Homogeneity of the ground state for fractional Laplacian on cones

Joint work with K. Bogdan and A. Stós

B. A. Siudeja

University of Oregon

Aug 2, 2013

Classical harmonic functions

Function u is harmonic if

$$\Delta u = 0.$$

Also, eigenfunction for zero eigenvalue.

Classical harmonic functions

Function u is harmonic if

$$\Delta u = 0.$$

Also, eigenfunction for zero eigenvalue.

Probabilistic approach

Let X_t be a stochastic process and A its generator (matrix or differential).

$$e^{tA}u(x) = E_x u(X_t).$$

Classical harmonic functions

Function u is harmonic if

$$\Delta u = 0.$$

Also, eigenfunction for zero eigenvalue.

Probabilistic approach

Let X_t be a stochastic process and A its generator (matrix or differential).

$$e^{tA}u(x) = E_x u(X_t).$$

Harmonic on D (τ_D - exit time):

$$Au = 0 \qquad \Longleftrightarrow \qquad u(x) = E_x(u(X(\tau_D))).$$

Hence harmonic functions have averaging property.

Maximum principle

Averaging property implies no local maxima/minima.

Maximum principle

Averaging property implies no local maxima/minima.
Also, global maximum and minimum are on the boundary of the domain.

Maximum principle

Averaging property implies no local maxima/minima.
Also, global maximum and minimum are on the boundary of the domain.

ZERO Dirichlet eigenvalue / Dirichlet harmonic function

If

$$\begin{aligned} \Delta u &= 0 \quad \text{on } D, \\ u &= 0 \quad \text{on } \partial D, \end{aligned}$$

then

$$u = 0 \quad \text{on } D.$$

Maximum principle

Averaging property implies no local maxima/minima.
Also, global maximum and minimum are on the boundary of the domain.

ZERO Dirichlet eigenvalue / Dirichlet harmonic function

If

$$\begin{aligned} \Delta u &= 0 \quad \text{on } D, \\ u &= 0 \quad \text{on } \partial D, \end{aligned}$$

then

$$u = 0 \quad \text{on } D.$$

Proof.

Obvious. □

Maximum principle

Averaging property implies no local maxima/minima.
Also, global maximum and minimum are on the boundary of the domain.

ZERO Dirichlet eigenvalue / Dirichlet harmonic function

If

$$\begin{aligned} \Delta u &= 0 \quad \text{on } D, \\ u &= 0 \quad \text{on } \partial D, \end{aligned}$$

then

$$u = 0 \quad \text{on } D.$$

Proof.

Obvious. Not for unbounded domains! □

Some 2D cones

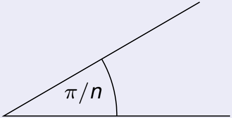
- Half-space: $u(x, y) = y$ is harmonic with zero boundary condition.

Some 2D cones

- Half-space: $u(x, y) = y$ is harmonic with zero boundary condition.
- First quadrant: $u(x, y) = xy$.

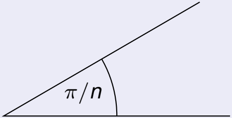
Some 2D cones

- Half-space: $u(x, y) = y$ is harmonic with zero boundary condition.
- First quadrant: $u(x, y) = xy$.
- Function $u(x, y) = \mathrm{Im} z^n$ is harmonic, and zero on the boundary of



Some 2D cones Half-space has infinite ZERO eigenspace!

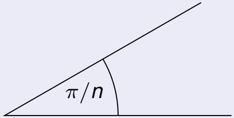
- Half-space: $u(x, y) = y$ is harmonic with zero boundary condition.
- First quadrant: $u(x, y) = xy$.
- Function $u(x, y) = \text{Im} z^n$ is harmonic, and zero on the boundary of



- n above does not need to be natural.

Some 2D cones Half-space has infinite ZERO eigenspace!

- Half-space: $u(x, y) = y$ is harmonic with zero boundary condition.
- First quadrant: $u(x, y) = xy$.
- Function $u(x, y) = \mathrm{Im} z^n$ is harmonic, and zero on the boundary of



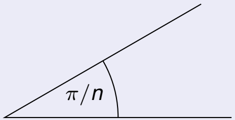
- n above does not need to be natural.

