Non-integrability criterion for homogeneous Hamiltonian systems via blowing-up theory of singularities

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Hamiltonian system and its integrability

- Hamiltonian system :

$$\frac{dq_j}{dt} = \frac{\partial H}{\partial p_j}(p,q), \quad \frac{dp_j}{dt} = -\frac{\partial H}{\partial q_j}(p,q) \qquad (j = 1, \dots, k) \quad (1)$$

  where $p = (p_1, \dots, p_k), q = (q_1, \dots, q_k)$, $H : \mathbb{R}^{2k} \to \mathbb{R}$.

- Hamiltonian system (1) is integrable $\Longleftrightarrow$ there are k first integrals $F_1(= H), F_2, \dots, F_k$ such that $dF_1, \dots, dF_k$ are linearly independent a.e. and that $\{F_i, F_j\} = 0$ for any $i, j = 1, \dots, k$.
- The dynamics of the integrable Hamiltonian systems are well understood because of the Liouville-Arnold theorem.
- The dynamics of the non-integrable Hamiltonian systems may be "chaotic".
- Problem: distinguish between integrable and non-integrable Hamiltonian systems.

Brief history

- Bruns (1887) proved $\nexists$ algebraic first integral in the 3BP.
- Poincaré(around 1890) proved $\nexists$ analytic first integral in the R3BP.

- Kovalevskaya(1889) discovered an integrable parameter in the rigid body model by focusing on the property of the singularity.
- Ziglin(1982 –) provided a criterion for non-integrability by using Monodoromy matrix.
- Yoshida(1986 –) provided a criterion for non-integrability of homogeneous Hamiltonian systems.
- Morales-Ruiz & Ramis (1999 –) extended the Ziglin analysis by using the differential Galois theory.
- Maciejewski (2011) proved meromorphic non-integrability of the P3BP for any masses by applying the Morales-Ramis theory.

Goal

- Goal: give a criterion of the non-integrability of the homogeneous Hamiltonian systems with two degrees of freedom from a new approach.

Homogeneous Hamiltonian system

Consider a homogeneous Hamiltonian system with two degrees of freedom:

$$H(\mathbf{p}, \mathbf{q}) = \frac{1}{2}\|\mathbf{p}\|^2 + U(\mathbf{q}) \qquad ((\mathbf{p}, \mathbf{q}) \in \mathbb{R}^2 \times \mathbb{R}^2)$$

where U is a homogeneous potential with degree $\beta(\in \mathbb{R})$:

$$U(\lambda\mathbf{q}) = \lambda^\beta U(\mathbf{q}) \qquad (\forall\mathbf{q} \in \mathbb{R}^2 \backslash \{\mathbf{0}\}, \forall\lambda > 0).$$

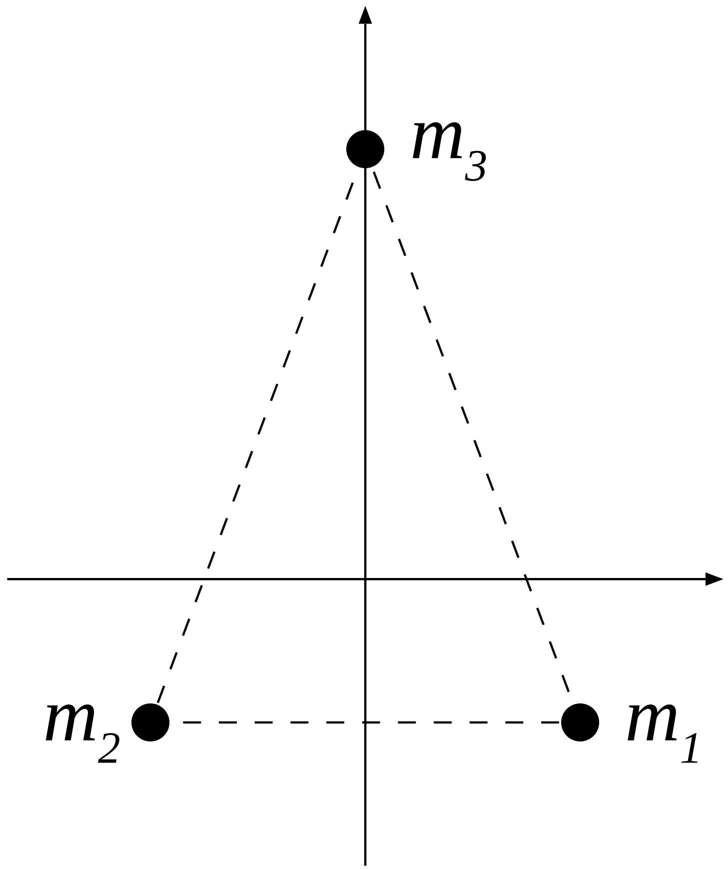
Let $V(\theta) = U(\cos\theta, \sin\theta)$.

