UNIVERSITY OF WISCONSIN–MADISON
Department of Mathematics

MESOCHRONIC ANALYSIS: COMPUTATION AND INTERPRETATION

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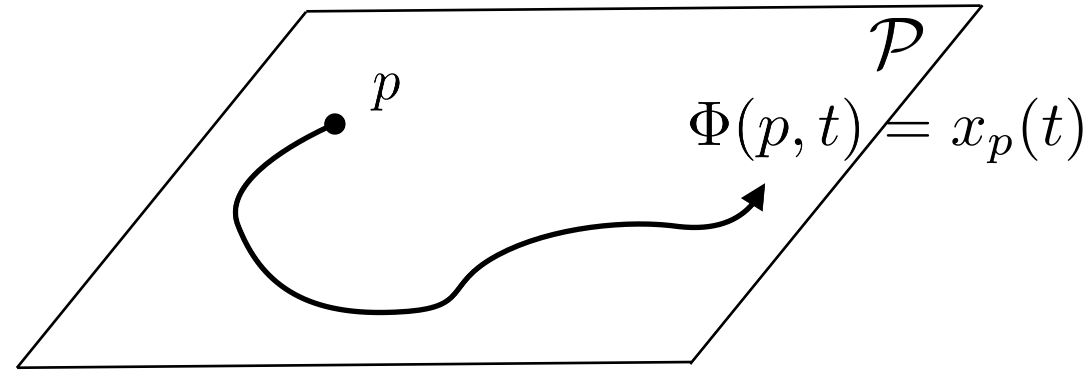
The term "mesochronic" means "time-averaged".

$$\dot{x}_p = f(t, x_p), \quad x_p(0) = p$$
$$(p, t) \mapsto x_p(t)$$



Flow map can be interpreted as a Lagrangian average of the velocity field.

Flow map

$$\Phi(p, T) = p + \boxed{\int_0^T f(\tau, x_p(\tau))d\tau}$$

Mesochronic velocity field

$$\tilde{f}(p, T) = \frac{1}{T}\boxed{\int_0^T f(\tau, x_p(\tau))d\tau}$$

$$\Phi(p, T) = p + T\tilde{f}(p, T)$$

Mesochronic Jacobian captures the linear deformation by the flow.

$$J_{\tilde{f}}(p, T) = \frac{J_\Phi(p, T) - \mathrm{Id}}{T} = \begin{bmatrix} \partial_1 \tilde{f}_1(p, T) & \partial_2 \tilde{f}_1(p, T) & \dots \\ \partial_1 \tilde{f}_2(p, T) & \partial_2 \tilde{f}_2(p, T) & \\ \vdots & & \ddots \end{bmatrix}$$

Flow Jacobian

Character of deformation: elliptic (rigid rotation), hyperbolic (stretching) or parabolic (shear).

Mesochronic Jacobian is evaluated using a numerical semi-Lagrangian method.

Mesochronic J. Advected J.

$$\dot{M}_p(t) = -\frac{M_p(t)}{t-t_0} + \frac{A_p(t)}{|t-t_0|} + A_p(t)\cdot M_p(t), \qquad M_p(t_0) = A_p(t_0)$$

1. Compute a particle trajectory (dynamics of the fluid flow).

$$\dot{x}_p = f(t, x_p), \qquad x_p(0) = p$$

[MacLachlan, Quispel, JPhysA, 2006]

2. Evaluate the advected Jacobian along the particle trajectory.

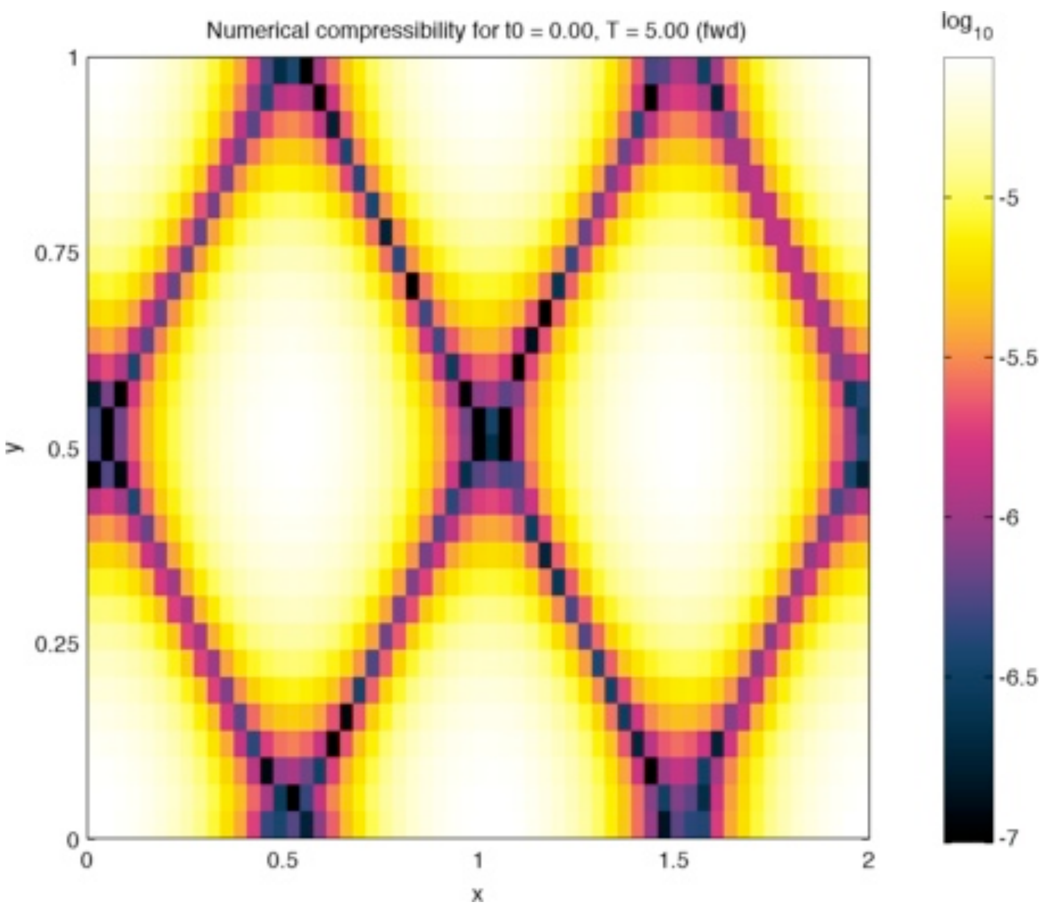
$$\partial_d F(p) \approx \frac{F(p+\varepsilon \hat{p}_d) - F(p-\varepsilon \hat{p}_d)}{2\varepsilon}$$

3. Solve the mesochronic Jacobian matrix ODE.

$$M_p(t) \approx M \left[\!\!\left[\frac{t-t_0}{h} \right]\!\!\right]$$

Accuracy proxy: numerical compressibility

$$\delta[\![n]\!] := \operatorname{tr} M[\![n]\!] + nh \det M[\![n]\!] \approx 0$$



