Orbits with increasing angular momentum in the elliptic restricted three body problem: combining two scattering maps

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New Perspectives on the N-body Problem

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The (planar) elliptic restricted three body problem (ER3BP).

We consider the motion of a particle q with zero mass under the attraction of two particles q_{S} and q_{J}, called *primaries*, with mass ratio μ which move in elliptic orbits with *eccentricity* e_0 around their center of mass.

Typical models:

- Sun–Jupiter–asteroid or comet: $e_0 = 0.048$
- Sun–Earth–Moon systems: $e_0 = 0.016$

We consider the motion of the particle q (comet) when it moves outside of the orbit of the primaries along nearly parabolic orbits.

The equations

The motion of the particle q (comet) is described by

$$\frac{d^2q}{dt^2} = -(1-\mu)\frac{q - q_{\mathrm{S}}(t,e_0)}{|q - q_{\mathrm{S}}(t,e_0)|^3} - \mu\frac{q - q_{\mathrm{J}}(t,e_0)}{|q - q_{\mathrm{J}}(t,e_0)|^3}.$$

This is a time-periodic Hamiltonian system (2 and 1/2 degrees of freedom) with Hamiltonian

$$H(q,p,t;e_0,\mu) = \frac{p^2}{2} - \frac{(1-\mu)}{|q - q_{\mathrm{S}}(t,e_0)|} - \frac{\mu}{|q - q_{\mathrm{J}}(t,e_0)|}.$$

Parameters: $0 < \mu, e_0 < 1$.

The two body problem:

When $\mu = 0$, the second primary does not appear in the equations and the first primary is fixed at the origin: $q_{\mathrm{S}}(t, e_0) = 0$

The first primary q_{S} and the third body q form the two-body problem with the Hamiltonian $H(q, p, t; e_0, 0) = H_0(q, p) = \dfrac{p^2}{2} - \dfrac{1}{|q|}$.

The two–body problem is integrable.

The ER3BP as a perturbation of the 2BP

Hamiltonian $H_\mu(q, p, t, e_0)$ is a *small 2π-periodic in time perturbation (if μ is small)* of the integrable two body problem.

The perturbation term is

$$\begin{aligned} \Delta H_\mu(q, p, t; e_0) &= H(q, p, t; e_0, \mu) - H_0(q, p) \\ &= (1-\mu)\left(\frac{1}{|q - q_{\mathrm{S}}(t, e_0)|} - \frac{1}{|q|}\right) \\ &+ \mu\left(\frac{1}{|q - q_{\mathrm{J}}(t, e_0)|} - \frac{1}{|q|}\right). \end{aligned}$$

Since $q_{\mathrm{J}}(t, e_0)$ moves along an ellipse with semi-major axis $1 - \mu$, in the case q being uniformly away from the unit ball both terms are of order of μ and tend to zero as $q \to \infty$.

Hamiltonian equations in polar coordinates

With Polar coordinates $q = (x, y) = (r\cos\alpha, r\sin\alpha),\ \alpha \in \mathbb{T},\ r \geq 0$ the Hamiltonian reads:

$$H(r, P_r, \alpha, P_\alpha, t; e_0, \mu) = \frac{P_r^2}{2} + \frac{P_\alpha^2}{2r^2} - U(r, \alpha, t; e_0, \mu)$$

where (r, P_r) and (α, P_α) are pairs of conjugate variables,

$$U(r, \alpha, t; e_0, \mu) = \frac{1-\mu}{|q - q_{\mathrm{S}}|} + \frac{\mu}{|q - q_{\mathrm{J}}|},$$

$$\begin{aligned} |q - q_{\mathrm{J}}|^2 &= r^2 - 2(1-\mu) r\, r_0 \cos(\alpha - f) + (1-\mu)^2 r_0^2, \\ |q - q_{\mathrm{S}}|^2 &= r^2 + 2\mu\, r\, r_0 \cos(\alpha - f) + \mu^2 r_0^2. \end{aligned}$$

$$r_0 = r_0(t; e_0) = \frac{1 - e_0^2}{1 + e_0 \cos f}, \qquad \frac{df}{dt} = \frac{(1 + e_0\cos f)^2}{(1 - e_0^2)^{3/2}}.$$

where $f = f(t; e_0)$ is the true anomaly.

