Delocalization of eigenvectors and convergence of the density of states on random regular graphs

Leander Geisinger

Spectral Theory of Laplace and Schroedinger Operators

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PRINCETON UNIVERSITY

Introduction: The Anderson model

The model describes the behavior of an electron in a random environment (e.g. in $\mathbb{Z}^d$).

- free electron: Laplace operator

$$(-\Delta\phi)(x) = \sum_{y\sim x} (\phi(x) - \phi(y))$$

$$\sigma(-\Delta) = \sigma_{\text{ac}}(-\Delta) = [0, 4d]\,, \qquad \text{generalized eigenfunctions:} \quad e^{ik\cdot x}$$

- environment: random potential $(V_x)_{x\in\mathbb{Z}^d}$ iid random variables

$$(V\phi)(x) = V_x\,\phi(x)$$

We assume that the distribution of V_x is absolutely continuous with bounded density ρ and with support $(-\rho_0, \rho_0)$. Almost surely:

$$\sigma(V) = \sigma_{\text{pp}}(V) = (-\rho_0, \rho_0)\,, \qquad \text{eigenfunctions:} \quad \delta_x \in \ell^2(\mathbb{Z}^d)$$

- Anderson model:

$$H = -\Delta + V \qquad \text{with} \qquad \sigma(H) = [-\rho_0, 4d + \rho_0] \quad \text{a.s.}$$

Introduction: Anderson localization

Fröhlich & Spencer '83, Aizenman & Molchanov '93

There is a constant $C(\rho) > 2d$ such that in the energy range

$$\{|E - 2d| \geq C(\rho)\}$$

the operator

$$H = -\Delta + V$$

has almost surely only pure point spectrum $\{\lambda_j\}_{j\in\mathbb{N}}$ with exponentially localized, $\ell^2(\mathbb{Z}^d)$-normalized eigenfunctions $\{\phi_j\}_{j\in\mathbb{N}}$:

$$\sigma(H) \cap \{|E - 2d| \geq C(\rho)\} = \overline{\{\lambda_j\}_{j\in\mathbb{N}}}$$

and

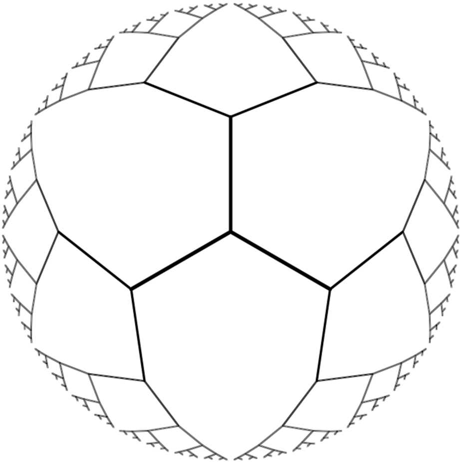
$$|\phi_j(x)| \leq C \exp\left(-c|x - x_j|\right) .$$

In particular, there is a finite $r > 0$ such that ϕ_j is localized in $B_r(x_j)$:

$$\left\| \phi_j|_{B_r(x_j)} \right\|^2 > \frac{1}{2} .$$

Consider a regular tree $\mathcal{T}$ of degree $K+1 \geq 3$ and

$$H_{\mathcal{T}} = -\Delta_{\mathcal{T}} + V\,.$$



Introduction: The Anderson model on a tree

Consider a regular tree $\mathcal{T}$ of degree $K+1 \geq 3$ and

$$H_{\mathcal{T}} = -\Delta_{\mathcal{T}} + V\,.$$

The spectra are given by

$$\begin{aligned} \sigma(-\Delta_{\mathcal{T}}) &= (K+1-2\sqrt{K}, K+1+2\sqrt{K}) \\ \sigma(H_{\mathcal{T}}) &= (K+1-2\sqrt{K}-\rho_0, K+1+2\sqrt{K}+\rho_0) \end{aligned} \qquad .$$

Klein '98, Aizenman & Warzel '13

- There is an interval $I_\rho \subset (K+1-2\sqrt{K}, K+1+2\sqrt{K})$ such that almost surely

$$I_\rho \cap \sigma_{\text{ac}}(H_{\mathcal{T}}) = I_\rho\,.$$

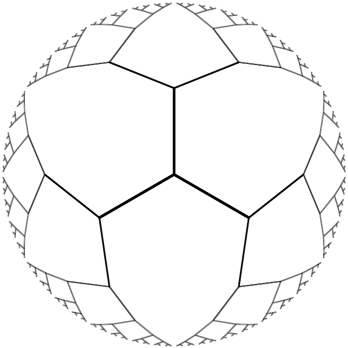
- For ρ_0 small enough there is almost surely a.c. spectrum up to the spectral edges.