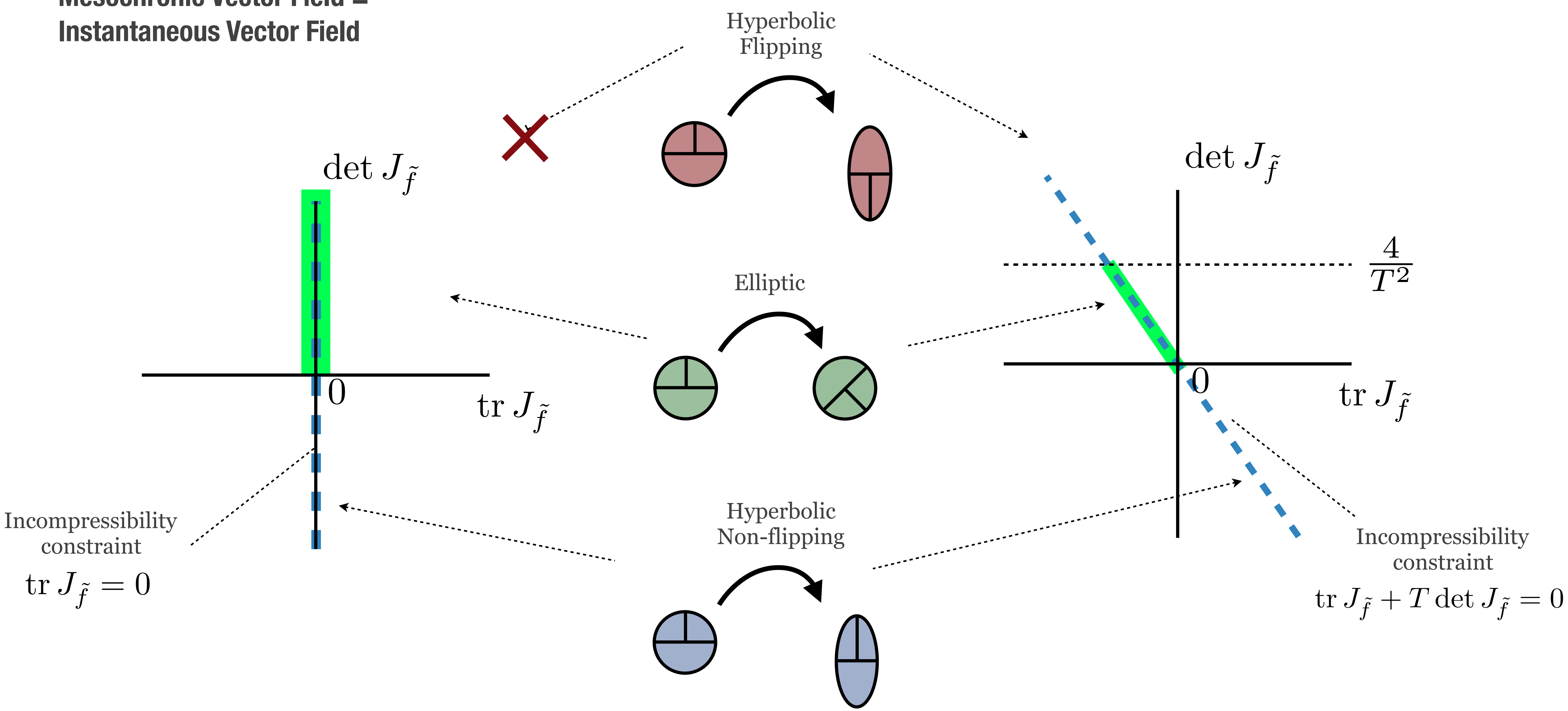
Deformation class of the flow is requires only one quantity for incompressible flows in 2D.

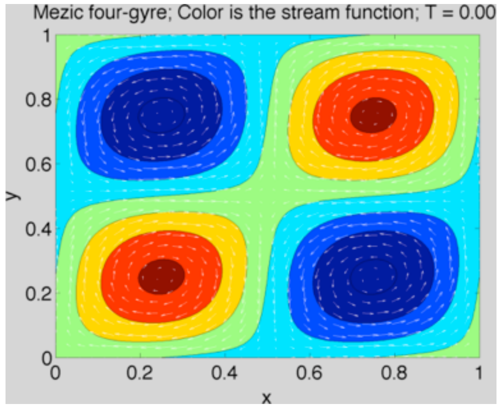
Okubo-Weiss: $T = 0^+$

Mesochronic Analysis: $T > 0$

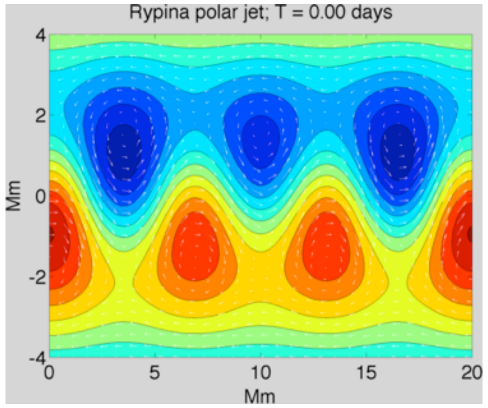
[Mezic, Loire, et al., Science, 2010]

Mesochronic Vector Field = Instantaneous Vector Field

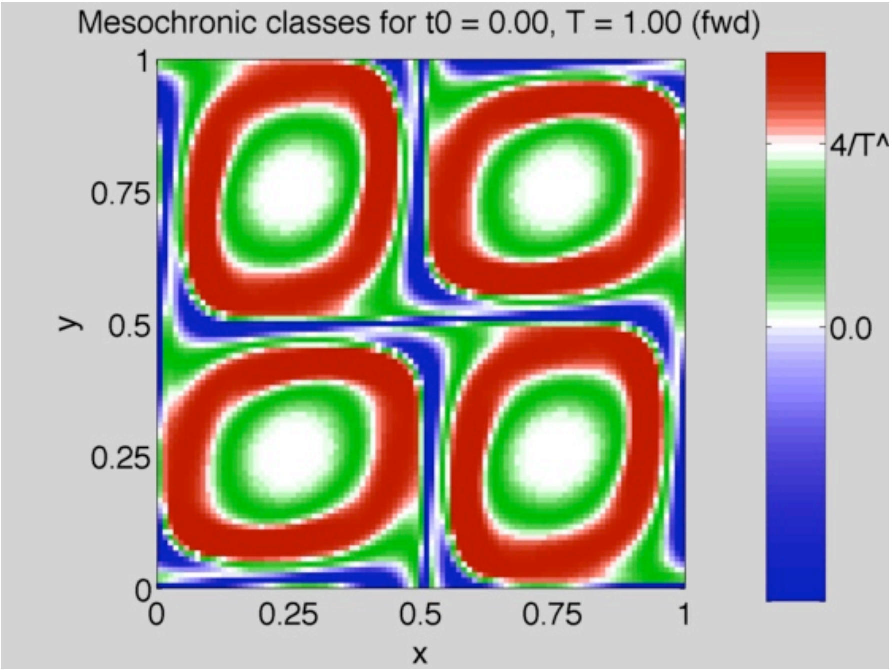


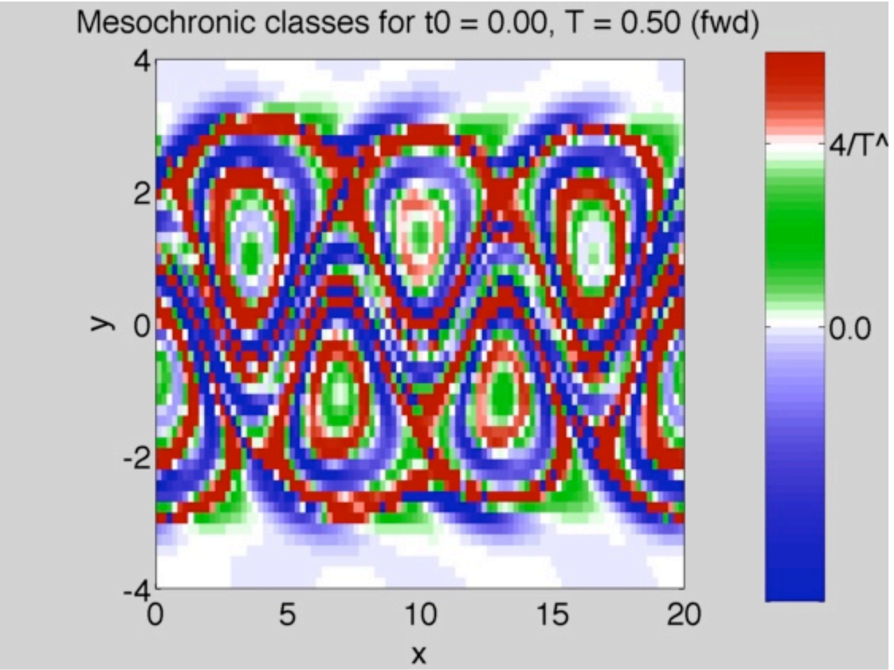


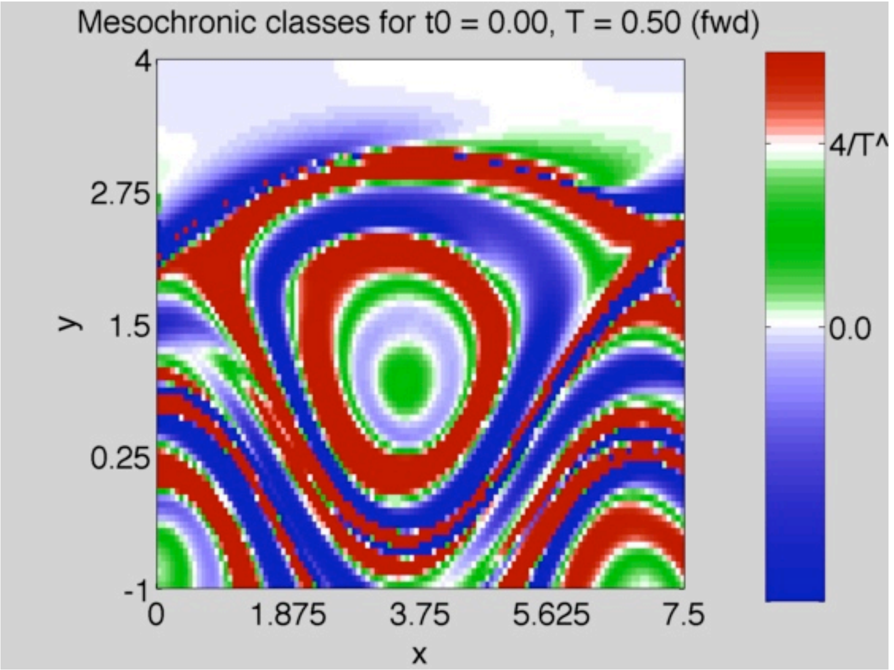
[Mezic et al., Science, 2010]