Ground state features

- Positive inside, unbounded

Some 2D cones Half-space has infinite ZERO eigenspace!

- Half-space: $u(x, y) = y$ is harmonic with zero boundary condition.
- First quadrant: $u(x, y) = xy$.
- Function $u(x, y) = \mathrm{Im} z^n$ is harmonic, and zero on the boundary of



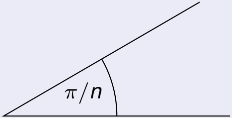
- n above does not need to be natural.

Ground state features

- Positive inside, unbounded
- Homogeneous $\left(u(ax) = a^{\beta} u(x)\right)$ of order $\beta = n$, when aperture $\Theta = \pi/n$.

Some 2D cones Half-space has infinite ZERO eigenspace!

- Half-space: $u(x, y) = y$ is harmonic with zero boundary condition.
- First quadrant: $u(x, y) = xy$.
- Function $u(x, y) = \mathrm{Im} z^n$ is harmonic, and zero on the boundary of



- n above does not need to be natural.

Ground state features

- Positive inside, unbounded
- Homogeneous $\left(u(ax) = a^\beta u(x)\right)$ of order $\beta = n$, when aperture $\Theta = \pi/n$.

In any dimension!

If $\Theta \to 0$ then $\beta \to \infty$.

Poisson kernel

If f is defined on the boundary then u defined below is harmonic

$$u(x) = \int_{\partial D} P(x,y) f(y) dy.$$

$P(x,y)$ - Poisson kernel.

Poisson kernel

If f is defined on the boundary then u defined below is harmonic

$$u(x) = \int_{\partial D} P(x, y) f(y) dy.$$

$P(x, y)$ - Poisson kernel.
Functions from previous slide cannot be represented this way!

Poisson kernel

If f is defined on the boundary then u defined below is harmonic

$$u(x) = \int_{\partial D} P(x, y) f(y) dy.$$

$P(x, y)$ - Poisson kernel.
Functions from previous slide cannot be represented this way!

Extended (Martin) boundary and Martin kernel

Define $\partial_M D$ so that "other" nonnegative harmonic functions have the representation

$$u(x) = \int_{\partial_M D} M(x, y) f(y) dy.$$

$M(x, y)$ - Martin kernel. For cones $\partial_M D$ contains only ∞ point.

Poisson kernel

If f is defined on the boundary then u defined below is harmonic

$$u(x) = \int_{\partial D} P(x,y)f(y)dy.$$

$P(x,y)$ - Poisson kernel.
Functions from previous slide cannot be represented this way!

Extended (Martin) boundary and Martin kernel

Define $\partial_M D$ so that “other” nonnegative harmonic functions have the representation

$$u(x) = \int_{\partial_M D} M(x,y)f(y)dy.$$

$M(x,y)$ - Martin kernel. For cones $\partial_M D$ contains only ∞ point.
Functions from previous slide are Martin kernels (with pole at infinity)!

Stable process and fractional Laplacian

Characteristic function

Isotropic α stable process X_t with $0 < \alpha < 2$ satisfies

$$E_0 e^{i\xi X_t} = e^{-t|\xi|^{\alpha}}.$$

Stable process and fractional Laplacian

Characteristic function

Isotropic α stable process X_t with $0 < \alpha < 2$ satisfies

$$E_0 e^{i\xi X_t} = e^{-t|\xi|^{\alpha}}.$$

Discontinuous process with density

$$p(t,x,y) \approx \min\left\{t^{-d/\alpha}, \frac{t}{|x-y|^{d+\alpha}}\right\}.$$

Stable process and fractional Laplacian

Characteristic function

Isotropic α stable process X_t with $0 < \alpha < 2$ satisfies

$$E_0 e^{i\xi X_t} = e^{-t|\xi|^{\alpha}}.$$

Discontinuous process with density

$$p(t,x,y) \approx \min\left\{ t^{-d/\alpha}, \frac{t}{|x-y|^{d+\alpha}} \right\}.$$

