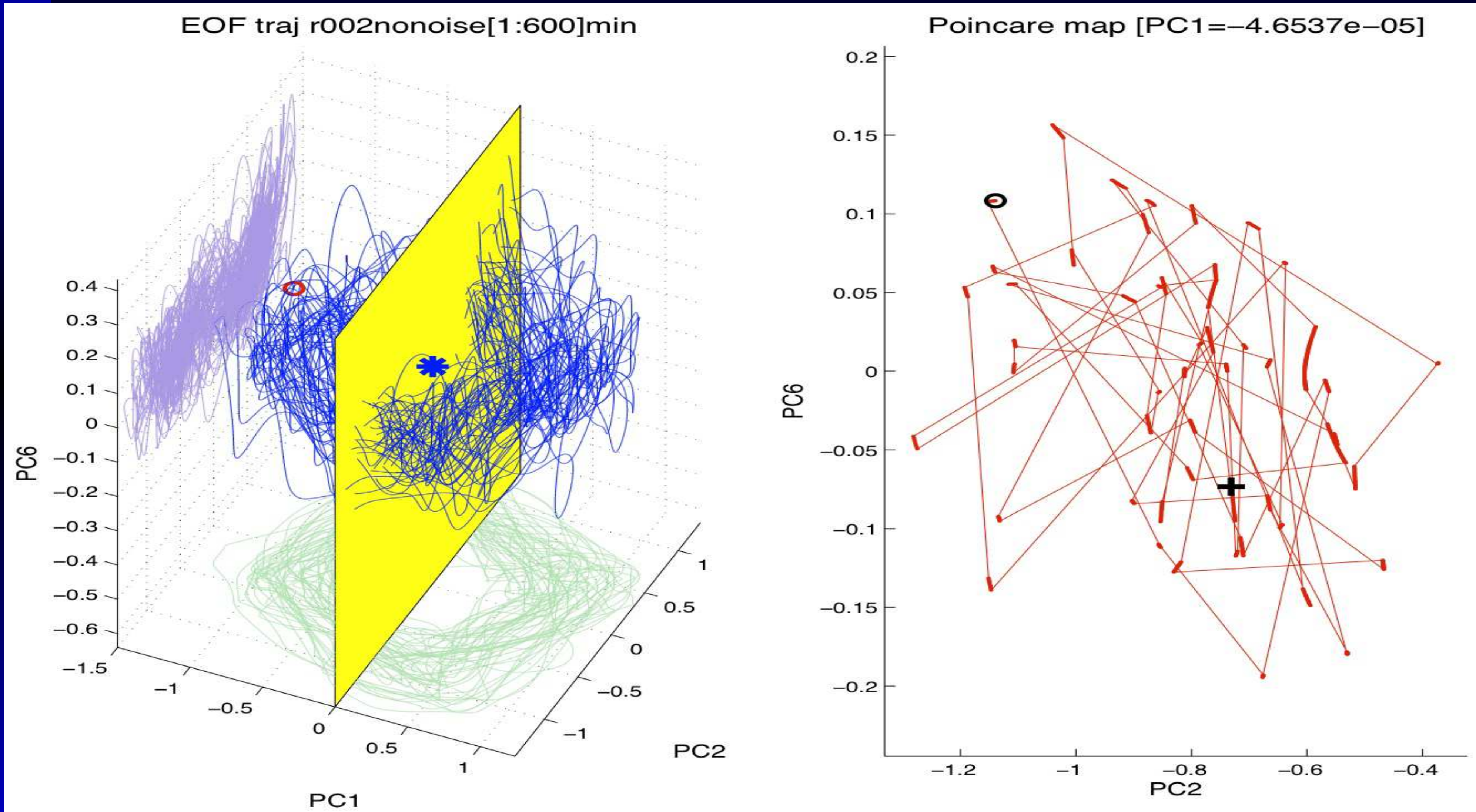
Aperiodic Wave Regime



Equilibrated Wave Regime: Projections



Aperiodic Wave Regime: Projections



The Implicit Particle Filter

- Dynamical model is an Ito SDE:

$$d\mathbf{x} = f(\mathbf{x})dt + GdW$$

- W is a Brownian motion with independent increments each increment having zero mean and variance dt