Example(The isosceles three-body problem)

Consider the isosceles three-body problem.

Assume that $m_1 = m_2, m_3 = \alpha m_1$.

This model is governed by the homogeneous Hamiltonian system with the potential energy



$$U(\mathbf{q}) = -\frac{1}{q_1} - \frac{4\alpha^{3/2}}{\sqrt{\alpha q_1^2 + (\alpha+2)q_2^2}}.$$

$\beta = -1.$

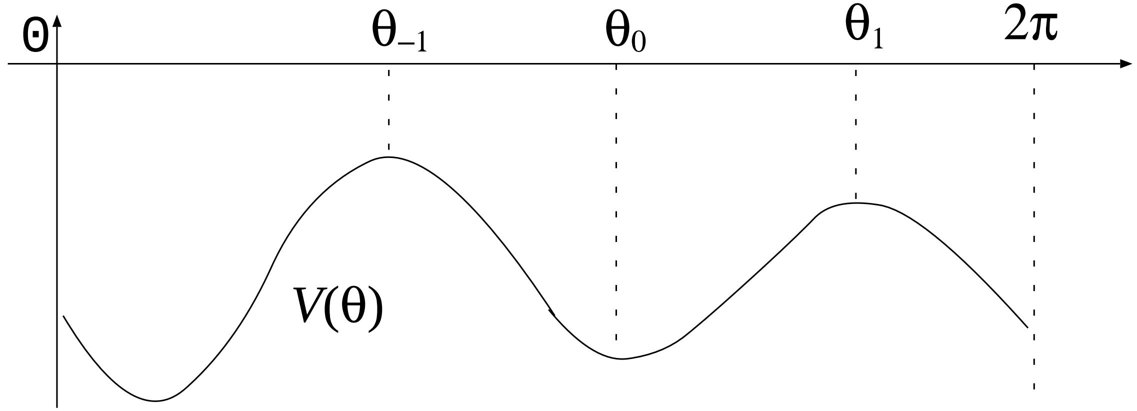
$$V(\theta) = -\sec\theta - \frac{4\alpha^{3/2}}{\sqrt{\alpha + 2\sin^2\theta}}.$$

Main result

Theorem Assume the following:

1. $\beta \in \mathbb{R}\backslash\{-2,0\}$;
2. $\exists\theta_{-1} < \exists\theta_0 < \exists\theta_1$ s.t. $\frac{\partial V}{\partial\theta}(\theta_l) = 0$;
3. $V(\theta) < 0$ on $[\theta_{-1}, \theta_1]$;
4. $\frac{\partial V}{\partial\theta}(\theta) \neq 0$ on $(\theta_{-1}, \theta_0) \cup (\theta_0, \theta_1)$;
5. $\frac{\partial^2 V}{\partial\theta^2}(\theta_{\pm 1}) < 0$;
6. $-\frac{1}{8}(\beta+2)^2 V(\theta_0) < \frac{\partial^2 V}{\partial\theta^2}(\theta_0)$.

Then the homogeneous Hamiltonian system has no meromorphic first integral independent from H.