Hamiltonian equations in polar coordinates

$P_\alpha := G$ is the *angular momentum*.

$$H(r, P_r, \alpha, G, s; e_0, \mu) = \frac{P_r^2}{2} + \frac{G^2}{2r^2} - U(r, \alpha, s; e_0, \mu)$$

We will work in the extended phase space

$(r, P_r, \alpha, G, s) \in \mathbb{R} \times \mathbb{R} \times \mathbb{T} \times \mathbb{R} \times \mathbb{T}$

The two body problem in polar coordinates

In the polar coordinates: $q = (x, y) = (r \cos \alpha, r \sin \alpha),\ \alpha \in \mathbb{T},\ r \geq 0,$
The Hamiltonian of the two body problem $\mu = 0$, becomes

$$H_0(r, P_r, \alpha, G) = \frac{P_r^2}{2} + \frac{G^2}{2r^2} - \frac{1}{r},$$

$h = H_0$ is the energy.

G and H_0 are both first integrals of motion.

If $h < 0$, motions are elliptic:

If $h = 0$ the motion is parabolic.

Increassing the angular momentum

Final goal: in the elliptic restricted three body (ERTBP) problem we want to see that the angular momentum of the third body $G(t)$ can have *large changes*

We have partial results when the eccentricity $e_0 > 0$ and $\mu > 0$ are small enough:

Given any $G_1, G_2 \gg 1$, there exist heteroclinic trajectories of the ERTBP whose angular momentum satisfies, for some $T > 0$:

$$G(0) < G_1 \qquad G(T) > G_2$$

Proven for $0 < \mu \ll e_0 \ll 1$ and any $1 \ll G_1, G_2 \leq 1/e_0$.

Previous results

For oscillatory motions or diffusion close to parabolic orbits:

Llibre-Simó 1980 (oscillatory motions in the RPC3BP for $0 < \mu \ll 1$)

Xia 1992 (for RPC3BP oscillatory motions for every $\mu \in (0, 1/2]$ except a finite number of values)

Guàrdia-Martín-Seara 2012 (idem for any $0 < \mu < 1$)

Galante-Kaloshin 2011(orbits initially bounded and which become oscillatory: $\mu = 10^{-3}$, realistic for the Jupiter-Sun)

Kaloshin and Gorodetski 2011 (results about the Hausdorff dimension of oscillatory motions for both the Sitnikov problem and the RPC3BP)

Xia 1993 (local diffusion in the ERTBP)

Martínez-Pinyol 1994 (Massive computations in the ERTBP)

Previous results

Other types of oscillatory motions or diffusion:

Llibre-Martínez-Simó 1985 (oscillatory motions close to L_2 in the CRTBP)

Bolotin 2006 (close to collision in the ERTBP)

Capiñski-Zgliczyñski 2011 (close to L_2 in the ERTBP)

Féjoz-Guàrdia-Kaloshin-Roldán 2012 (close to resonances in the ERTBP)

