Motion in Random Potentials

Andreas Knauf

Department Mathematik
Universität Erlangen-Nürnberg (Germany)

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General question

- Transport properties and ergodic theory of the classical flow generated by Hamiltonian function

$$H(p,q) = \tfrac{1}{2}\|p\|^2 + V(q),$$

  with (random) potential V on $\mathbb{R}^d$.
- In particular: *asymptotic velocities* for initial conditions $x_0 = (p_0, q_0)$

$$\overline{v}^{\pm}(x_0) := \lim_{T\to\pm\infty} \frac{q(T,x_0) - q_0}{T}.$$

- Quantum mechanical counterpart has been intensively studied for more than 40 years.
- A.K. and Christoph Schumacher:
  Classical motion in random potentials.
  Ergodic Theory and Dynamical Systems **33**, 1–37, 2013

Known: Motion in *periodic* 2D coulombic potentials (of finite horizon) is diffusive:
For all energies E above a threshold energy and all probability measures μ of initial conditions x_0 on $H^{-1}(E)$ (of finite second moment and absolutely continuous w.r.t. Liouville measure)

$$\lim_{t\to\infty} \frac{q(t,x_0)}{\sqrt{t}} \stackrel{\mathcal{D}}{=} N(0,D) \qquad \text{(bivariate normal distribution).}$$

A.K.: Ergodic and Topological Properties of Coulombic Periodic Potentials. *Commun. Math. Phys.* **110**, 89-112 ('87)

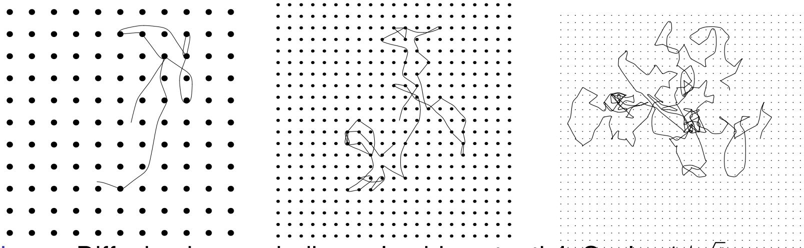


Figure: Diffusion in a periodic coulombic potential. Scale: $1/\sqrt{t}$
Left: time $t=1$, mid: $t=4$, right: $t=16$

Periodic vs random scatterers

Motion in *periodic* 2D **Lorentz gas** (of finite horizon)

- is diffusive.

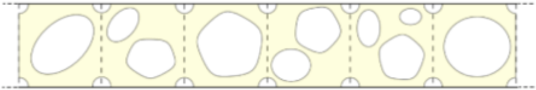
  L.A. Bunimovich, N.I. Chernov, Ya.G. Sinai: Statistical properties of two-dimensional hyperbolic billiards. *Russ. Math. Surv.* **46**, 47–106 (1991)

- is recurrent (with probability one it returns infinitely often in any prescribed neighborhood)

  J.-P. Conze: Sur un critère de récurrence en dimension 2 pour les marches stationnaires, applications. *Ergodic Theory Dynam. Systems* **19**, 1233–1245 (1999)

  K. Schmidt: On joint recurrence. C. R. Acad. Sci. Paris **327**, 837–842 (1998)

One expects recurrence for the *random* case, too,
iff spatial dimension $d \leq 2$.
Known for (effective) dimension $d = 1$:



Giampaolo Cristadoro, Marcello Seri, Marko Lenci:
Recurrence for quenched random Lorentz tubes, *Chaos* **20**, (2010)

Random potentials: The lattice case

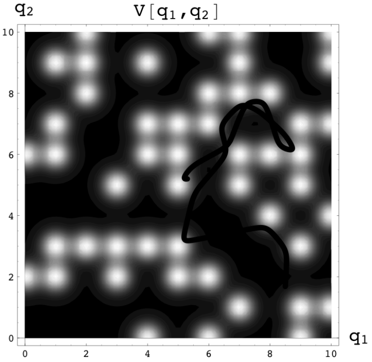
$$H_\omega : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R} \quad , \quad H_\omega(p,q) = \tfrac{1}{2}\|p\|^2 + V_\omega(q)$$

with $\omega \in \Omega := \{1,\ldots,N\}^{\mathcal{L}}$ for a lattice $\mathcal{L} \subseteq \mathbb{R}^d$,