Generator - fractional Laplacian

Nonlocal pseudo-differential operator:

$$\Delta^{\alpha/2} u(x) = C_{\alpha,d} \int_{\mathbb{R}^d} \frac{u(x)-u(y)}{|x-y|^{d+\alpha}} dy$$

Stable process and fractional Laplacian

Characteristic function

Isotropic α stable process X_t with $0 < \alpha < 2$ satisfies

$$E_0 e^{i\xi X_t} = e^{-t|\xi|^{\alpha}}.$$

Discontinuous process with density

$$p(t,x,y) \approx \min\left\{ t^{-d/\alpha}, \frac{t}{|x-y|^{d+\alpha}} \right\}.$$

Generator - fractional Laplacian

Nonlocal pseudo-differential operator:

$$\Delta^{\alpha/2} u(x) = C_{\alpha,d}\, PV \int_{\mathbb{R}^d} \frac{u(x)-u(y)}{|x-y|^{d+\alpha}} dy =$$

$$= C_{\alpha,d} \lim_{\varepsilon\to 0} \int_{B(x,\varepsilon)^c} \frac{u(x)-u(y)}{|x-y|^{d+\alpha}} dy$$

Green function for $\mathbb{R}^d \setminus \{0\}$, harmonic.

$$G(x,y) = C_{\alpha,d}|x|^{-d+\alpha}$$

Note that $\alpha = 2$ gives Brownian case in dimensions $d \geq 3$.
Decay on domains is also similar to Brownian case: $\delta_{\partial D}^{\alpha/2}(x)$.

Green function for $\mathbb{R}^d \setminus \{0\}$, harmonic.

$$G(x, y) = C_{\alpha,d}|x|^{-d+\alpha}$$

Note that $\alpha = 2$ gives Brownian case in dimensions $d \geq 3$.
Decay on domains is also similar to Brownian case: $\delta_{\partial D}^{\alpha/2}(x)$.

Expected exit time from a ball (radius r), superharmonic.

$$\varphi(x) = E_x(\tau_B) = C_{\alpha,d}(r^2 - |x|^2)^{\alpha/2}.$$

Note that $\Delta^{\alpha/2}\varphi(x) = -1$ on B. Take $\alpha = 2$ for Brownian case.

Green function for $\mathbb{R}^d \setminus \{0\}$, harmonic.

$$G(x,y) = C_{\alpha,d}|x|^{-d+\alpha}$$

Note that $\alpha = 2$ gives Brownian case in dimensions $d \geq 3$.
Decay on domains is also similar to Brownian case: $\delta_{\partial D}^{\alpha/2}(x)$.

Expected exit time from a ball (radius r), superharmonic.

$$\varphi(x) = E_x(\tau_B) = C_{\alpha,d}(r^2 - |x|^2)^{\alpha/2}.$$

Note that $\Delta^{\alpha/2}\varphi(x) = -1$ on B. Take $\alpha = 2$ for Brownian case.

Poisson kernel for a ball

$$P(x,y) = C_{\alpha,d}\left(\frac{r^2 - |x|^2}{|y|^2 - r^2}\right)^{\alpha/2} |x-y|^{-d}.$$

There is one extra factor compared to Brownian case. Due to jumps natural boundary equals $D^c \setminus \partial D$, instead of ∂D.

Martin boundary

$\partial D \subset \partial_M D$ for stable processes. Plus there is ∞ point for cones.

Martin boundary

$\partial D \subset \partial_M D$ for stable processes. Plus there is ∞ point for cones.

Half-space

Function $\max(x_d^{\alpha/2}, 0)$ is α-harmonic on $D = \{x_d > 0\}$ and zero outside. Take $\alpha = 2$ and $d = 2$ to get our first example $f(x, y) = y$.

Martin boundary

$\partial D \subset \partial_M D$ for stable processes. Plus there is ∞ point for cones.

Half-space

Function $\max(x_d^{\alpha/2}, 0)$ is α-harmonic on $D = \{x_d > 0\}$ and zero outside. Take $\alpha = 2$ and $d = 2$ to get our first example $f(x, y) = y$. Curiously, for $D = \{x_d \neq 0\}$ and $\alpha > 1$ we get a different harmonic function with homogeneity exponent $\alpha - 1$!