The Discrete Model

- The discretized SDE

$$\mathbf{x}_{j+1} = \mathbf{x}_j + f(\mathbf{x}_j)\Delta t + (\Delta t)^{1/2} G b_{j+1}$$

where $b_j \sim N(0, I);\ E(b_j b_k^T) = I\delta_{jk}$

- The deterministic forecast model:

$$\mathbf{x}_{j+1}^f = \mathbf{x}_j + f(\mathbf{x}_j)\Delta t$$

- Observations:

$$\mathbf{z}_{j+1} = H\mathbf{x}_{j+1} + b_{j+1}^o$$

- $E(b_{j+1}^o b_{k+1}^{oT}) = R\delta_{jk}$

The Implicit Particle Filter

$$\mathbf{x}_{j+1} - \mathbf{x}_j - \Delta t f(\mathbf{x}_j) \equiv \mathbf{x}_{j+1} - \mathbf{x}^f_{j+1} \sim N(0, \Delta t G G^T)$$

For the i^{th} particle, the pdf of the state $x^{(i)}$ conditioned on an observation $\mathbf{z}$ at time t_{j+1} is $\propto exp(-\mathbf{F}^{(i)})$, where

$$\mathbf{F}^{(i)} = (\mathbf{x} - \mathbf{x}^{(i)f}_{j+1})^T (\Delta t G G^T)^{-1} (\mathbf{x} - \mathbf{x}^{(i)f}_{j+1})/2 + (\mathbf{z} - H\mathbf{x})^T R^{-1} (\mathbf{z} - H\mathbf{x})/2$$

Obs & model noise assumed independent

The Implicit Particle Filter

after a bit of algebra $\cdots$

$$
\begin{aligned}
\mathbf{F} &= \phi + (\mathbf{x} - m)^T (P^a)^{-1} (\mathbf{x} - m)/2 \\
\phi &= min(\mathbf{F}) \\
&= (\mathbf{z} - H\mathbf{x}^f)^T (HQH^T + R)^{-1} (\mathbf{z} - H\mathbf{x}^f)/2 \\
m &= \mathbf{x}^f_{j+1} + K(\mathbf{z} - H\mathbf{x}^f_{j+1}) \\
Q &= \Delta t G G^T \\
K &= QH^T (HQH^T + R)^{-1} \\
P^a &= (I - KH)Q
\end{aligned}
$$

The Implicit Particle Filter

- Consider the i^{th} particle, at state $\mathbf{x}_j$ at time t_j
- Its location at time t_{j+1}, conditioned on the observation $\mathbf{z}$, is a random variable with pdf $\propto exp(-\mathbf{F}^{(i)})$

$$
\begin{aligned}
\mathbf{F}^{(i)} &= (\mathbf{x} - \mathbf{x}_{j+1}^{(i)f})^T (\Delta t G G^T)^{-1} (\mathbf{x} - \mathbf{x}_{j+1}^{(i)f})/2 \\
&\quad + (\mathbf{z} - H\mathbf{x})^T R^{-1} (\mathbf{z} - H\mathbf{x})/2 \\
&\equiv \phi + (\mathbf{x} - m)^T (P^a)^{-1} (\mathbf{x} - m)/2
\end{aligned}
$$

The Implicit Particle Filter

$$\begin{aligned} \mathbf{F}^{(i)} &= (\mathbf{x}-\mathbf{x}_{j+1}^{(i)f})^T(\Delta t G G^T)^{-1}(\mathbf{x}-\mathbf{x}_{j+1}^{(i)f})/2 \\ &\quad +(\mathbf{z}-H\mathbf{x})^T R^{-1}(\mathbf{z}-H\mathbf{x})/2 \\ &\equiv \phi + (\mathbf{x}-m)^T(P^a)^{-1}(\mathbf{x}-m)/2 \end{aligned}$$

- ϕ and P^a are derived from algebra that is formally identical to the Kalman filter
- This is nothing more or less than minimization of a positive definite quadratic form, known long before Kalman's famous article was published in 1960.