- single site potentials $W_j : \mathbb{R}^d \to \mathbb{R} \quad (j \in \{1,\ldots,N\})$ (with $W_j(q) = \mathcal{O}(\|q\|^{-d-\varepsilon})$ as $q \to \infty$), and
- random potential

$$V_\omega : \mathbb{R}^d \to \mathbb{R},$$
$$V_\omega(q) = \sum_{z\in\mathcal{L}} W_{\omega(z)}(q-z).$$

- $\mathcal{L}$-ergodic probability measure β on Ω,
- Application: crystals with impurities/foreign atoms, alloys



Random potentials: The Poisson case

- prescribe intensities ρ_j for the single site potentials W_j, $j = 1, \ldots, N$
- *marked Poisson process* on $\mathbb{R}^d$

$$\tilde{\Omega} := \{\omega \mid \omega \text{ Borel measure on } \mathbb{R}^d \times \{1, \ldots, N\} \text{ with } \omega(K \times \{j\}) \in \mathbb{N}_0 \text{ if } K \subseteq \mathbb{R}^d \text{ is compact}\},$$

Lebesgue

- with measure

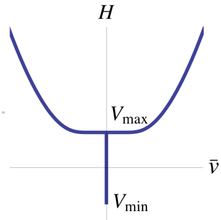
$$\beta\big(\{\omega \in \tilde{\Omega} \mid \omega(K \times \{j\}) = m\}\big) = \frac{\big(\rho_j \lambda^d(K)\big)^m}{m! \exp\big(\rho_j \lambda^d(K)\big)} \quad (m \in \mathbb{N}_0).$$

- Poisson potential

$$V : \tilde{\Omega} \times \mathbb{R}^d \longrightarrow \mathbb{R} \quad , \quad (\omega, q) \longmapsto \int_{\mathbb{R}^d \times J} W_j(q - x)\, d\omega(x, j).$$

Dynamics: results

- Due to ergodic theorem, asymptotic velocities $\overline{v}^{\pm}_{\omega}(x) = \lim_{T\to\pm\infty} \frac{q_\omega(T,x)}{T}$ exist for $\beta \otimes \lambda^{2n}$–a.e. (ω, x), and
$$\overline{v}^{+}_{\omega}(x) = \overline{v}^{-}_{\omega}(x) =: \overline{v}_{\omega}(x).$$
- Joint distribution ν_ω of energy and asymptotic velocity on $\mathbb{R}^d \times \mathbb{R}$ exists and is β–a.s. independent of ω
- mirror symmetry $(H, \overline{v}) \mapsto (H, -\overline{v})$
- $d = 1$: Dichotomy:
  - Either the energy E is higher than the supremum of V_ω, then the motion is *ballistic* (positive speed), or
  - E is lower than the supremum of V_ω, then the motion is almost surely *bounded*.



Intermezzo: critical values of the Hamiltonian

- **Problem:** By conservation of energy, one has to decompose phase space into energy shells in order to do ergodic theory. These carry decent measures only for regular energy values.
- critical values of H_ω $\quad = \quad$ critical values of V_ω
- The closure of the set $\mathrm{CVal}_\omega \subseteq [V_{\min}, V_{\max}]$ of critical values is β–a.s. ω–independent.
- Example of exponentially decaying ($W_j(q) = \mathcal{O}(N^{-2|q|})$) single-site potentials with $\overline{\mathrm{CVal}_\omega} = [V_{\min}, V_{\max}]$!
- But for faster exponential decay $\lambda^1(\overline{\mathrm{CVal}_\omega}) = 0$.