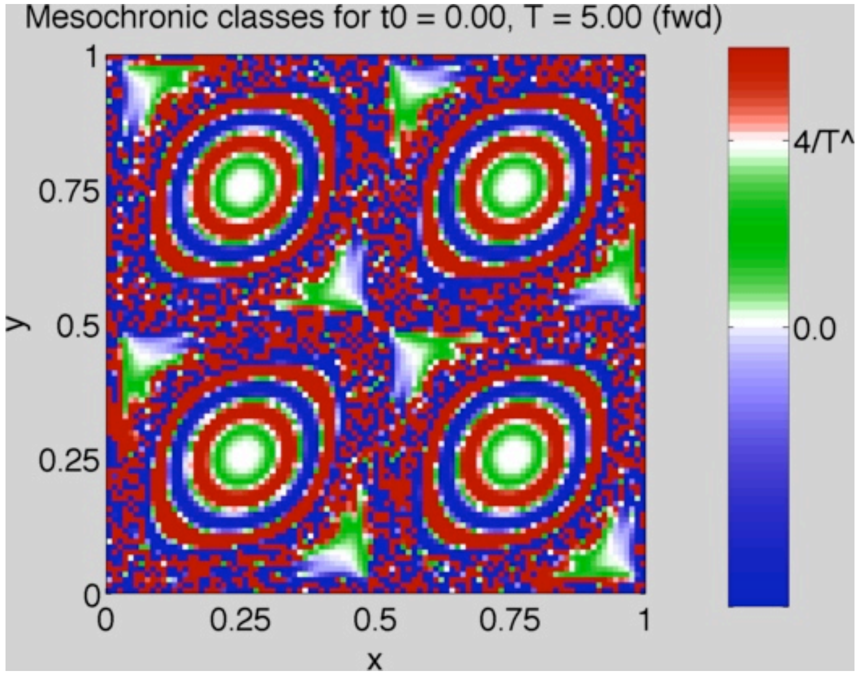


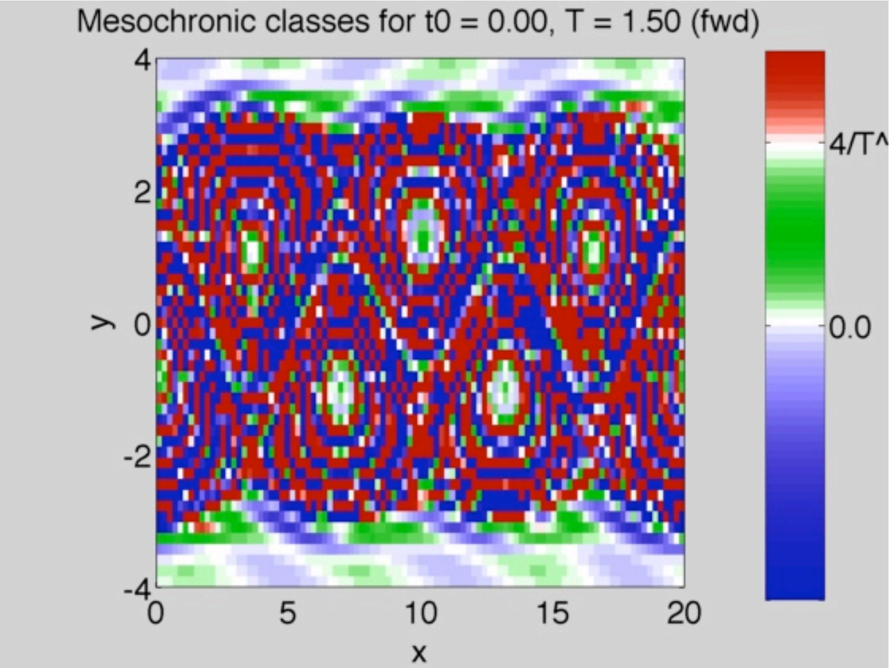
[Rypina et al., JAtmSci, 2007]