Martin boundary

$\partial D \subset \partial_M D$ for stable processes. Plus there is ∞ point for cones.

Half-space

Function $\max(x_d^{\alpha/2}, 0)$ is α-harmonic on $D = \{x_d > 0\}$ and zero outside. Take $\alpha = 2$ and $d = 2$ to get our first example $f(x, y) = y$. Curiously, for $D = \{x_d \neq 0\}$ and $\alpha > 1$ we get a different harmonic function with homogeneity exponent $\alpha - 1$!

Homogeneity exponent β for a cone of aperture Θ satisfies

$$\lim_{\Theta \to 0} \beta(\Theta) = \alpha \ (\infty \text{ for } \alpha = 2).$$

Martin boundary

$\partial D \subset \partial_M D$ for stable processes. Plus there is ∞ point for cones.

Half-space

Function $\max(x_d^{\alpha/2}, 0)$ is α-harmonic on $D = \{x_d > 0\}$ and zero outside. Take $\alpha = 2$ and $d = 2$ to get our first example $f(x, y) = y$. Curiously, for $D = \{x_d \neq 0\}$ and $\alpha > 1$ we get a different harmonic function with homogeneity exponent $\alpha - 1$!

Homogeneity exponent β for a cone of aperture Θ satisfies

$$\lim_{\Theta \to 0} \beta(\Theta) = \alpha \ (\infty \text{ for } \alpha = 2).$$

Theorem (K. Bogdan, B.S., A. Stós)

Homogeneity exponent β for a cone of aperture Θ satisfies

$$\beta = \alpha - C_{\alpha,d}\Theta^{d+\alpha-1} + O(\Theta^{d+\alpha-1+1\wedge\alpha})$$

Homogeneity lemma

Suppose we can find super and sub harmonic functions that are homogeneous (and decay appropriately). Then the harmonic function we seek will be homogeneous of order between the other two.

Homogeneity lemma

Suppose we can find super and sub harmonic functions that are homogeneous (and decay appropriately). Then the harmonic function we seek will be homogeneous of order between the other two.

"Spherical coordinates" for homogeneous functions (Bañuelos, Bogdan 2004)

If u is γ-homogeneous then

$$\Delta^{\alpha/2} u(x) = \Delta^{\alpha/2}_{\mathbb{S}^{d-1}} u(x) + R_\gamma[u|_{\mathbb{S}^{d-1}}](x),$$

where spherical part is independent of γ.

Homogeneity lemma

Suppose we can find super and sub harmonic functions that are homogeneous (and decay appropriately). Then the harmonic function we seek will be homogeneous of order between the other two.

“Spherical coordinates” for homogeneous functions (Bañuelos, Bogdan 2004)

If u is γ-homogeneous then

$$\Delta^{\alpha/2}u(x) = \Delta^{\alpha/2}_{\mathbb{S}^{d-1}}u(x) + R_\gamma[u|_{\mathbb{S}^{d-1}}](x),$$

where spherical part is independent of γ.

General strategy

- Find 0-homogeneous function φ (no R_γ part to deal with).
- Extend $\varphi|_{\mathbb{S}^{d-1}}$ to be γ-homogeneous and call that u_γ.
- Find γ so that u_γ is sub-/superharmonic.

Any function on the sphere can be extended to any homogeneity order, but what to put on the sphere? Ideally: eigenfunction.

Any function on the sphere can be extended to any homogeneity order, but what to put on the sphere? Ideally: eigenfunction.

Surprisingly easy answer

(Almost) the only explicit thing we have, expected exit time φ from a ball (though it is defined on flat space).

Any function on the sphere can be extended to any homogeneity order, but what to put on the sphere? Ideally: eigenfunction.

Surprisingly easy answer

(Almost) the only explicit thing we have, expected exit time φ from a ball (though it is defined on flat space).

Naive approach

Project to the sphere and extend, then find fractional Laplacian.

Any function on the sphere can be extended to any homogeneity order, but what to put on the sphere? Ideally: eigenfunction.

Surprisingly easy answer

(Almost) the only explicit thing we have, expected exit time φ from a ball (though it is defined on flat space).

Naive approach

Project to the sphere and extend, then find fractional Laplacian.