Remark

In the case of $\beta = -2$, the Hamiltonian system is always integrable. Because a function

$$G(\mathbf{p}, \mathbf{q}) = (\mathbf{q} \cdot \mathbf{p})^2 - 2\|\mathbf{q}\|^2 H(\mathbf{p}, \mathbf{q})$$

is a first integral independent from H.

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McGehee coordinates

We mainly consider the case of $\beta < 0$.

McGehee coordinates: (r, θ, v, w) and τ

$$\mathbf{q} = r(\cos\theta, \sin\theta),$$
$$\mathbf{p} = r^{\beta/2}(v(\cos\theta, \sin\theta) + w(-\sin\theta, \cos\theta))$$
$$dt = r^{1-\beta/2} d\tau.$$

Then the canonical equations become

$$\frac{dr}{d\tau} = rv \tag{2}$$
$$\frac{d\theta}{d\tau} = w \tag{3}$$
$$\frac{dv}{d\tau} = -\frac{\beta}{2}v^2 + w^2 - \beta V(\theta) \tag{4}$$
$$\frac{dw}{d\tau} = -\left(\frac{\beta}{2} + 1\right) vw - \frac{\partial V}{\partial \theta}(\theta) \tag{5}$$

$\mathbf{q} = 0$ is singularity but $r = 0$ is not singular in these differential equations (2)-(5).

Energy and Collision manifold

In these coordinates the total energy is

$$h = r^{\beta}\left(\frac{v^2+w^2}{2}+V(\theta)\right). \tag{6}$$

We fix $h \neq 0$ and regard r as a function of (θ, v, w).

We consider the 3-dimensional dynamics.

The set

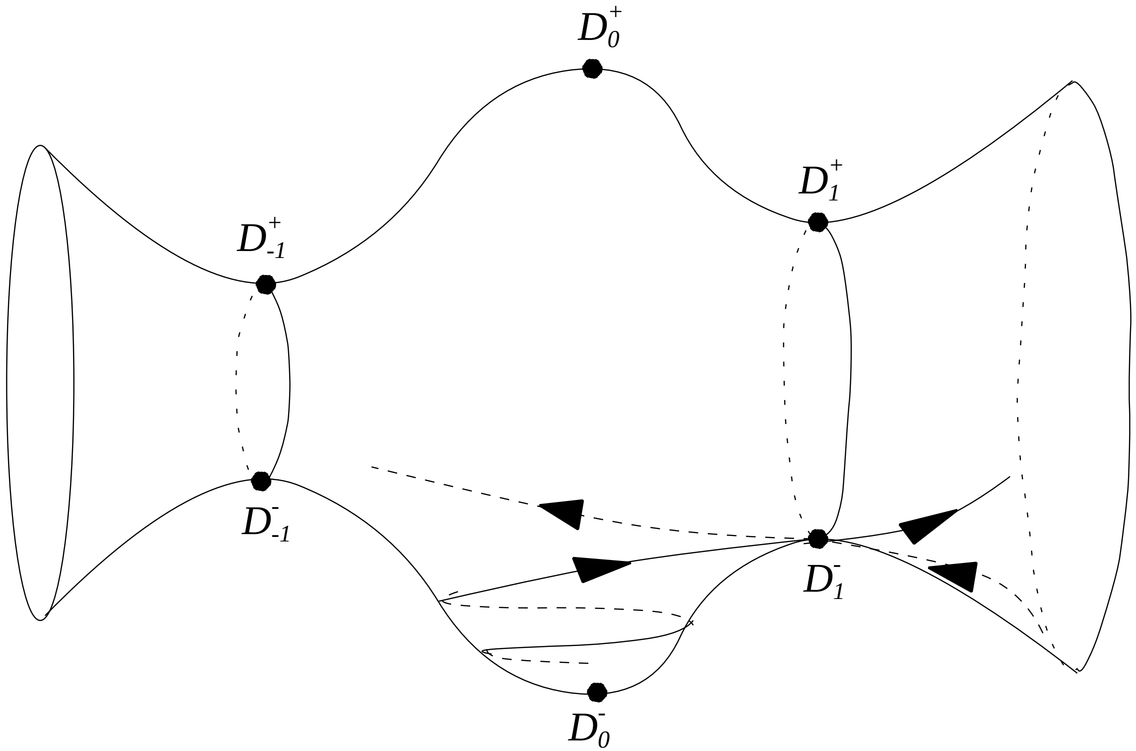
$$\mathcal{M} = \left\{(\theta, v, w) \mid \frac{v^2+w^2}{2}+V(\theta)=0\right\}$$