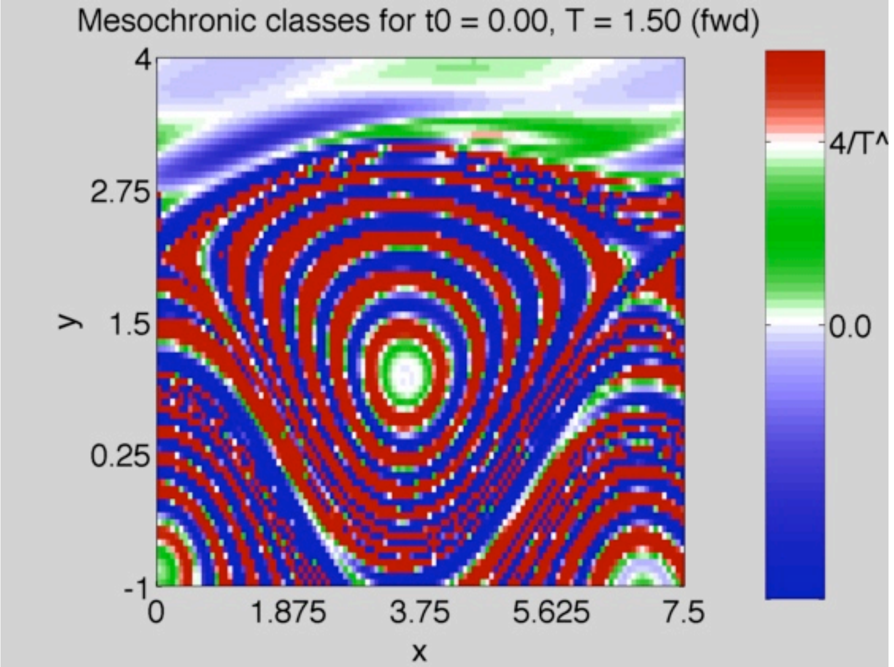






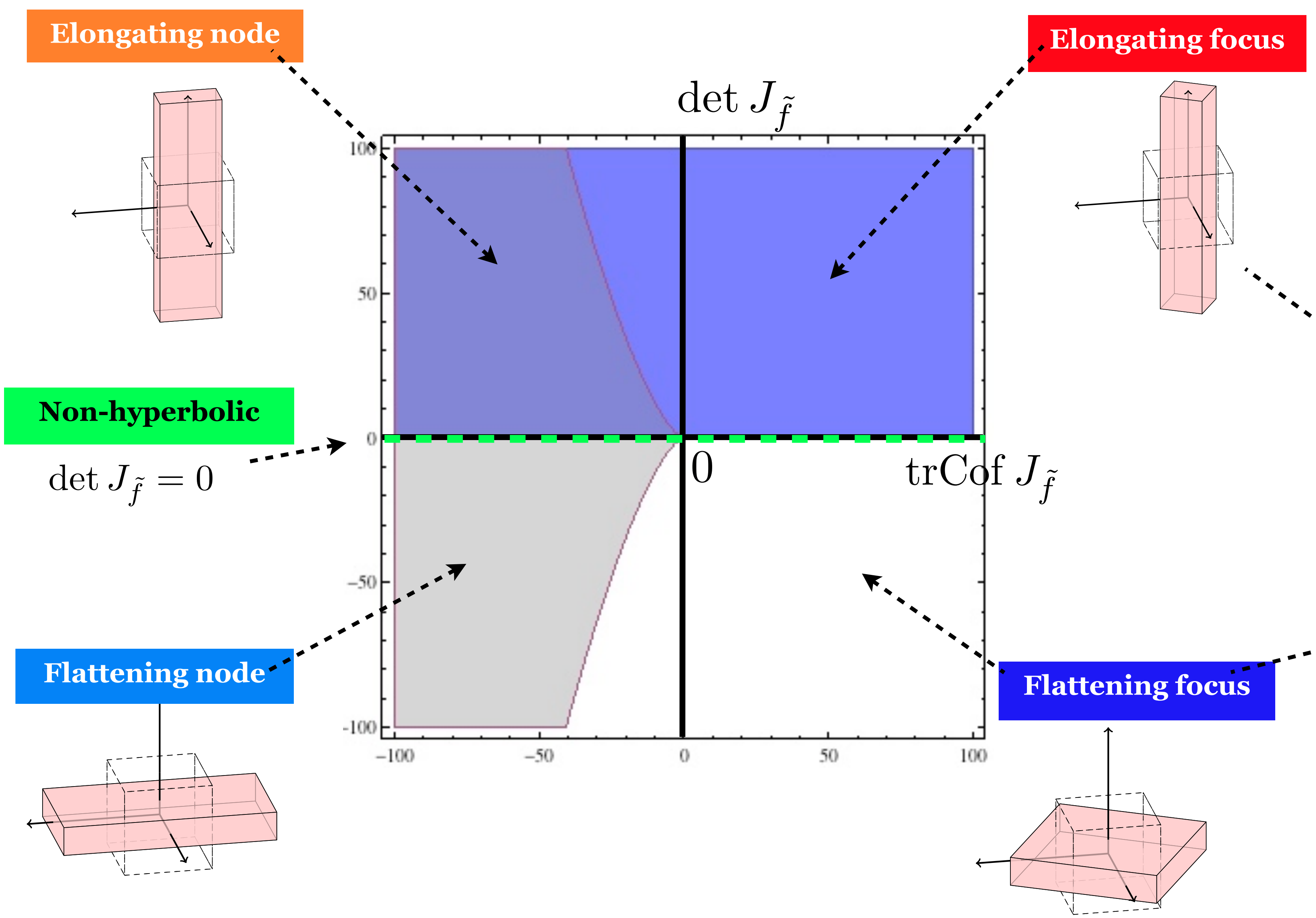


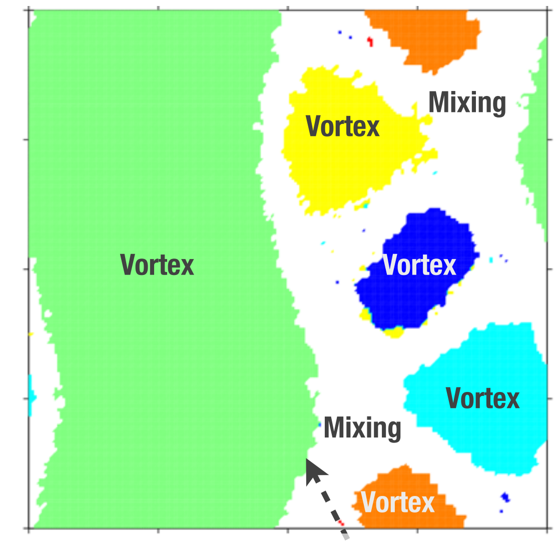




Deformation class of the flow requires two quantities for incompressible flows in 3D.

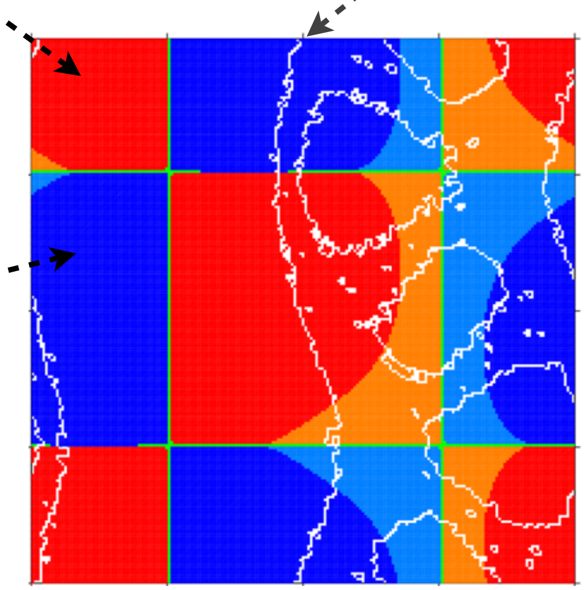
$T = 0^+$ Okubo-Weiss-Chong:





ABC Flow (z=0 slice)