Two notions of ergodicity of time evolution

- $H : \Omega \times \mathbb{R}^d_p \times \mathbb{R}^d_q \to \mathbb{R} \quad , \quad H(\omega, p, q) = \frac{1}{2}\|p\|^2 + V_\omega(q)$ generates motion which is trivial on Ω
- Thus never ergodicity on $H^{-1}(E)$.
- Lattice $\mathcal{L}$ acts on phase space *and* on Ω, leaving H invariant.
  Thus motion on $(\Omega \times \mathbb{R}^{2d})/\mathcal{L}$, generated by Hamiltonian $\widehat{H}$.
- Motion on compactified energy surface $\widehat{H}^{-1}(E)$ may be ergodic.
- Motion on energy surfaces $H_\omega^{-1}(E)$ may be ergodic, too.
- If motion on $H_\omega^{-1}(E)$ is ergodic for β–almost all ω, then motion on $\widehat{H}^{-1}(E)$ is ergodic, too.
- but not vice versa.

Consequences of ergodicity

If the flow on the regular energy surface $\widehat{H}^{-1}(E)$ is ergodic, then:

- the asymptotic velocity satisfies $\overline{v}_\omega(x) = 0$ almost surely; but
- the motion is unbounded for almost every initial condition on $H_\omega^{-1}(E)$ and for β–a.e. ω.

Dynamics: further (negative) results

- $d \geq 2$, smooth bounded potentials:
  for no energy E (larger nor smaller the supremum of V_ω) the motion can be uniformly hyperbolic.

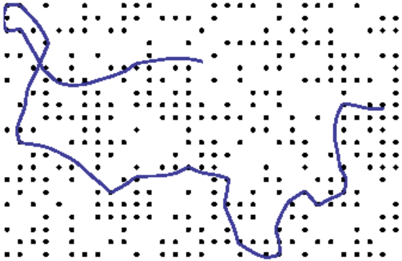
  G. Paternain and M. Paternain: On Anosov Energy Levels of Convex Hamiltonian Systems, *Mathematische Zeitschrift* **217**, 367–376 (1994)

  So we do not expect the motion to be ergodic in general.
- For the Poisson potentials, complete dynamics exists for all $\omega \in \Omega \subseteq \tilde{\Omega}$ with full measure ($\beta(\Omega) = 1$).
  But: for any energy E, the motion on the energy surface $H_\omega^{-1}(E)$ is β–a.s. not ergodic !

Dynamics: Coulombic potentials

$d = 2$, random coulombic potentials (say, with single site potentials $W_0(q) = 0$ and *Yukawa* potentials $W_j(q) = -Z_j \frac{e^{-\mu_j \|q\|}}{\|q\|}$):
For lattice-ergodic probability measures β (with $\beta(\{\omega = 0\}) = 0$)

- the motion is topologically transitive for all $E > E_\circ$ (even if it is not uniformly hyperbolic)
- the periodic orbits are dense.
- the compactified motion is then ergodic.



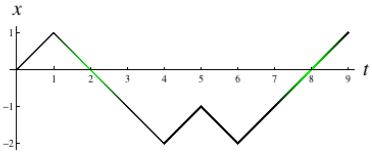
Motion in a random Coulombic potential

- In his thesis Christoph Schumacher constructed a *geometric* Markov partition (in the sense of C. Series, with Poincaré surfaces projecting to configuration space trajectories), which is adapted to the lattice action.
  `www.opus.ub.uni-erlangen.de/opus/volltexte/2010`
- We try to show a central limit theorem, that is, diffusion of the particle in the plane.
  Difficulty: Even with uniform hyperbolicity, the correlations do not decay exponentially, like in random motion in random environment.

Slow correlation decay in random media

Herbert Spohn's example

- random motion on phase space $\mathbb{R} \times \{-1, 1\}$
- starting at $(x_0, v_0) := (0, 1)$
- zig-zag motion $x(t) = x([t]) + (t - [t])v([t])$
- For times $t \in \mathbb{N}$: Probability
  - $\mathbb{P}\big(\{v(t+1) = v(t)\}\big) = \frac{1}{2}$ if $x(t) \neq 0$;
  - $\mathbb{P}\big(\{v(t+1) = v(t)\}\big) = 1$ if $x(t) = 0$ (no scatterer at $x = 0$).