This is *not* the Ensemble Kalman Filter

- We never use sample statistics from the collection of particles
- The *only* interaction among particles occurs at resampling
- We make no assumptions about sample moments In fact *the sample moments need not exist*
- We have examples in which the implicit particle filter significantly outperforms the EnKF

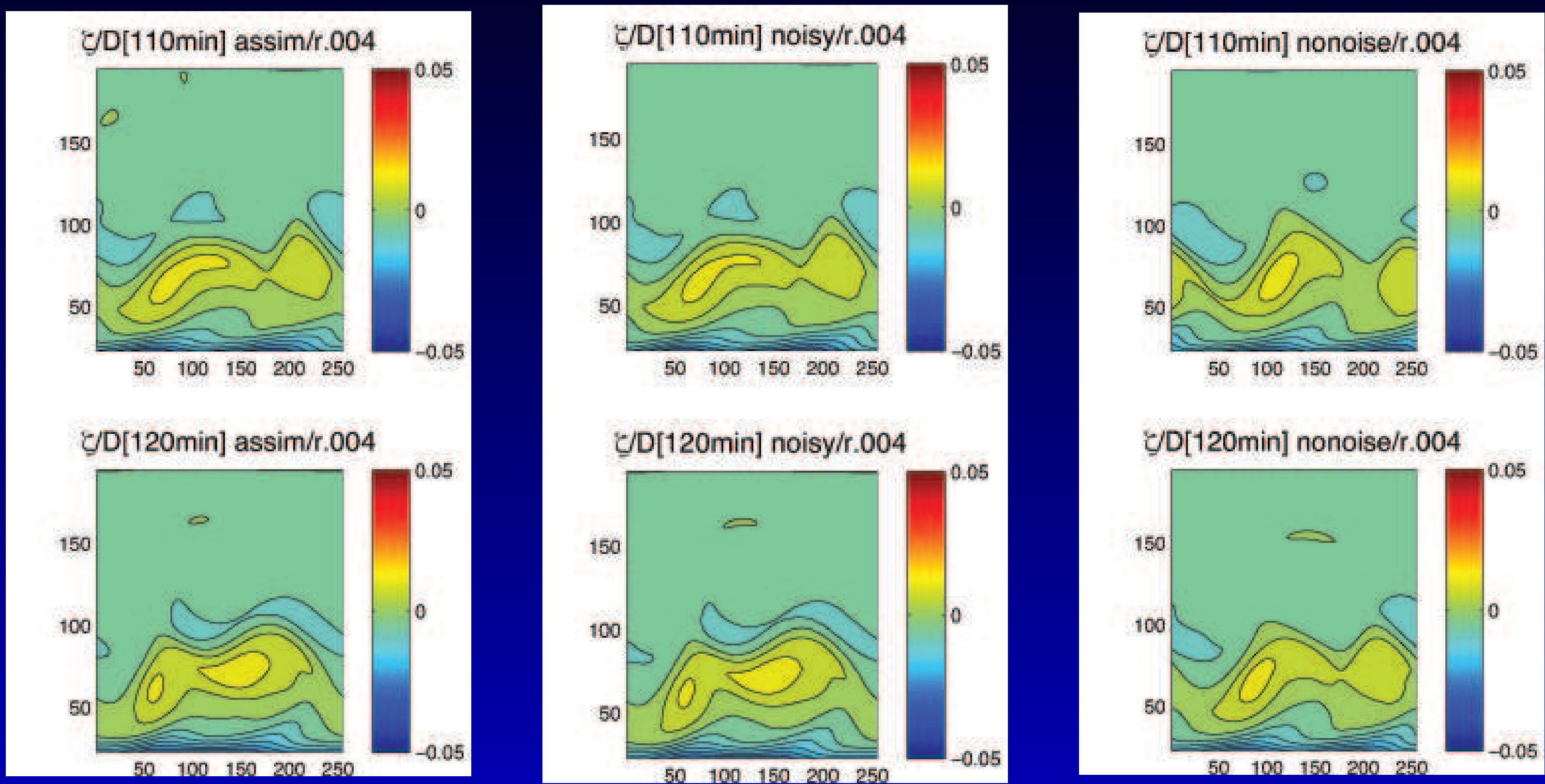
Recipe for The Implicit Particle Filter

For each particle:

1. Generate a random vector ξ_i, of state dimension, drawn from $N(0, I)$
2. Calculate m_i, the most probable state given the initial value of $\mathbf{x}^{(i)}$ and the minimizer of $F^{(i)}$
3. Choose the updated state $\mathbf{x}^{(i)}$ of the i^{th} particle so that

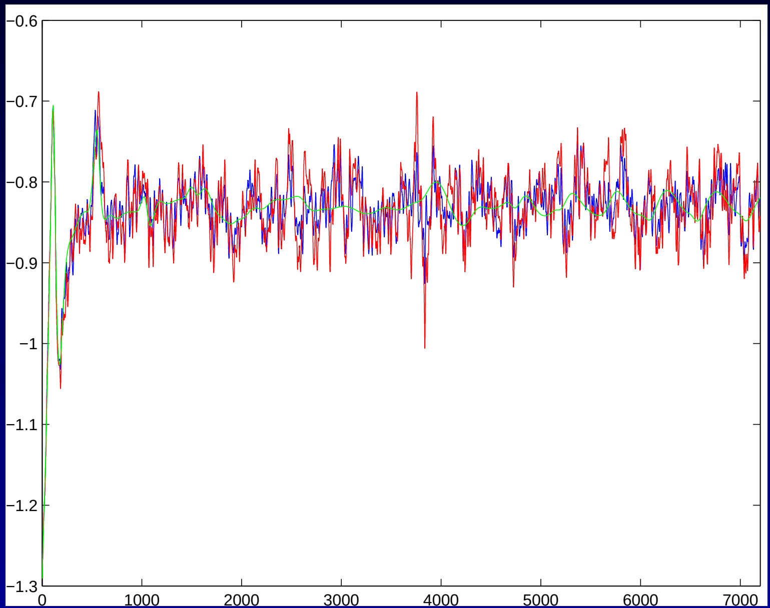
$$(\mathbf{x}^{(i)} - m_i)^T (P^a)^{-1} (\mathbf{x}^{(i)} - m_i) \;=\; \xi_i \cdot \xi_i$$

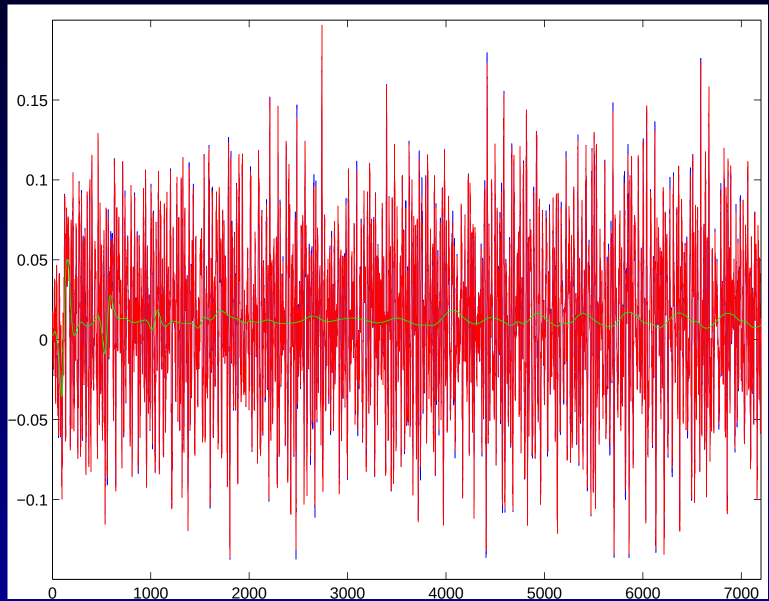
Assimilation Results, High Drag Case, 10 Particles