Boundaries of invariant regions

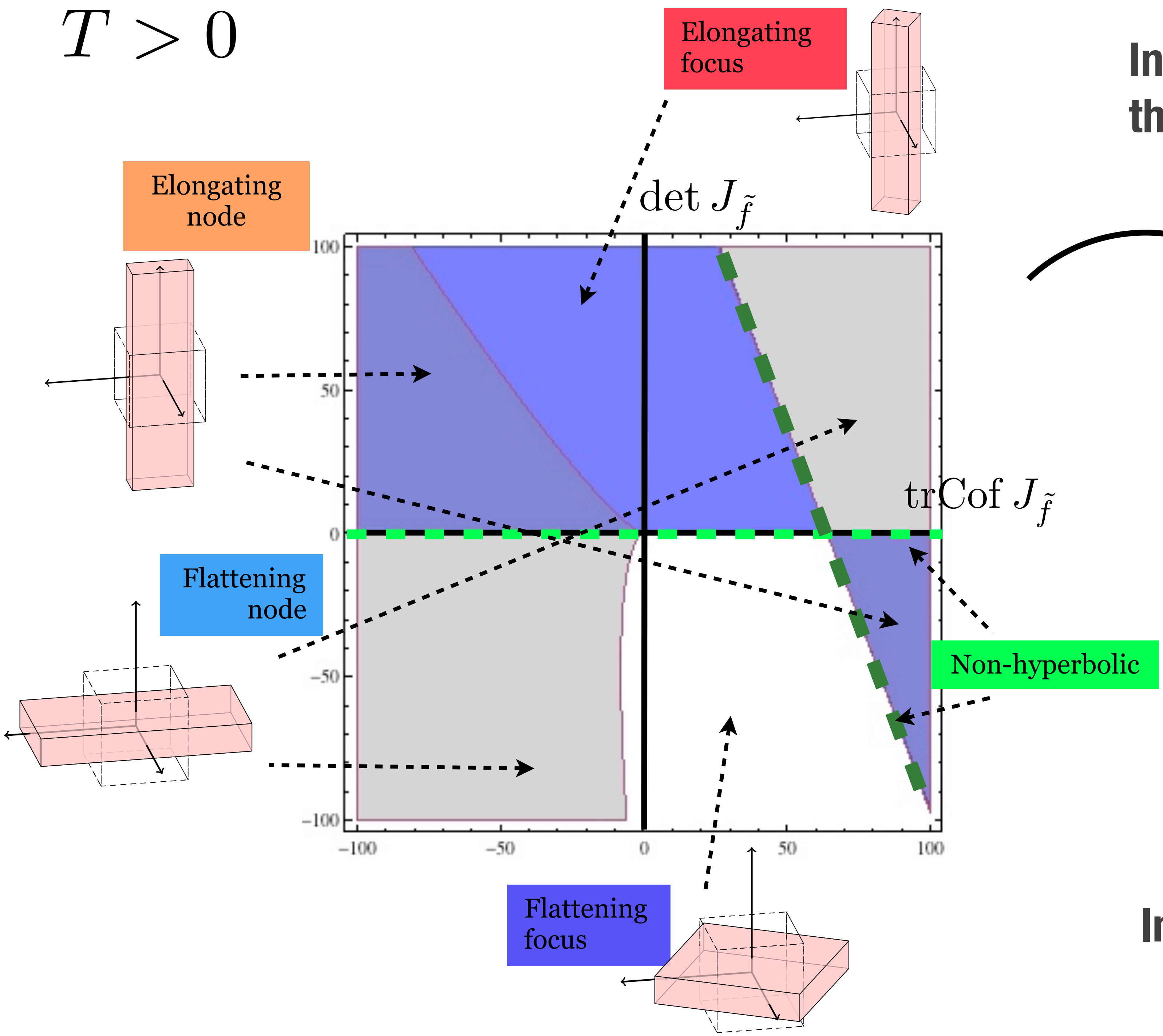


Okubo-Weiss-Chong

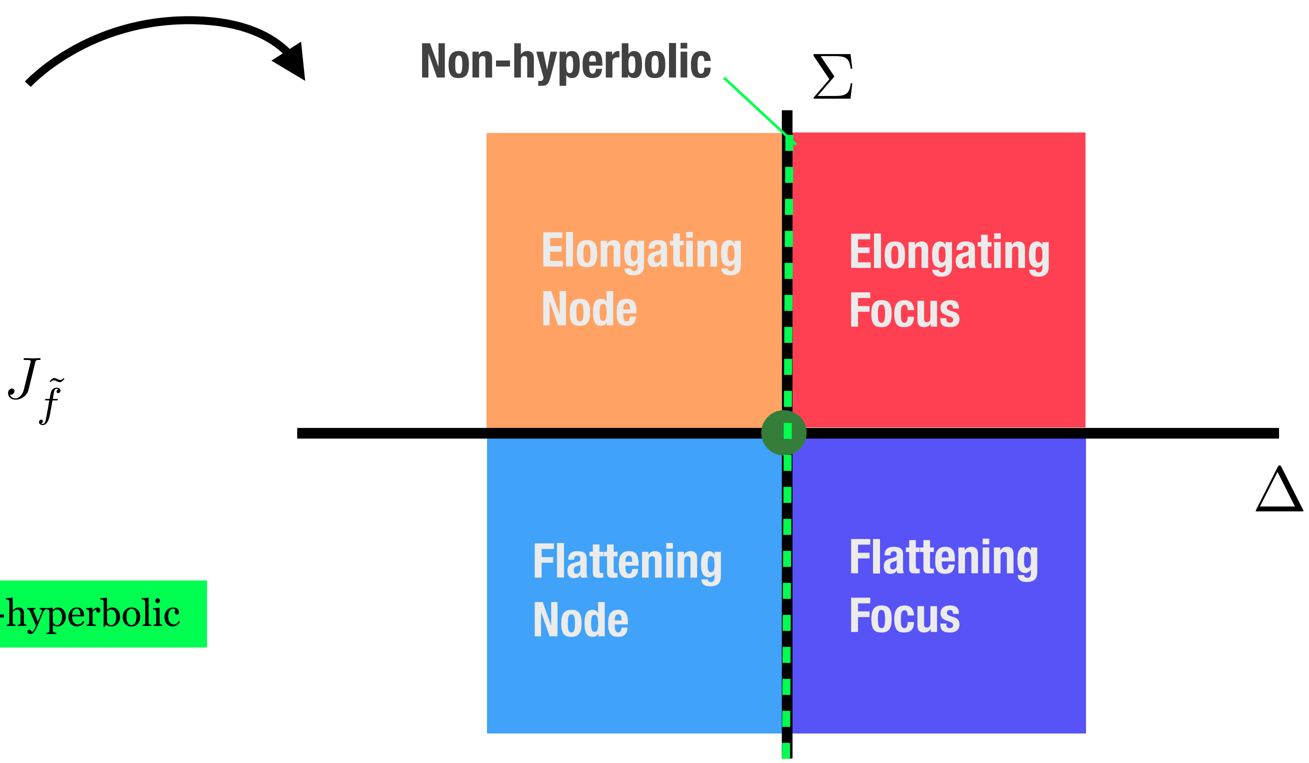
Criterion yields **non-intuitive results even for steady flows:** boundaries do not match understanding of invariant structures.

Introducing non-zero time intervals increases complexity.

$T > 0$



Introduce two new quantities $\Sigma\,\Delta$ that separate hyperbolic classes:



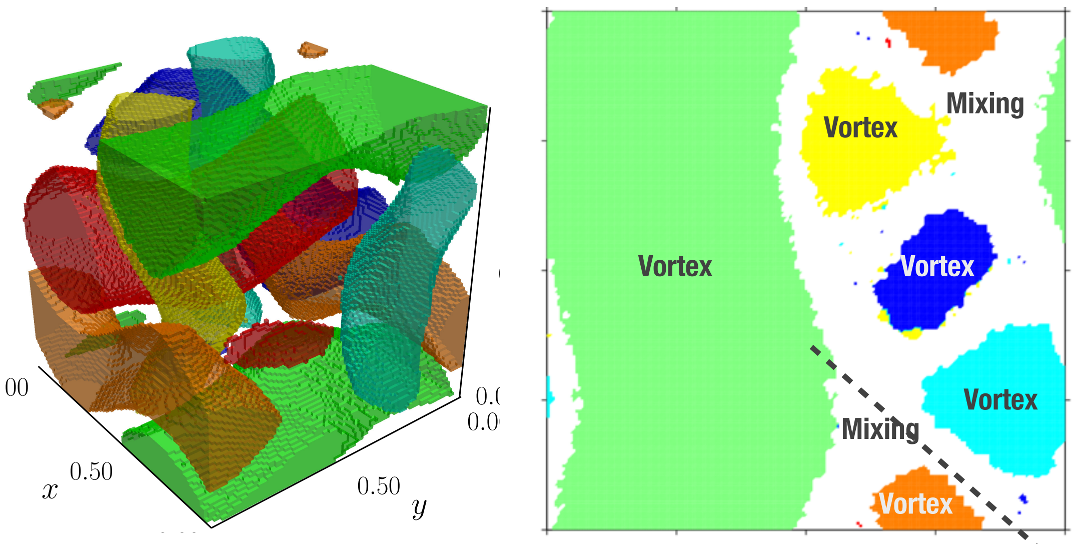
Incompressibility:

$$\mathrm{tr}\, J_{\tilde{f}} + T\, \mathrm{trCof}\, J_{\tilde{f}} + T^2 \det J_{\tilde{f}} = 0$$

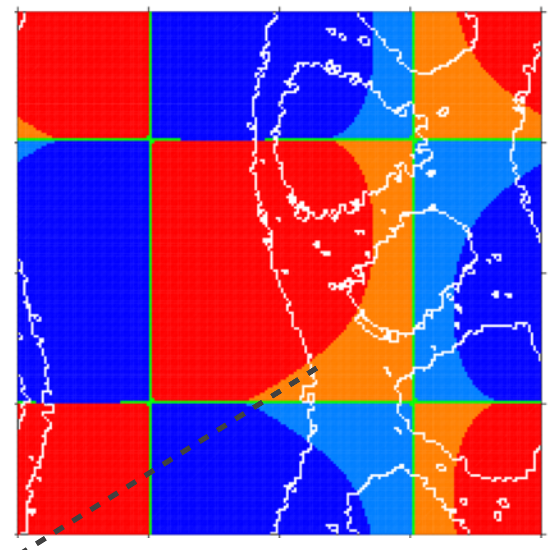
[Collaboration w/ S. Siegmund, TU Dresden and Doan Thai Son, Imperial College, London]

Results for ABC flow match the intuition.

Invariant sets ($z=0$ slice)