is invariant. In the case of the n-body problem, $\mathcal{M}$ is called collision manifold.

Since we fix the energy, as $r \to \mathbf{0}(\mathbf{q} \to \mathbf{0})$, the orbit converges to $\mathcal{M}$ in the McGehee coordinates.

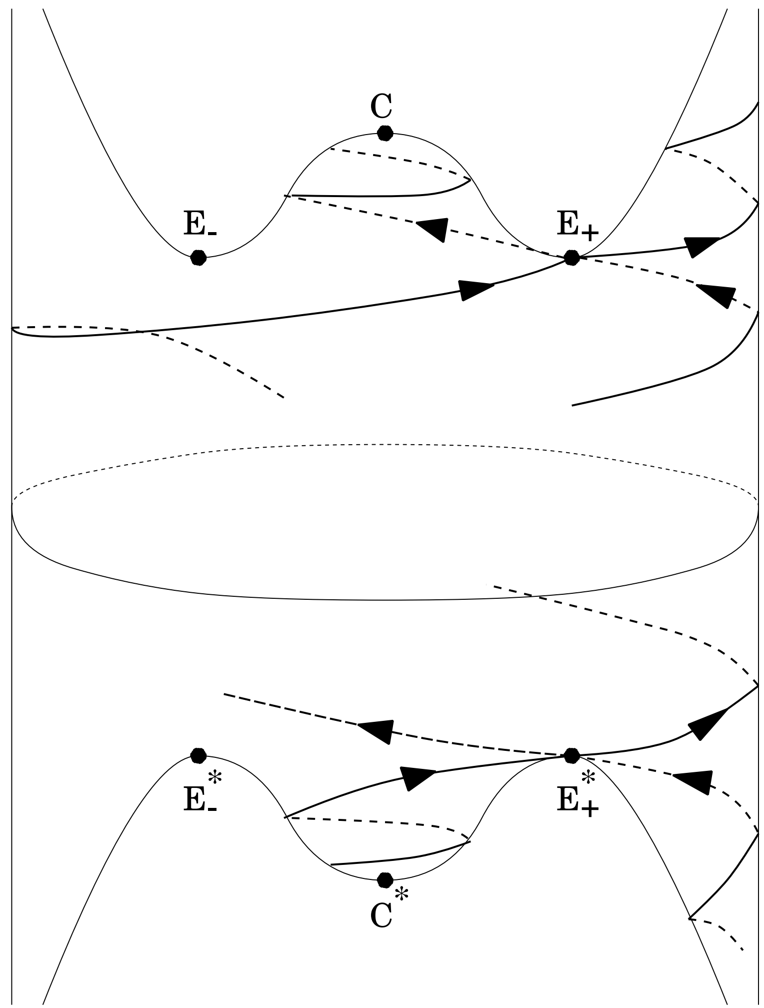
Equilibrium points

Recall that θ_l are a critical point of V, i.e. $\frac{\partial V}{\partial \theta}(\theta_l) = 0$.
Then $D_l^{\pm} = (\theta_l, \pm\sqrt{-2V(\theta_l)}, 0) \in \mathcal{M}$ are equilibrium points.



The case of the isosceles three-body problem

The invariant manifold (collision manifold) $\mathcal{M}$ for the isosceles three-body problem is like this figure:



Case of $\beta > 0$

In the case of $\beta > 0$, we replace r with $R = r^{-1}$.
The equation $\frac{dr}{d\tau} = rv$ becomes $\frac{dR}{d\tau} = -Rv$.
We can define an invariant manifold corresponding to $R \to 0$ and we can discuss a similar argument as the case of $\beta < 0$.