$\Longrightarrow$ $\sigma_{\text{ac}}(H_{\mathcal{T}})$ *is not empty.*

Results: Finite regular graphs

Let $\mathcal{G}_{n,K}$ denote the set of all simple, $K+1$-regular graphs with n vertices.

Pick $G_n \in \mathcal{G}_{n,K}$ at random with uniform distribution.



Lemma (tree approximation)

For $x \in G_n$ let

$$B^{(n)}(x) = \{y \in G_n \ : \ d(x,y) < \log_K(n)/2\}$$

$$F_n = \{x \in G_n \ : \ B_n(x) \text{ is acyclic}\}.$$

Then, for all $\epsilon > 0$,

$$|F_n| \geq (1-\epsilon)|G_n|$$

holds asymptotically almost surely as $n \to \infty$.

Results: Delocalizaiton of eigenvectors

Consider the operator

$$H_n = -\Delta_n + V$$

on $\ell^2(G_n)$ and let $(\lambda_j^{(n)})_{j=1}^n$ and $(\phi_j^{(n)})_{j=1}^n$ denote the eigenvalues and $\ell^2(G_n)$-normalized eigenvectors.

Theorem (Delocalization)

Choose $G_n \in \mathcal{G}_{n,K}$ and $x_0 \in G_n$ uniformly at random. Put

$$c_j^{(n)}(x_0) = \begin{cases} 0 & \text{if } \lambda_j^{(n)} \notin \sigma_{\text{ac}}(H_{\mathcal{T}}) \\ |\phi_j^{(n)}(x_0)|^2 & \text{if } \lambda_j^{(n)} \in \sigma_{\text{ac}}(H_{\mathcal{T}}) . \end{cases}$$

Then for any fixed $r > 0$ the estimate

$$\sum_{j=1}^{n} c_j^{(n)}(x_0) \left\| \phi_j^{(n)} \Big|_{B_r(x_0)} \right\|^2 \leq C(K,\rho) \frac{|B_r(x_0)|}{\sqrt{\log_K(n)}}$$

holds asymptotically almost surely as $n \to \infty$.

In particular, eigenvectors with eigenvalues in $\sigma_{\text{ac}}(H_{\mathcal{T}})$ are not localized.

RESULTS: DELOCALIZAITON OF EIGENVECTORS

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holds asymptotically almost surely as $n \to \infty$.

In particular, eigenvectors with eigenvalues in $\sigma_{\text{ac}}(H_{\mathcal{T}})$ are not localized.

Ideas of proof: Convergence of the density of states

To study spectral properties of $-\Delta_n$ on a regular graph G_n define

$$\mu_{n,x}(I) = (\delta_x, \chi_I(-\Delta_n)\delta_x)_{\ell^2(G_n)} = \sum_{\lambda_j \in I} |\phi_j(x)|^2 , \qquad I \subset \mathbb{R} , \qquad x \in G_n .$$

For fixed $I \subset \mathbb{R}$:

$$\frac{1}{n} \sum_{x \in G_n} \mu_{n,x}(I) \longrightarrow \sigma_{\mathcal{T}}(I) ,$$

where $\sigma_{\mathcal{T}}(I) = (\delta_x, \chi_I(-\Delta_{\mathcal{T}})\delta_x)_{\ell^2(\mathcal{T})}$ is independent of $x \in \mathcal{T}$, supported in $(K+1-2\sqrt{K}, K+1+2\sqrt{K})$, and a.c. with bounded density.