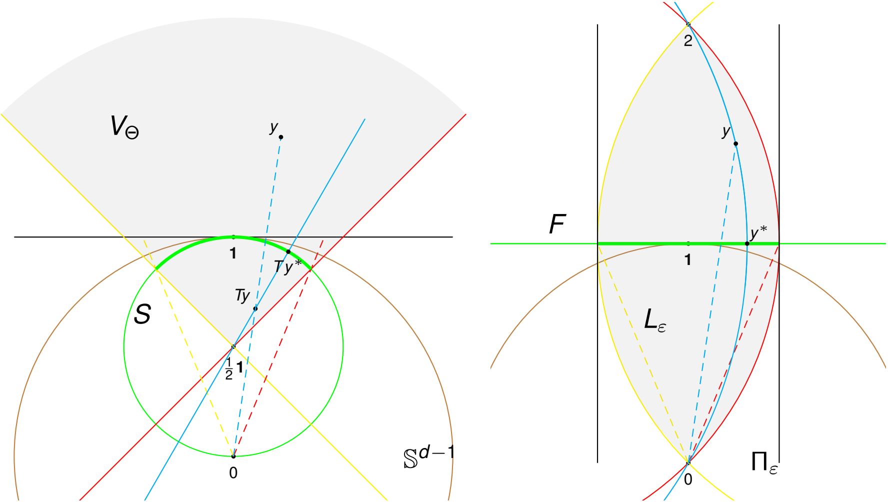
Unfortunately to do the calculations we have to pull back to flat space and this operation destroys PV limit.
Spherical cap can be mapped onto a ball, but centers will not match.

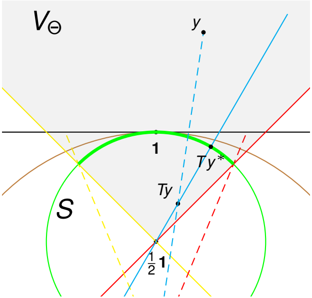
Inversion: $Tx = x/|x|^2$, Kelvin transform: $Kf(x) = G(x)f(Tx)$.

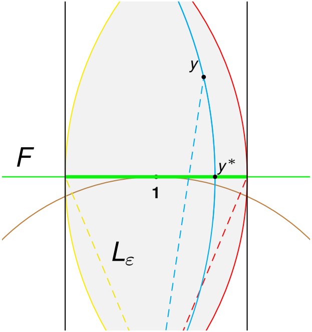
Then: $$\Delta^{\alpha/2}(Kf)(x) = |x|^{-2\alpha}G(x)[\Delta^{\alpha/2}f](Tx).$$

Inversion: $Tx = x/|x|^2$, Kelvin transform: $Kf(x) = G(x)f(Tx)$.

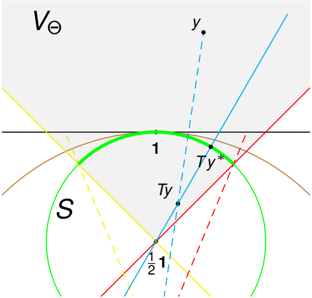
Then: $\Delta^{\alpha/2}(Kf)(x) = |x|^{-2\alpha}G(x)[\Delta^{\alpha/2}f](Tx)$.

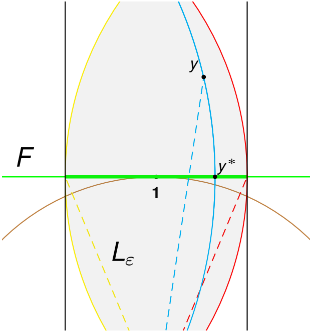






- $\Delta^{\alpha/2}\varphi = -1$ (exit time, lower and full dimension)
- $\Delta^{\alpha/2}G = 0$ (Green function, full dimension)
- $G(y) \approx 1$ on F

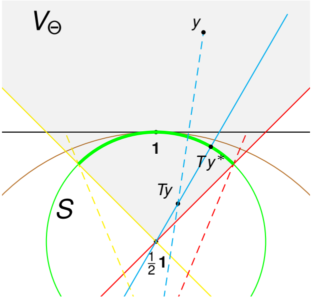


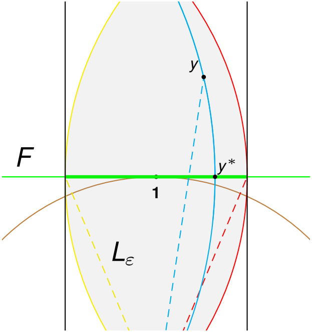


- $\Delta^{\alpha/2}\varphi = -1$ (exit time, lower and full dimension)
- $\Delta^{\alpha/2}G = 0$ (Green function, full dimension)
- $G(y) \approx 1$ on F

Construction:

- Put φG on F to get just φ on S.