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Proof (homogeneous property)

We give the outline of the proof for $-2 < \beta < 0$. The other cases are similar (some signs change in the computation).

Assume that $\Phi(\mathbf{p}, \mathbf{q})$ is a moromorphic first integral where $(\mathbf{p}, \mathbf{q})$ are the original coordinates.

From the homogeneous property
(if $(\mathbf{p}(t), \mathbf{q}(t))$ is a solution, so is $(c^{\beta/2}\mathbf{p}(c^{\beta/2-2}t), c\mathbf{q}(c^{\beta/2-2}t))$ for any constant $c > 0$),
we can assume that Φ satisfies $\Phi(c^{\beta/2}\mathbf{p}, c\mathbf{q}) = c^{\rho}\Phi(\mathbf{p}, \mathbf{q})$ without loss of generality.

In the McGehee coordinates, this property corresponds to the fact that Φ can be represented as $\Phi = r^{\rho} g(\theta, v, w)$.

Proof(Coordinates)

We use the coordinates (θ, z, w) where $z = \frac{v^2+w^2}{2} + V(\theta)$. These are analytic near the equilibrium points. The energy is

$$h = r^{\beta} z. \tag{7}$$

We consider the Laurent series of g at $z = 0$ with respect to z:

$$g = \sum_{k=\mu}^{\infty} \gamma_k(\theta, w) z^k \qquad (\gamma_\mu \not\equiv 0).$$

From (7), we get $\Phi = (\dfrac{h}{z})^{\frac{\rho}{2\beta}} \displaystyle\sum_{k=\mu}^{\infty} \gamma_k(\theta, w) z^k$.

The lowest order of z is $\mu - \frac{\rho}{2\beta}$.

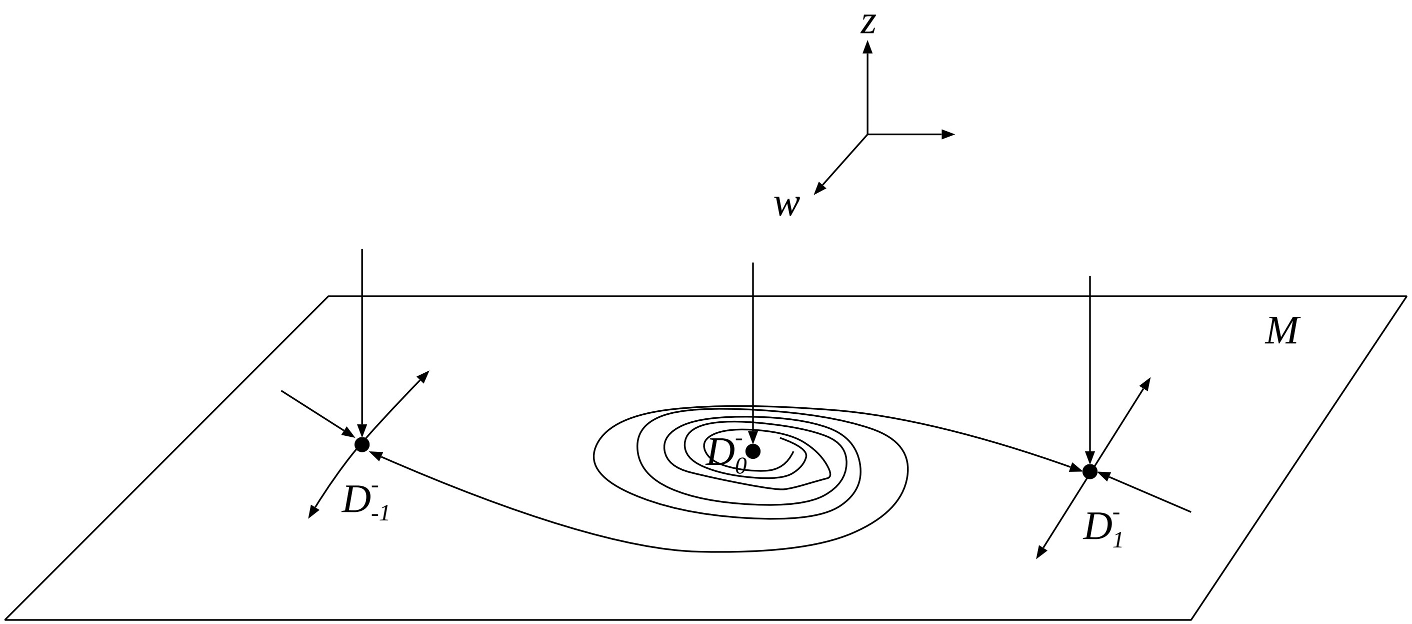
Proof(the case of $\mu - \frac{\rho}{2\beta} < 0$)