Theorem (Rate of convergence)

Asymptotically almost surely, we have

$$\sup_{t \in \mathbb{R}} \frac{1}{n} \sum_{x \in G_n} |\mu_{n,x}((-\infty, t]) - \sigma_{\mathcal{T}}((-\infty, t])| \leq C_d \frac{1}{\log_K(n)} .$$

If $B^{(n)}(x) = \{y \in G_n \ : \ d(x,y) < \log_K(n)/2\}$ is acyclic, then for $k = 0, 1, \ldots, \lfloor \log_K(n) \rfloor$

$$\int \lambda^k d\mu_{n,x} = (\delta_x, (-\Delta_n)^k \delta_x)_{\ell^2(G_n)} = (\delta_x, (-\Delta_{\mathcal{T}})^k \delta_x)_{\ell^2(\mathcal{T})} = \int \lambda^k d\sigma_{\mathcal{T}} .$$

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Ideas of proof: Consequences

For $z = E + i\eta \in \mathbb{C}_+$ put

$$\Gamma_n(x,z) = (\delta_x, (-\Delta_n - z)^{-1}\delta_x)_{\ell^2(G_n)} = \int (\lambda - z)^{-1} d\mu_{n,x}$$

$$\Gamma_{\mathcal{T}}(z) = (\delta_x, (-\Delta_{\mathcal{T}} - z)^{-1}\delta_x)_{\ell^2(\mathcal{T})} = \int (\lambda - z)^{-1} d\sigma_{\mathcal{T}}\,.$$

Corollary (convergence of the Green function)

Assume that $B^{(n)}(x) = \{y \in G_n \,:\, d(x,y) < \log_K(n)/2\}$ *is acyclic. Then*

$$|\mathrm{Im}\Gamma_n(x, E+i\eta) - \mathrm{Im}\Gamma_{\mathcal{T}}(E+i\eta)| \le C_d \frac{1}{\eta \log_K(n)}\,.$$

Fix $x_0 \in G_n$ and note that for all $m = 1, \ldots, n$ and $\eta > 0$:

$$\sum_{x\in B_r(x_0)} |\phi_m(x)|^2 = \eta \sum_{x\in B_r(x_0)} \frac{\eta}{(\lambda_m - \lambda_m)^2 + \eta^2} |\phi_m(x)|^2$$

$$\le \eta \sum_{x\in B_r(x_0)} \sum_{j=1}^{n} \frac{\eta}{(\lambda_m - \lambda_j)^2 + \eta^2} |\phi_j(x)|^2 = \eta \sum_{x\in B_r(x_0)} \mathrm{Im}\Gamma_n(x, \lambda_m + i\eta)\,.$$

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$$\leq \eta \sum_{x \in B_r(x_0)} \sum_{j=1}^{n} \frac{\eta}{(\lambda_m - \lambda_j)^2 + \eta^2} |\phi_j(x)|^2 = \eta \sum_{x \in B_r(x_0)} \mathrm{Im}\Gamma_n(x, \lambda_m + i\eta)\,.$$

Ideas of proof: Delocalization

Fix $x_0 \in G_n$ and note that for all $m = 1, \ldots, n$ and $\eta > 0$:

$$\left\| \phi_m|_{B_r(x_0)} \right\|^2 = \sum_{x \in B_r(x_0)} |\phi_m(x)|^2 \leq \sum_{x \in B_r(x_0)} \eta \mathrm{Im}\Gamma_n(x, \lambda_m + i\eta)\,.$$

If $B^{(n)}(x) = \{y \in G_n \, : \, d(x,y) < \log_K(n)/2\}$ is acyclic for all $x \in B_r(x_0)$ then

$$\left\| \phi_m|_{B_r(x_0)} \right\|^2 \leq |B_r(x_0)| \left(\eta \,\mathrm{Im}\Gamma_{\mathcal{T}}(\lambda_m + i\eta) + \frac{C_d}{\log_K(n)} \right)\,.$$

Since $\sigma_{\mathcal{T}}$ is a.c. the function $\mathrm{Im}\Gamma_{\mathcal{T}}(\lambda_m + i\eta)$ is bounded as $\eta \downarrow 0$.
Thus

$$\left\| \phi_m|_{B_r(x_0)} \right\|^2 \leq C_d |B_r(x_0)| \frac{1}{\log_K(n)}\,.$$

$\Longrightarrow$ Asymptotically almost surely, none of the ϕ_m is localized in $B_r(x_0)$.