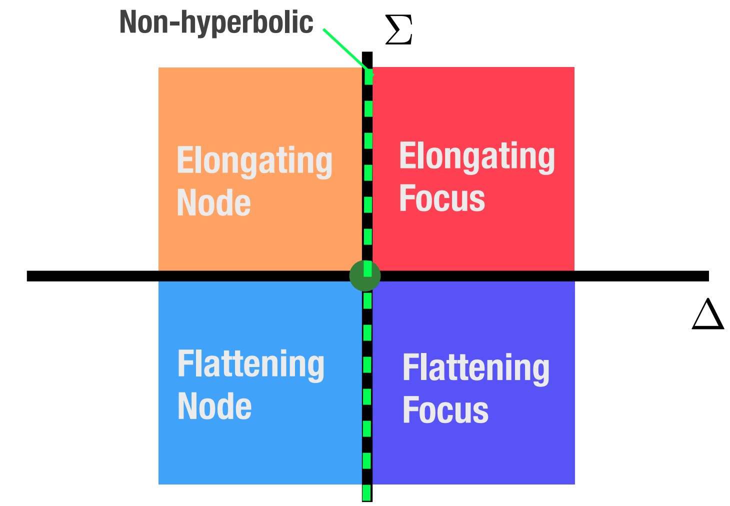
Okubo-Weiss-Chong



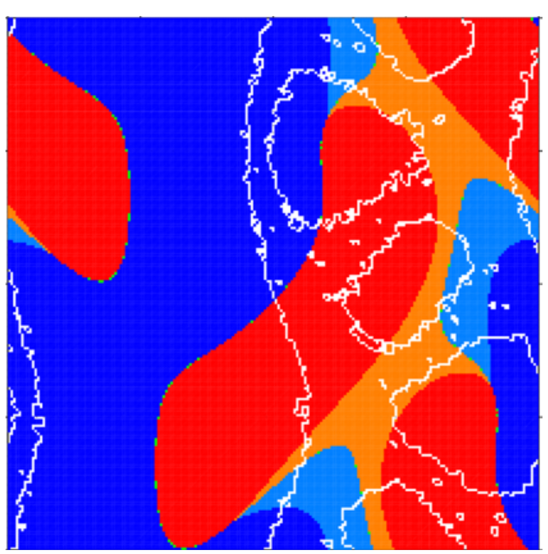
Boundaries of invariant regions

$T=0^+$

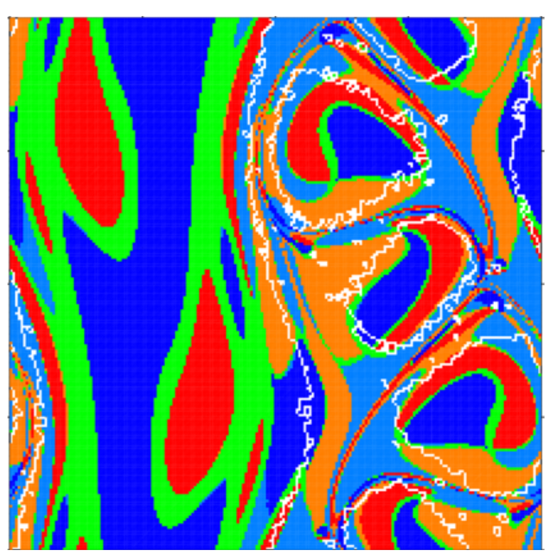
Mesochronic Classes



Hyperbolicity dominates at short time scales.

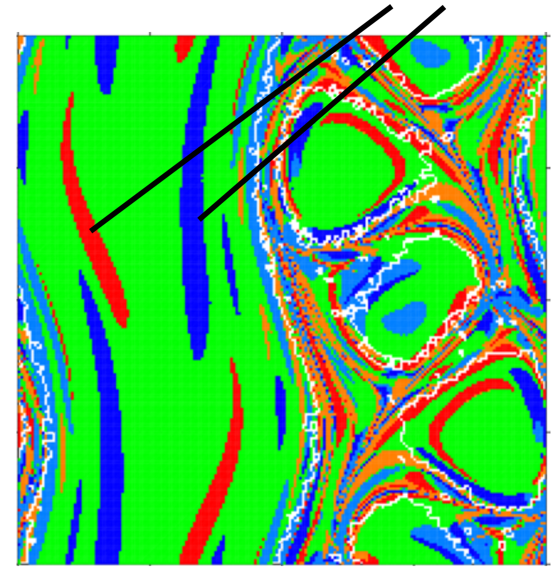


$T=1$



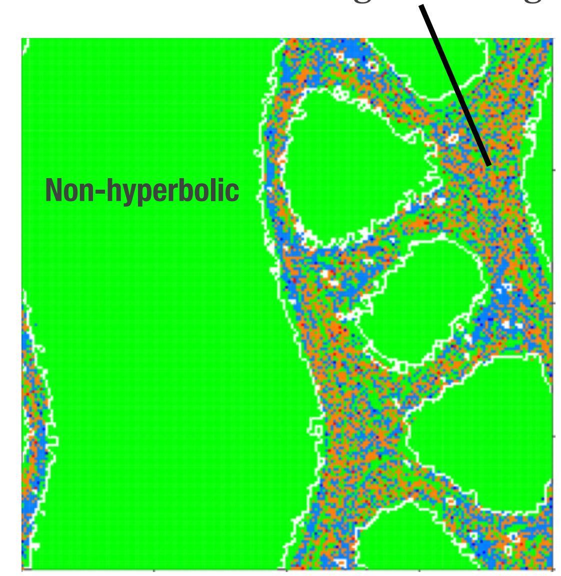
$T=5$

Hyperbolicity with rotation



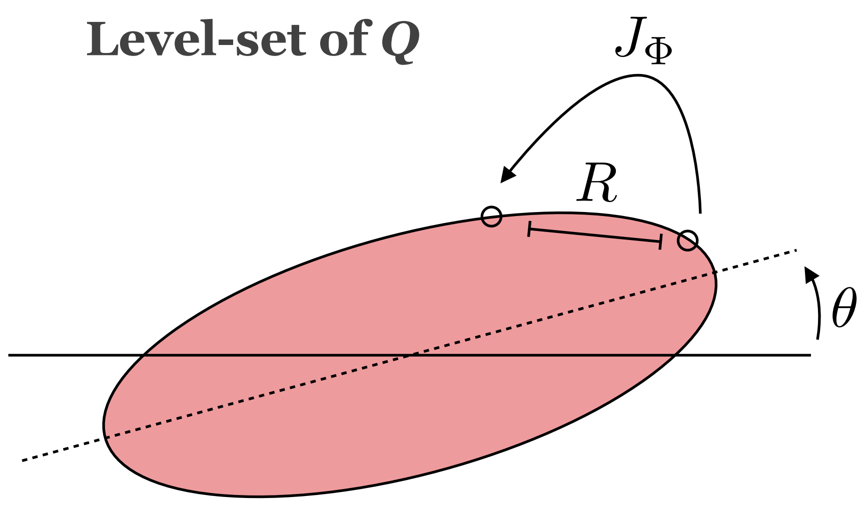
$T=10$

Mixture: non-hyperbolic, flattening and elongating



$T=50$

Mesochronic Classes relates to Greene criteria for KAM tori breakdown.



$$J_\Phi = \begin{bmatrix} a+d & c+b \\ c-b & a-d \end{bmatrix} \qquad Q(x,y) = (b-c)x^2 + 2d\,xy + (b+c)y^2$$

Ellipticity

$$E := \frac{b^2-(c^2+d^2)}{b^2+(c^2+d^2)}$$

Residue

$$R := \tfrac{1}{2}(1-a)$$

Orientation

$$\theta := \tfrac{1}{2}\arctan\tfrac{c}{d}$$

$$\operatorname{sign} R(1-R) = \operatorname{sign} E$$

Greene: Convergence of R over periodic orbits as they approach orbits with irrational winding numbers indicates a structurally stable KAM surface.

Greene: For strong hyperbolicity, residue behaves like an eigenvalue, but is a real analytic function of the perturbation.

$$\ln R \sim \ln|\lambda|$$

Connection to mesochronic Jacobian:

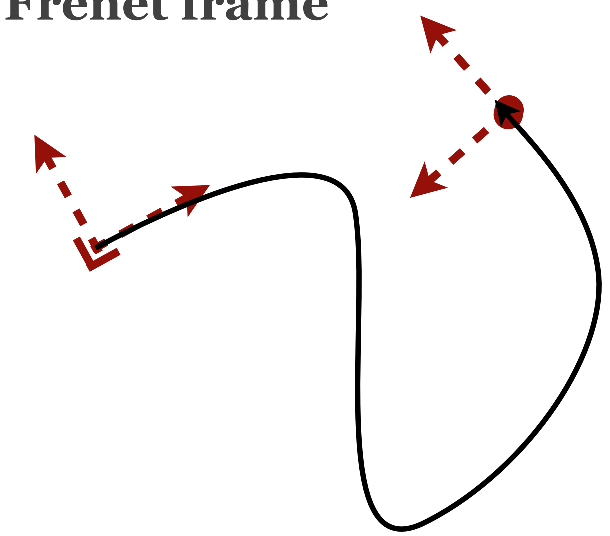
$$J_{\tilde f} = \frac{1}{T}\begin{bmatrix} a+d-1 & c+b \\ c-b & a-d-1 \end{bmatrix} \qquad R = \frac{T^2}{4}\det J_{\tilde f}$$

Haller-Iacono shear and stretch are mesochronic quantities in the Frenet frame.