We first consider the case of $\mu - \frac{\rho}{2\beta} < 0$.

<u>Lemma</u>: γ_μ *is zero on* $W^u(D_l^-)$.

$W^u(D_0^-)$ is an open set of $\mathcal{M}$. Hence $\gamma_\mu \equiv 0$.

This contradicts the assumption.

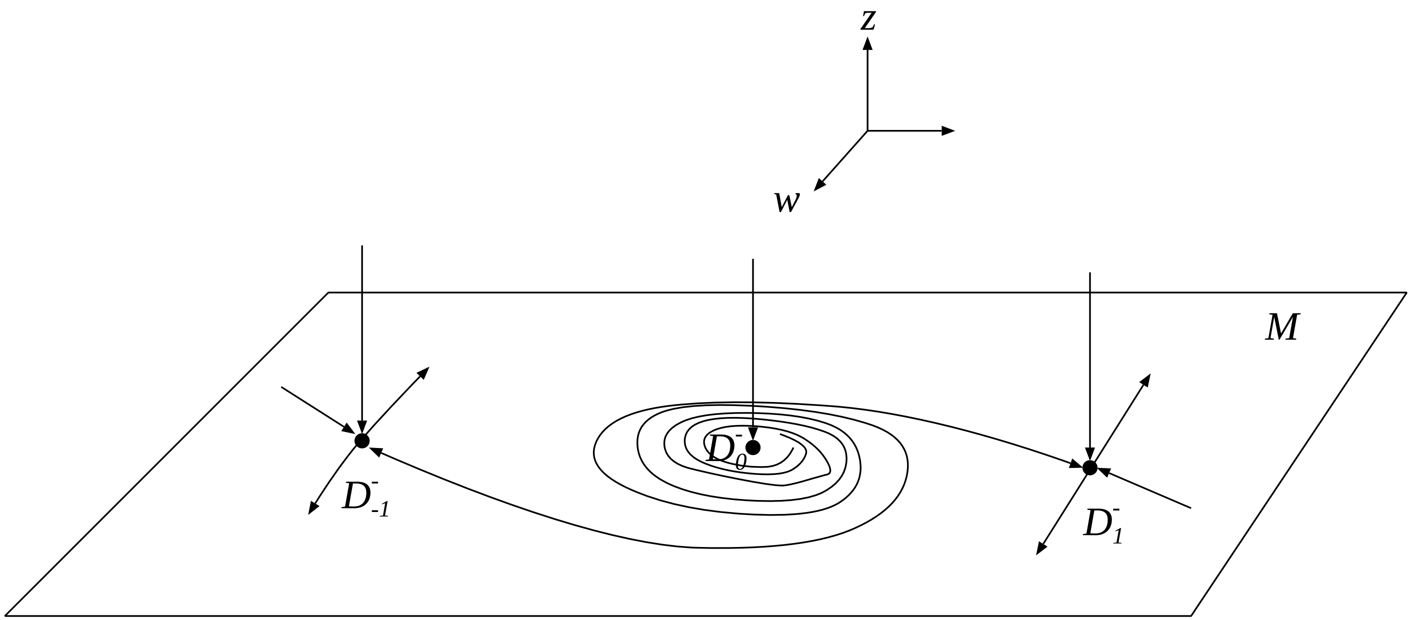


Proof(the case of $\mu - \frac{\rho}{2\beta} > 0$)

We consider the case of $\mu - \frac{\rho}{2\beta} > 0$.