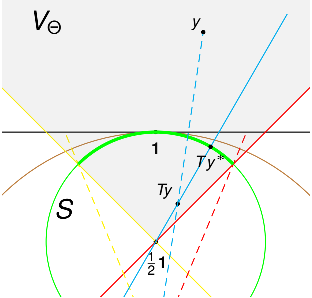




- $\Delta^{\alpha/2}\varphi = -1$ (exit time, lower and full dimension)
- $\Delta^{\alpha/2}G = 0$ (Green function, full dimension)
- $G(y) \approx 1$ on F

Construction:

- Put φG on F to get just φ on S.
- Extend to 0-homogeneous on V_Θ.

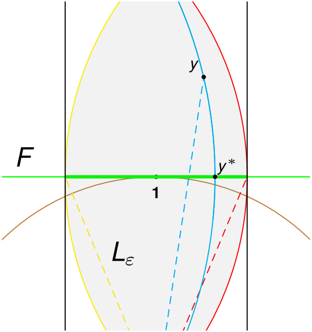


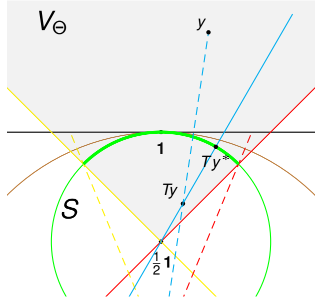
- $\Delta^{\alpha/2}\varphi = -1$ (exit time, lower and full dimension)
- $\Delta^{\alpha/2}G = 0$ (Green function, full dimension)
- $G(y) \approx 1$ on F

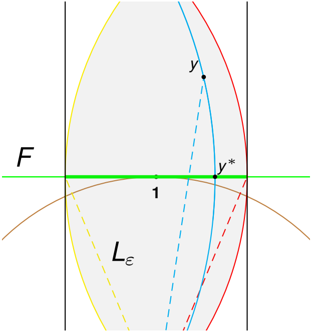
Construction:

- Put φG on F to get just φ on S.
- Extend to 0-homogeneous on $V_\ominus$.
- Push to L_ε to get

$$u(y) = G(y)\varphi(y^*).$$







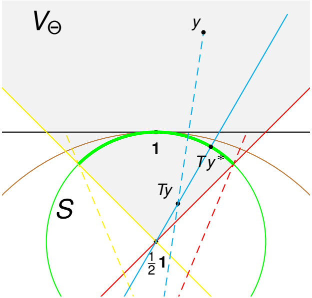
- $\Delta^{\alpha/2}\varphi = -1$ (exit time, lower and full dimension)
- $\Delta^{\alpha/2}G = 0$ (Green function, full dimension)
- $G(y) \approx 1$ on F

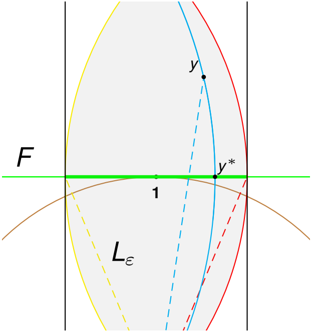
Construction:

- Put φG on F to get just φ on S.
- Extend to 0-homogeneous on $V_\ominus$.
- Push to L_ε to get

$$u(y) = G(y)\varphi(y^*).$$

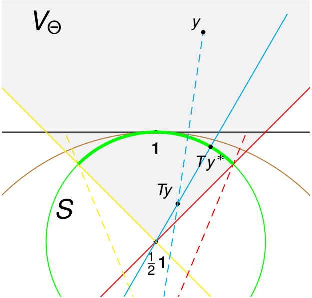
- Now find fractional Laplacian on F. It is roughly the same as spherical Laplacian on S.