H. van Beijeren, H. Spohn: Transport Properties of the One - Dimensional Stochastic Lorentz Model: I. Velocity, Autocorrelation. J. Stat. Phys. **31**, 231–254 (1983)

Velocity autocorrelation $\mathbb{E}\big(v(0)v(t)\big)$

- if scatterer at $x = 0$, too: $\mathbb{E}\big(v(0)v(t)\big) = 0$ for $t \geq 1$
- else: $|\mathbb{E}\big(v(0)v(t)\big)| \sim t^{-3/2}$.

Local regularisation

The Hamiltonian flow is incomplete at Coulombic singularities.

Example (Kepler problem: $V(q) = -\frac{1}{\|q\|}$)

- Up to time parametrization geodesics of the Maupertuis-Jacobi metric

$$\big(E - V(q)\big) g_{Euclid} \quad \text{on} \quad \mathbb{R}^2 \setminus \{0\}$$

are the energy E trajectories of the Kepler problem.

- At the singularity the metric develops a cone with opening angle $\frac{\pi}{3}$.
- Levi-Civita regularisation: The Riemann surface

$$\{(q, Q) \in \mathbb{C}^2 \mid q = Q^2\}$$

covers $\mathbb{C}$ via $(q, Q) \mapsto q$, and the lifted Maupertuis-Jacobi metric can be smoothly completed.

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Global regularisation

- To regularise all singularities of V_ω simultaneously we use the natural generalisation:

$$M := \{(q, Q) \in \mathbb{C}^2 \mid f(q) = Q^2\},$$

with $f\colon \mathbb{C} \to \mathbb{C}$ holomorphic,
$f(z) = 0$, $f'(z) \neq 0$ ($z \in \mathbb{C}$ position of Coulomb singularity).

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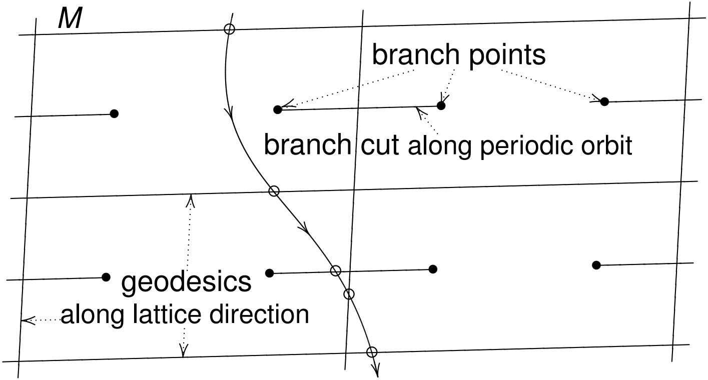
Problems and Strategy

- Problem 1:
  Due to arbitrarily large regions without Coulomb potentials, the negative curvature is not bounded away from zero.
- Strategy:
  But the Riemann surface $(\mathbf{M}^*_\omega, \mathbf{g}^*_\omega)$ is a *visibility manifold*, that is, for every $\varepsilon > 0$, seen from $p \in \mathbf{M}^*_\omega$ every geodesic of distance $> r(p, \varepsilon)$ encloses an angle $< \varepsilon$.

- Problem 2:
  - The energy surface $H^{-1}_\omega\{E\}$ has infinite invariant measure.
  - Compactification by lattice action leads to finite measure but nonhyperbolic system.
- Strategy:
  - Set up symbolic dynamics
  - via geometric Markov partition, *i.e.* adapted to the lattice.

Geometric Poincaré Sections

To encode the winding geodesic we can record
the pieces of the web N of geodesics with piercing points $\circ$.



Problem: resulting shift space not Markov!

Theorem (Existence of geometric Markov partition)

There exists a Markov partition for the geodesic flow on M whose atoms project to the net N.

- The proof is constructive and uses ideas from

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Thank you!

(Hopefully this presentation was not too chaotic!)