<u>Lemma</u>: γ_μ *is zero on* $W^s(D_l^-)$

From assumption 6: $(-\frac{1}{8}(\beta+2)^2 V(\theta_0) < \frac{\partial^2 V}{\partial \theta^2}(\theta_0))$, the dynamics near D_0^- on $\mathcal{M}$ is unstable focus. $W^s(D_1^-)$ is a spiral curve near D_0^-. Hence $\gamma_\mu \equiv 0$.

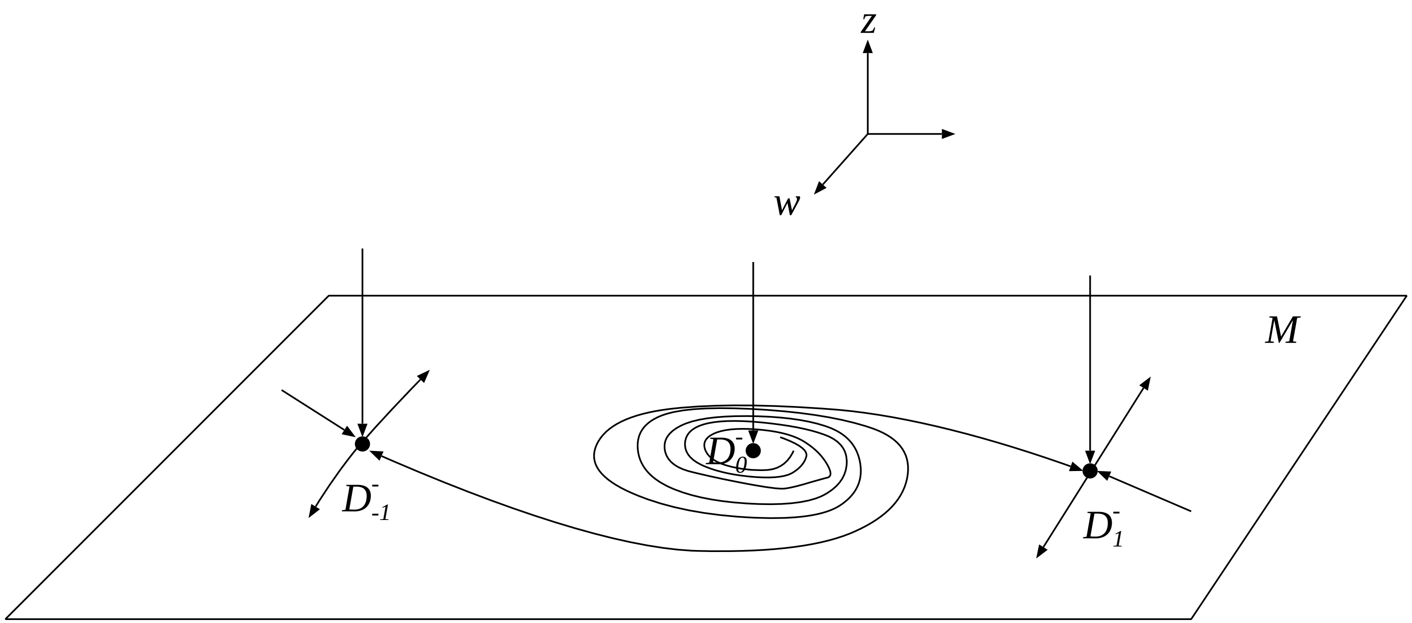


Proof(the case of $\mu - \frac{\rho}{2\beta} = 0$)

In the case of $\mu - \frac{\rho}{2\beta} = 0$,

Lemma: γ_μ *is a constant on* $W^{s/u}(D_l^-)$.