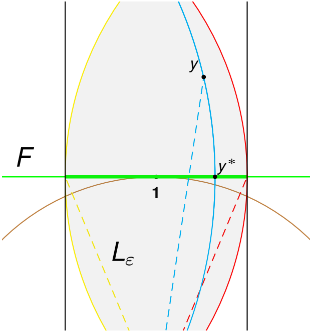




Product rule

$$\Delta^{\alpha/2}[G\varphi](x) = -1G(x) + 0\varphi(x) + \\ + \int_{\mathbb{R}^d} \frac{(G(x)-G(y))(\varphi(x)-\varphi(y))}{|x-y|^{d+\alpha}}dy$$



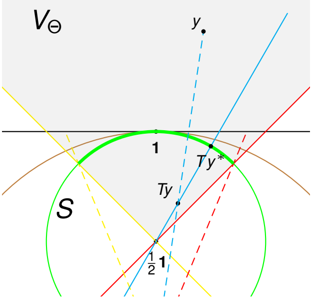


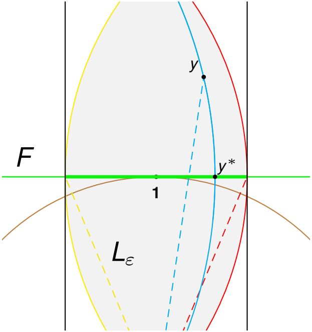
Product rule

$$\Delta^{\alpha/2}[G\varphi](x) = -1G(x) + 0\varphi(x) + \\ + \int_{\mathbb{R}^d} \frac{(G(x)-G(y))(\varphi(x)-\varphi(y))}{|x-y|^{d+\alpha}} dy$$

y^* deformation ($x = x^* \in F$)

$$\Delta^{\alpha/2}[G\varphi - u](x) = \Delta^{\alpha/2}[G(\varphi - \varphi^*)](x) \approx \\ \approx \int_{\mathbb{R}^d} \frac{\varphi(y) - \varphi(y^*)}{|x-y|^{d+\alpha}} dy.$$





Product rule

$$\Delta^{\alpha/2}[G\varphi](x) = -1G(x) + 0\varphi(x) + \\ + \int_{\mathbb{R}^d} \frac{(G(x) - G(y))(\varphi(x) - \varphi(y))}{|x-y|^{d+\alpha}} dy$$

y^* deformation ($x = x^* \in F$)

$$\Delta^{\alpha/2}[G\varphi - u](x) = \Delta^{\alpha/2}[G(\varphi - \varphi^*)](x) \approx \\ \approx \int_{\mathbb{R}^d} \frac{\varphi(y) - \varphi(y^*)}{|x-y|^{d+\alpha}} dy.$$

Both integrals are small, hence

$$\Delta^{\alpha/2}u(x) \approx -1 \text{ (as for exit time alone).}$$

We already know that

$$\Delta_{\mathbb{S}^{d-1}}^{\alpha/2} u = -1 + O(\Theta^{1\wedge\alpha}).$$

Radial part R_γ.

Now we find γ so that the radial part is just over 1, or just below.

We already know that

$$\Delta^{\alpha/2}_{\mathbb{S}^{d-1}} u = -1 + O(\Theta^{1\wedge\alpha}).$$

Radial part R_γ.

Now we find γ so that the radial part is just over 1, or just below.

We show that

$$R_\gamma[u] \approx F(\Theta, \alpha - \gamma),$$

where F is an ugly function.

We already know that

$$\Delta_{\mathbb{S}^{d-1}}^{\alpha/2} u = -1 + O(\Theta^{1\wedge\alpha}).$$

Radial part R_γ.

Now we find γ so that the radial part is just over 1, or just below.

We show that

$$R_\gamma[u] \approx F(\Theta, \alpha - \gamma),$$

where F is an ugly function.

Then we take

$$\gamma = \alpha - C\Theta^{d+\alpha-1}(1 \pm \kappa\Theta^{1\wedge\alpha})$$

and we get sub-/superharmonic functions.