[Haller, Iacono, PRE, 2003]

$$\dot{\xi}_p(t) = \overbrace{J_f(x_p(t))}^{\text{Advected Jacobian}} \xi_p(t)$$

Frenet frame



$$\eta_p(0) := L_{t_0}^{t_0}(p)\xi(0), \quad \eta_p(t) := L_{t_0}^{t}(p)$$

$$\dot{\eta}_p = A_{t_0}^t(p)\eta_p,$$

Steady Flows

$$A_{t_0}^t(p) := \begin{bmatrix} S_\parallel^{(t)} & S_\circ^{(t)} \\ 0 & -S_\parallel^{(t)} \end{bmatrix} + \cancel{\begin{bmatrix} 0 & -b(t) \\ b(t) & 0 \end{bmatrix}}$$

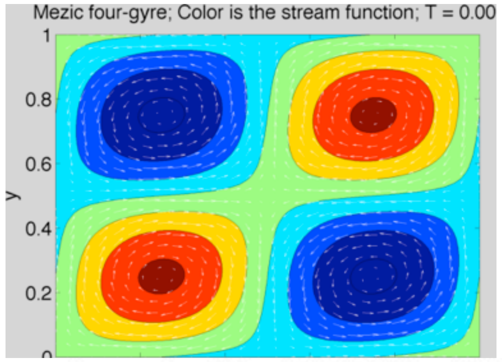
Steady state flow map is triangularized

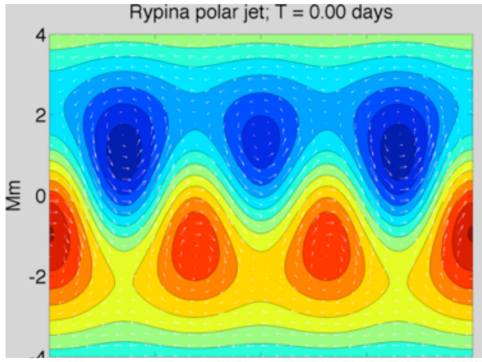
$$\eta_p(t) = \begin{bmatrix} \exp\left(-\lambda_{t_0}^t\right) & \mu_{t_0}^t \exp \lambda_{t_0}^t \\ 0 & \exp \lambda_{t_0}^t \end{bmatrix} \cdot \eta_p(0)$$

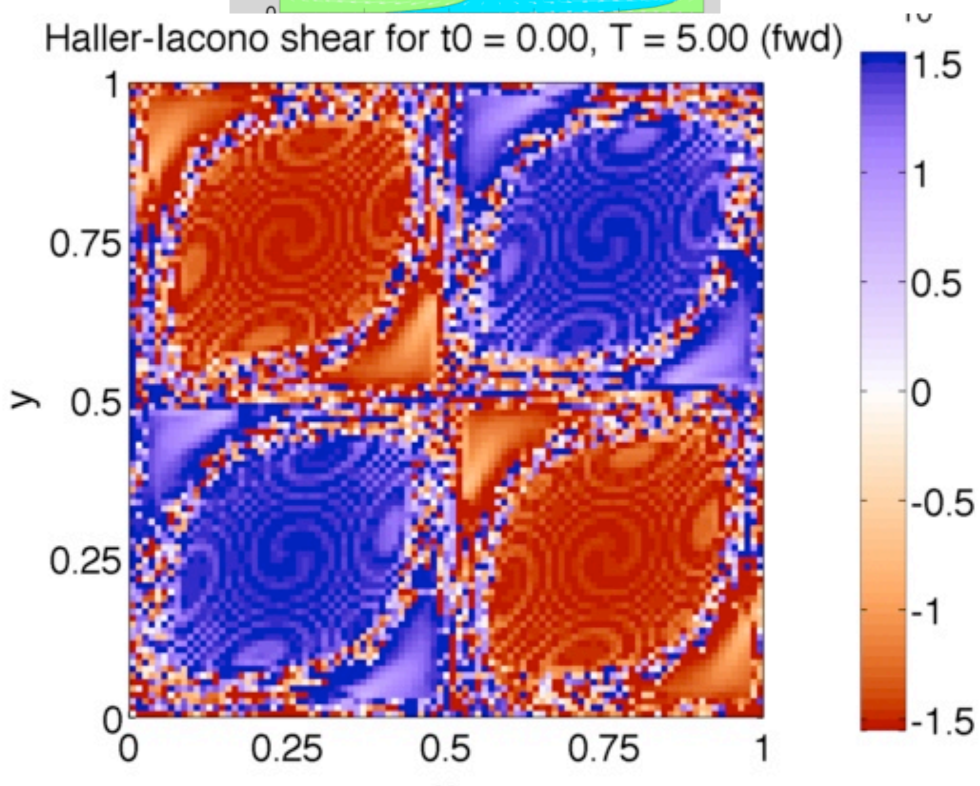
Computed from ROS tensor and v.f.

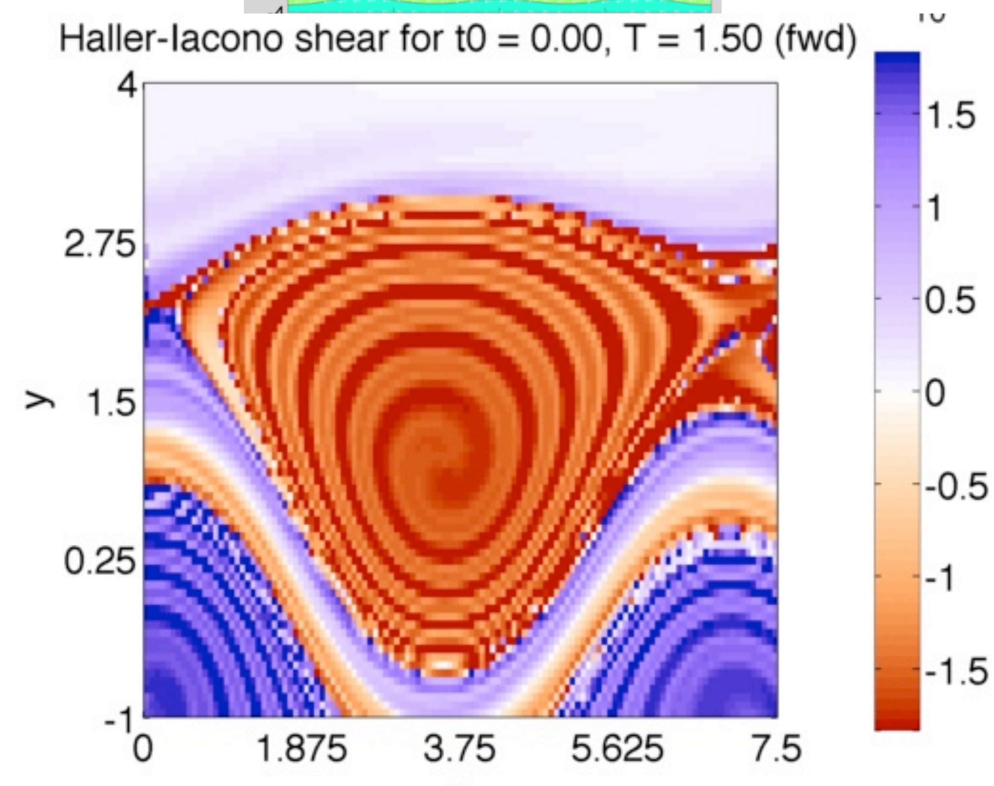
Stretching $$\lambda_{t_0}^t(p) := \int_{t_0}^t -S_\parallel^{(s)}(p)ds$$

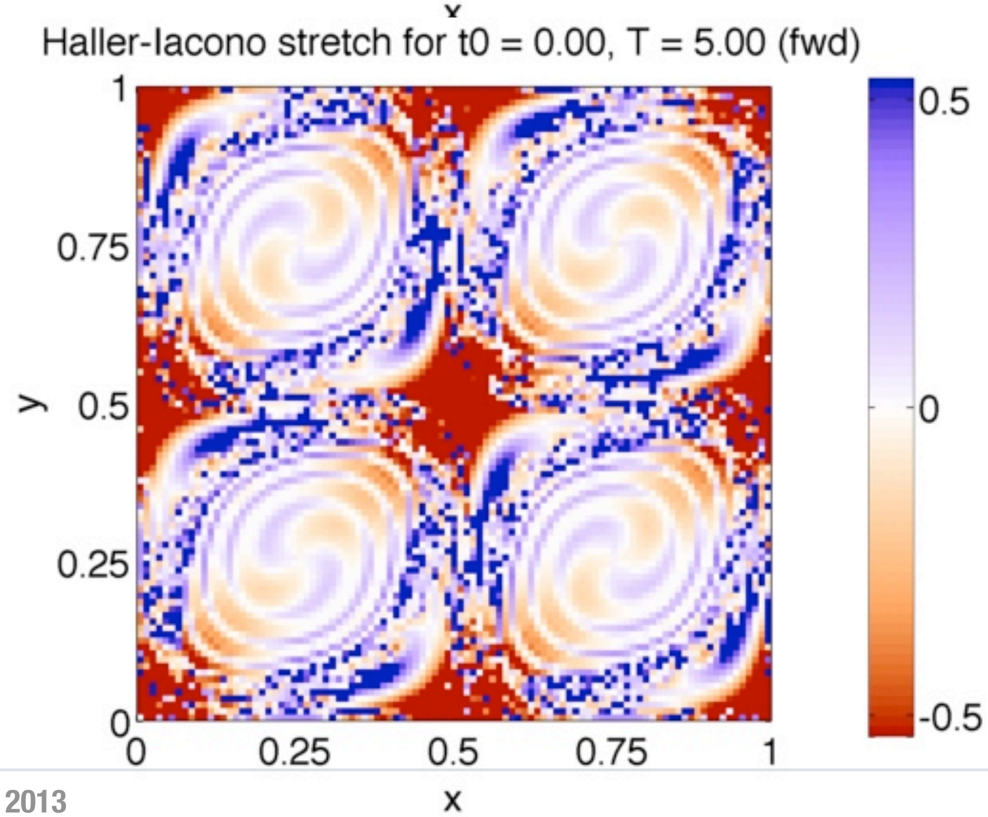
Shearing $$\mu_{t_0}^t(p) := \int_{t_0}^t S_\circ^{(s)}(p) \exp\left[-2\lambda_s^t(p)\right] ds.$$