Therefore $\gamma_\mu \equiv c$. If Φ is not constant, by considering $\Phi - c$, this case can be reduced to the case of $\mu - \frac{\rho}{2\beta} > 0$.



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Non-integrability of the isosceles three-body problem

The function V of this problem is

$$V(\theta) = -\sec\theta - \frac{4\alpha^{3/2}}{\sqrt{\alpha + 2\sin^2\theta}}.$$

By applying our theorem, we obtain the following:

Theorem 2

Assume that $\alpha < \frac{55}{4}$. Then the isosceles three-body problem is non-integrable. i.e. there is no moromorphic first integral independent from the energy.

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Yoshida coefficient

We call a point $\mathbf{c} \in \mathbb{R}^2$ the Darboux point if $\nabla U(\mathbf{c}) = \mathbf{c}$. In the case of n-body problem, $\mathbf{c}$ is called a central configuration. The eigenvalues of the Hessian matrix $D^2U(\mathbf{c})$ at the Darboux point $\mathbf{c}$ are called the Yoshida coefficients.

Since $U(\mathbf{c})$ is homogeneous with degree β, one of Yoshida coefficients is $\beta - 1$.

The other Yoshida coefficient is

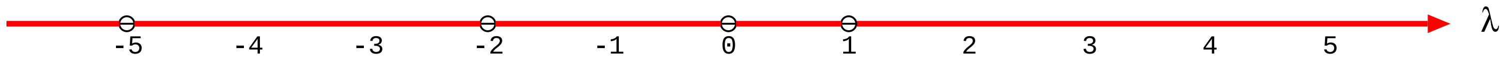
$$\lambda = \beta^{-1} V(\theta_c)^{-1} \frac{\partial^2 V}{\partial \theta^2}(\theta_c) + 1$$

in the polar coordinates where $\frac{\partial V}{\partial \theta}(\theta_c) = 0$.

Yoshida coefficient and integrability

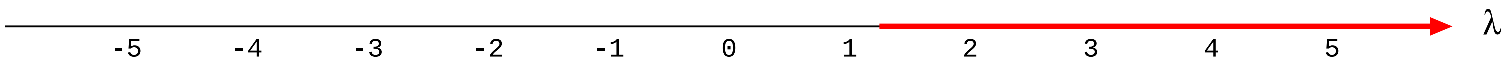
The Morales-Ramis theorem(the differential Galois theory) proves non-integrability if one of the Yoshida coefficient is not in a certain set of rational numbers. For example, in the case of $\beta = -1$, according to the Moreles-Ramis theorem, the homogeneous Hamiltonian system is non-integrable if λ is not in

$$\{-\frac{1}{2}p(p-3) \mid p \in \mathbb{Z}\} = \{1, 0, -2, -5, -9, \dots\}.$$



In our theorem the assumption 6 is

$$-\frac{1}{8}(\beta+2)^2 > (\lambda-1)\beta \qquad (\lambda > 9/8 \text{ if } \beta = -1).$$



In the case of the isosceles three-body problem,

- Our theorem: non-integrability for $\alpha < \frac{55}{4}$
- M-R theory:non-integrability for any α.

Our theorem v.s. Morales-Ramis theory

- Our theorem can be applied to $\beta \in \mathbb{R} \backslash \{-2, 0\}$ while M-R theory can be applied to $\beta \in \mathbb{Z} \backslash \{-2, 0\}$.
- In the case of integer β, M-R theory is stronger.
- Our theorem can be applied to two degrees of freedom while M-R theory can be applied to any degrees of freedom.
- Our function class of first integrals is bigger: we prove the non-existence of first integral which is meromorphic as a real function, while M-R theory prove the non-existence of first integrals which is meromorphic as a complex function.
- Our proof is simpler and based on dynamics (the behavior of stable and unstable manifolds). M-R's method is far from the theory of the dynamics.

Thank you for your attention.