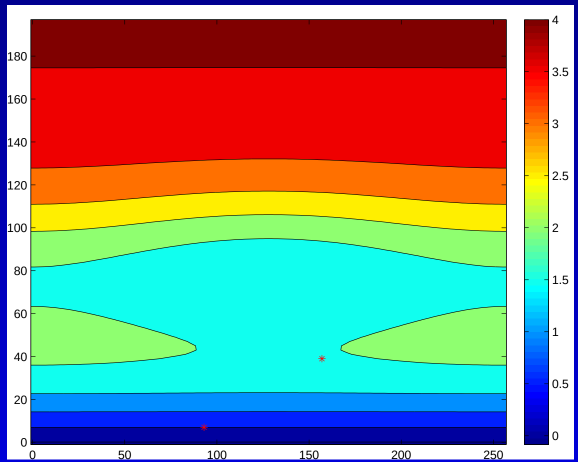


Potential vorticity. l to r: filtered, reference, noise free; Top to bottom: $1hr\ 50min,\ 2hr$

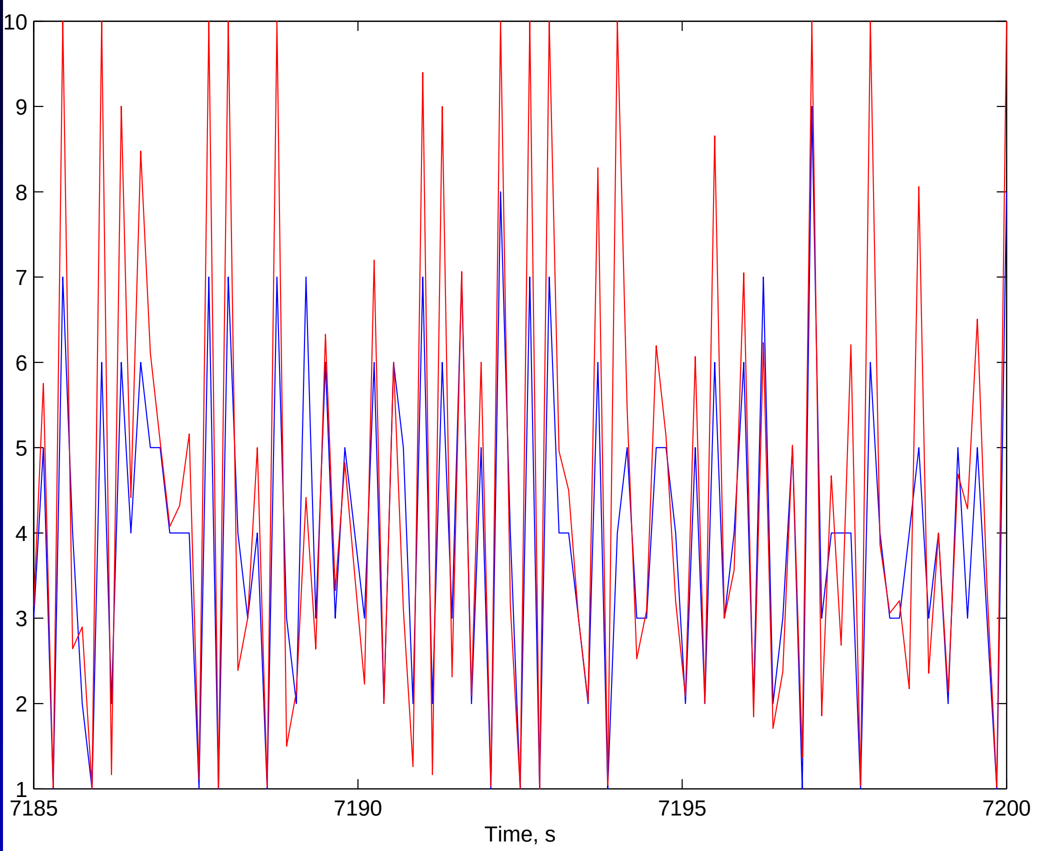
Assimilation Results, High Drag Case





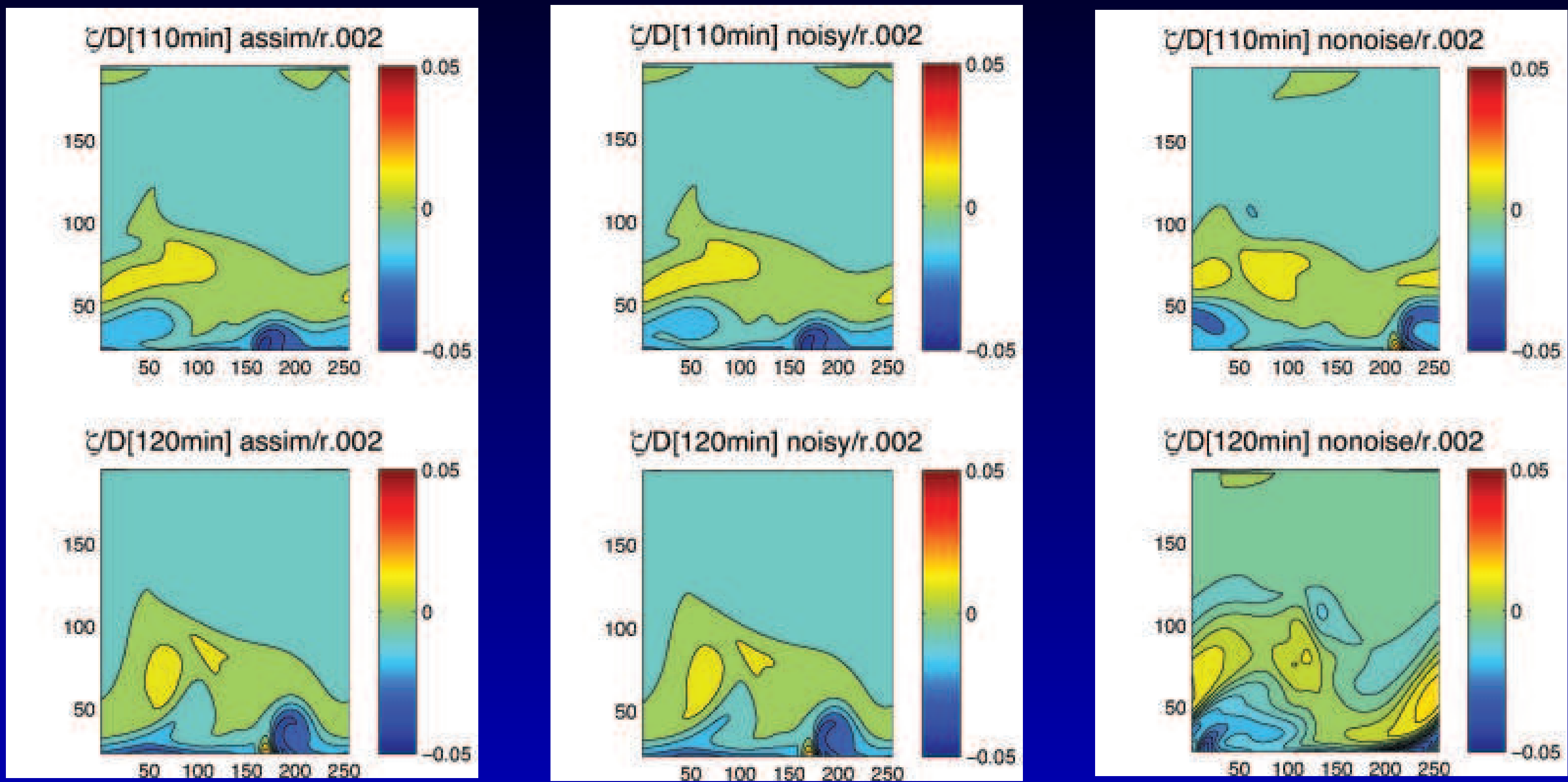


Particle Count, High Drag Case