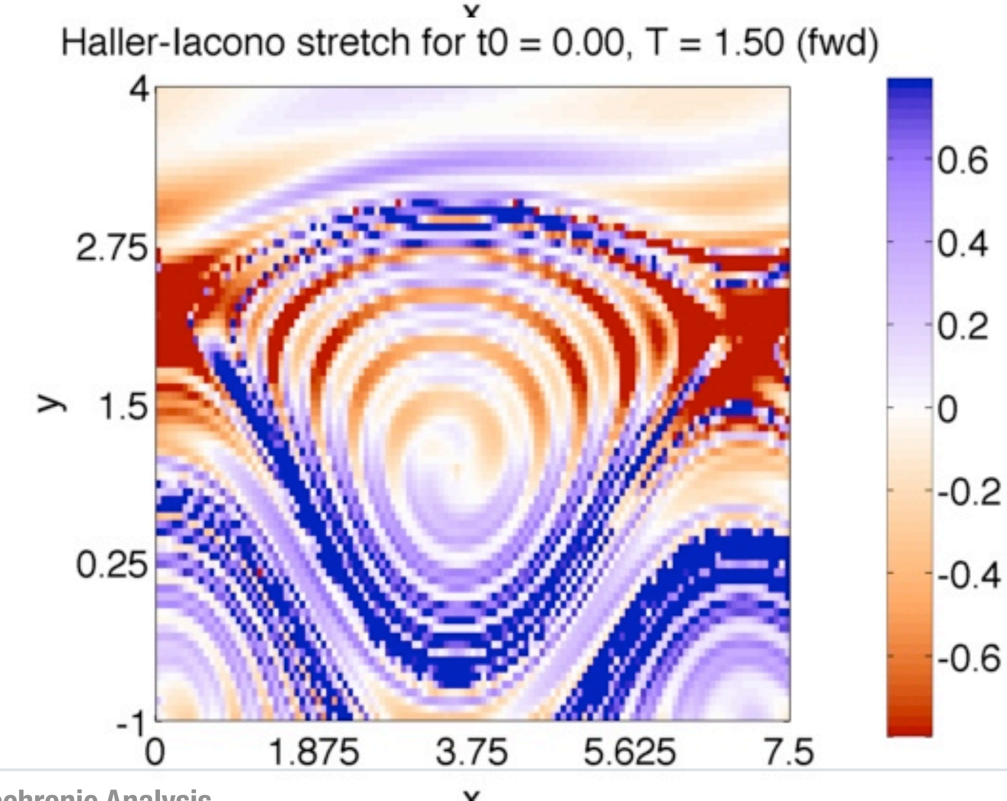

Mezic four-gyre; Color is the stream function; T = 0.00
y


Rypina polar jet; T = 0.00 days
Mm

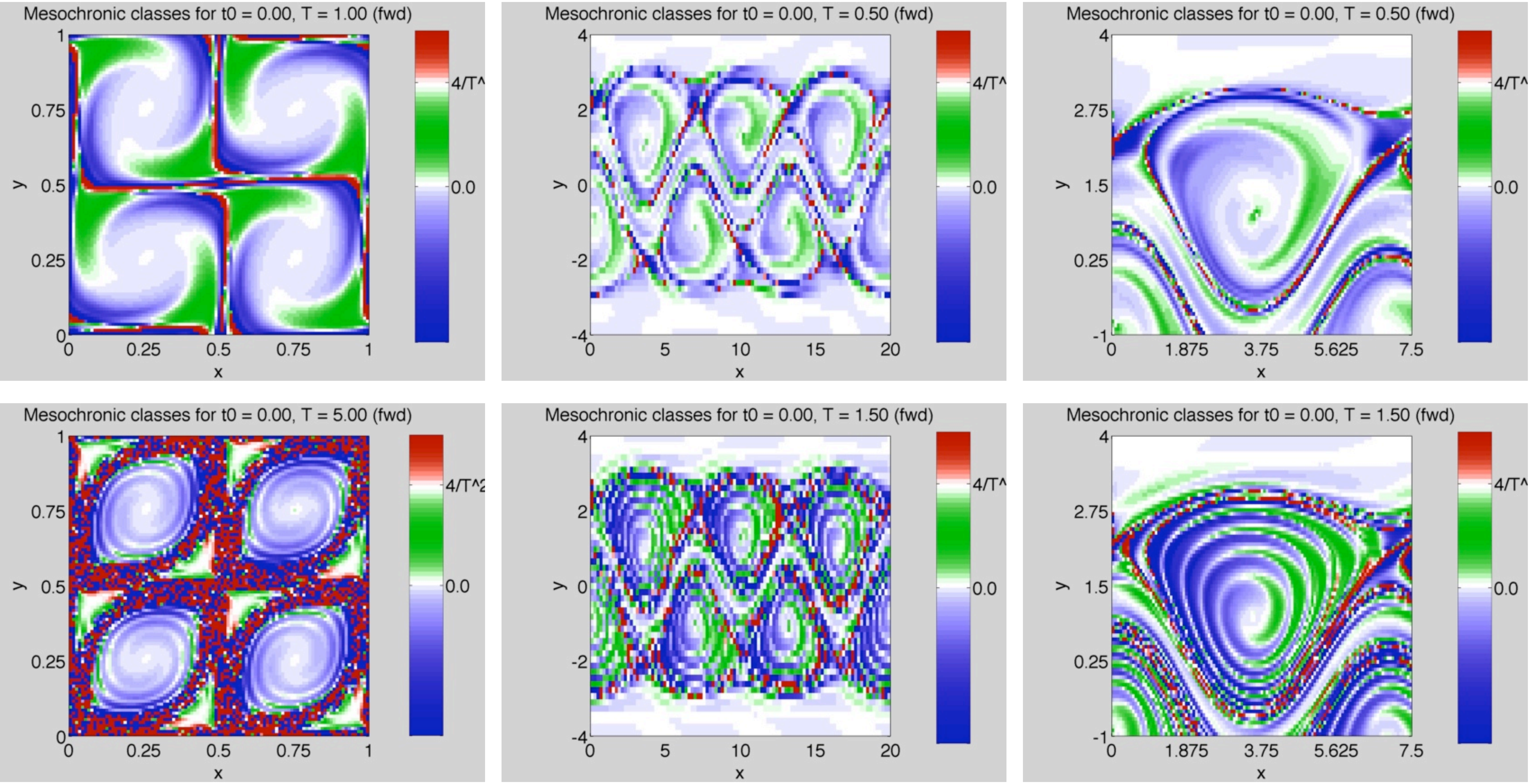

Haller-Iacono shear for t0 = 0.00, T = 5.00 (fwd)
y
x
0
0.25
0.5
0.75
1
1.5
1
0.5
0
-0.5
-1
-1.5


Haller-Iacono shear for t0 = 0.00, T = 1.50 (fwd)
y
x
4
2.75
1.5
0.25
-1
0
1.875
3.75
5.625
7.5
1.5
1
0.5
0
-0.5
-1
-1.5


Haller-Iacono stretch for t0 = 0.00, T = 5.00 (fwd)
y
x
0
0.25
0.5
0.75
1
0.5
0
-0.5


Haller-Iacono stretch for t0 = 0.00, T = 1.50 (fwd)
y
x
4
2.75
1.5
0.25
-1
0
1.875
3.75
5.625
7.5
0.6
0.4
0.2
0
-0.2
-0.4
-0.6

The Frenet frame mesochronic classes.



To-Do:

- Understand how the classes are advected as material or “dye”.
- Understand the importance of values of quantities, not just class.
- Understand bifurcation of structures with the change of time-interval endpoints.
- The 2D code available for download,
  3D needs some polishing but coming soon.

https://bitbucket.org/mbudisic/mesochronic-toolbox