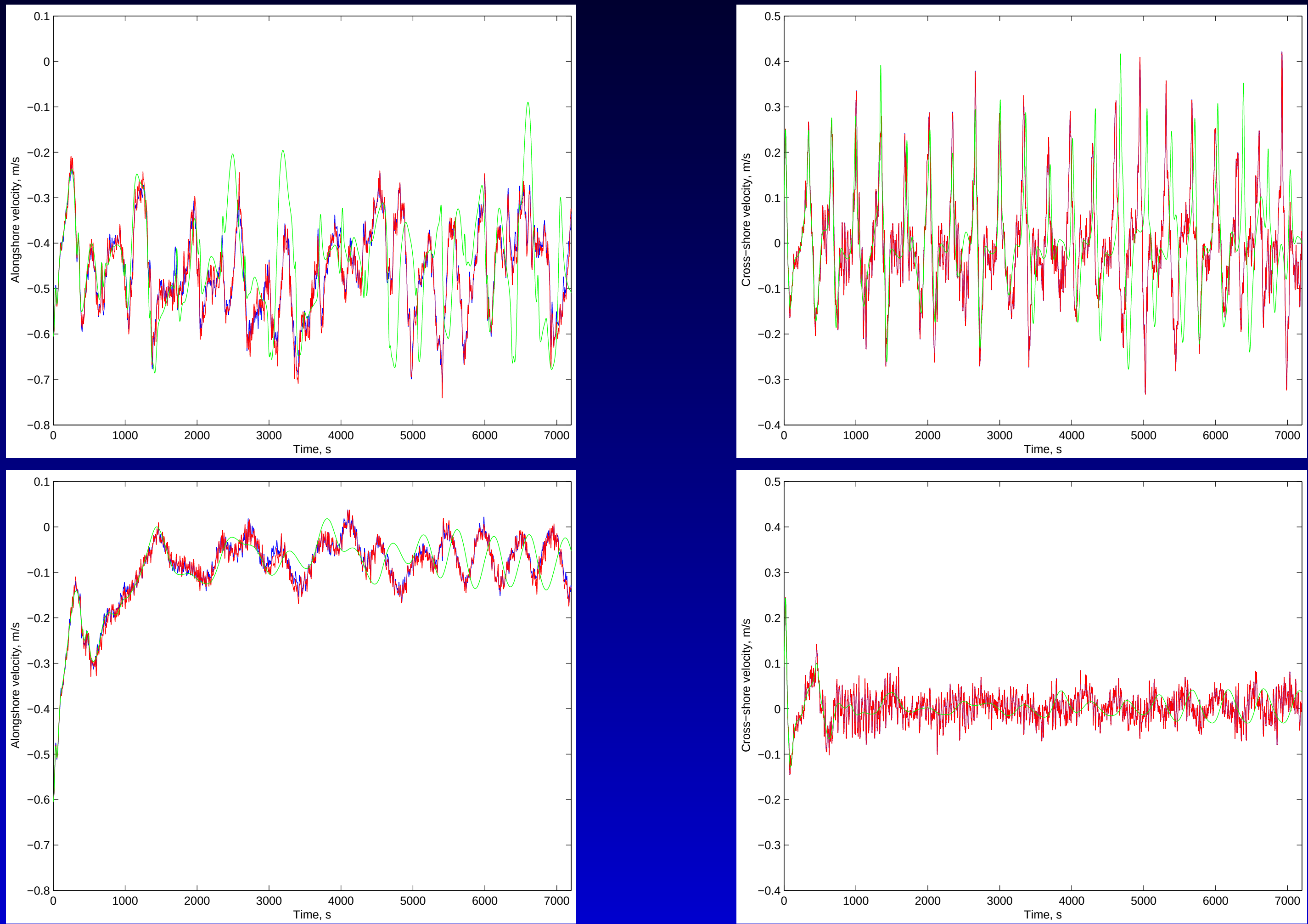
10 particle run. Red curve: Effective number of particles $= 1/\sum w_i^2$. Blue curve: number of particles after resampling.

Assimilation Results, Low Drag Case, 50 Particles



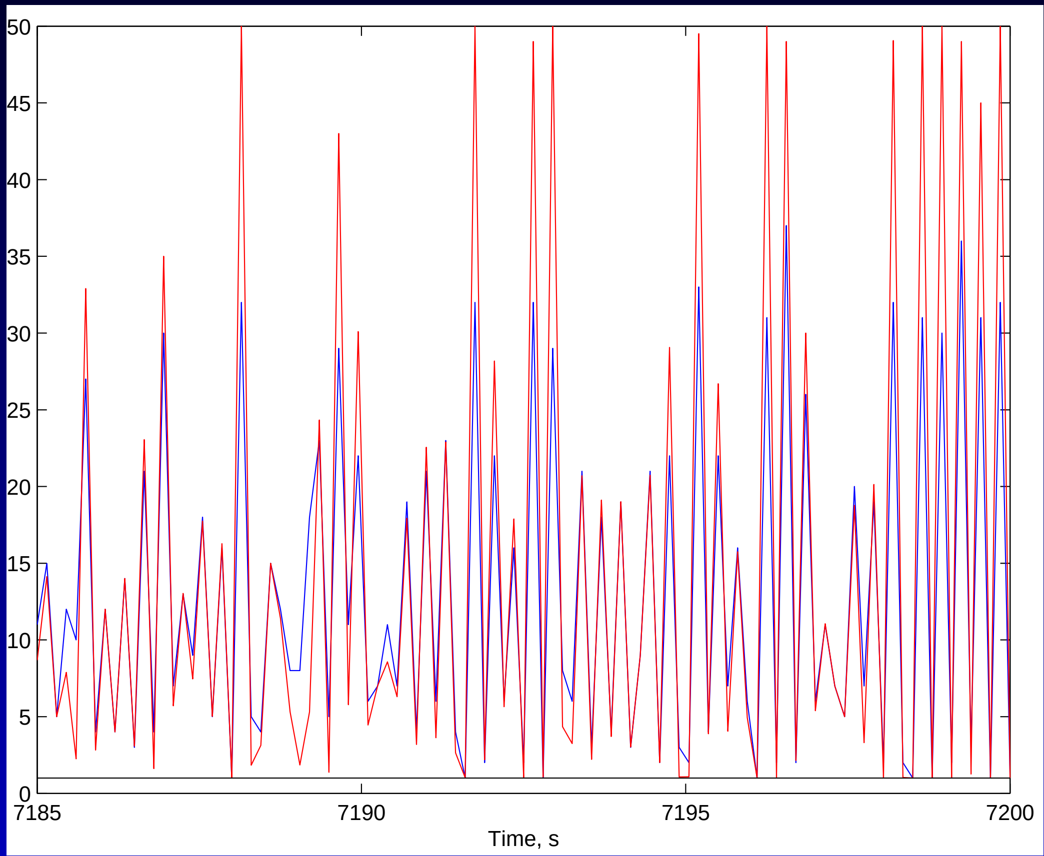
Potential vorticity. l to r: filtered, reference, noise free; Top to bottom: $1hr\ 50min,\ 2hr$

Low Drag Case, Point Comparisons



Blue curves: reference; Red curves: filter output; Green curves: noise free system

Particle Count, Low Drag Case



50 particle run. Red curve: Effective number of particles. Blue curve: number of particles after resampling. Black line: N=1

Conclusions

- The good news:
  - The implicit particle filter, (in this case, the optimal importance filter) can be implemented efficiently on models of geophysical interest
  - The resulting analysis looks good
- The bad news: We are still cursed by dimensionality!
- Next steps, no particular order
  - Sparse observations in time
  - Direct appeal to dynamical structure
  - Parameter estimation