One limit: the two body problem: $\mu = 0$

A priori unstable structure

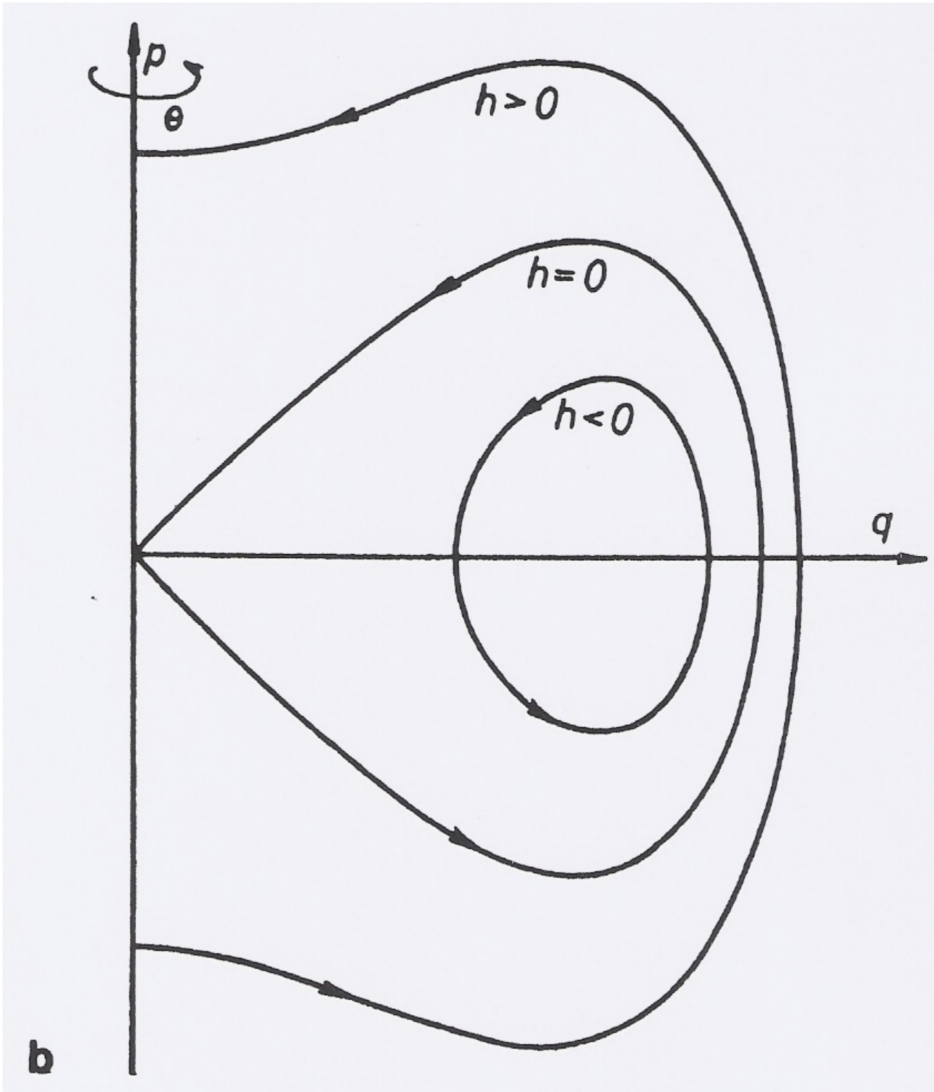
Introducing $x^2 := 1/r$, $y := P_r$, we get new Hamiltonian equations:

$$\begin{aligned} \dot{x} &= -\frac{x^3}{2}\frac{\partial \mathcal{H}_0}{\partial y} & \dot{\alpha} &= \frac{\partial \mathcal{H}_0}{\partial G} \\ \dot{y} &= \frac{x^3}{2}\frac{\partial \mathcal{H}_0}{\partial x} & \dot{G} &= -\frac{\partial \mathcal{H}_0}{\partial \alpha} = 0 \quad \dot{s} = 1 \end{aligned}$$

with Hamiltonian $\mathcal{H}_0(x, y, G) = \dfrac{y^2}{2} + \dfrac{G^2 x^4}{8} - \dfrac{x^2}{2}$, and Poisson bracket

$$\{f, g\} = -\frac{x^3}{2}\left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial g}{\partial x}\frac{\partial f}{\partial y}\right) + \frac{\partial f}{\partial \alpha}\frac{\partial g}{\partial G} - \frac{\partial g}{\partial \alpha}\frac{\partial f}{\partial G}$$

which has the separatrix loop $\gamma_G = \{\mathcal{H}_0(x, y, G) = 0\}$ to the origin.


p
θ
h>0
h=0
h<0
q
b

One limit: the two body problem: $\mu = 0$

A priori unstable structure: An invariant "normally parabolic" cylinder.

Main features we will use:

- The 3 dimensional manifold:

$$\tilde{\Lambda}_\infty = \{x = y = 0,\ (\alpha, G, s) \in \mathbb{T} \times \mathbb{R}_+ \times \mathbb{T}\}$$

  is invariant.

- $\tilde{\Lambda}_\infty = \bigcup_{\alpha,G} \tilde{\Lambda}_{\alpha,G}$, being $\tilde{\Lambda}_{\alpha,G}$ periodic orbits.

- The inner dynamics on $\tilde{\Lambda}_\infty$ is trivial:

$$(\alpha, G, s) \to (\alpha, G, s + t)$$

- $\tilde{\Lambda}_\infty$ has stable and unstable manifolds.

One limit: the two body problem: $\mu = 0$

A priori unstable structure: An invariant homoclinic manifold to $\tilde{\Lambda}_\infty$.

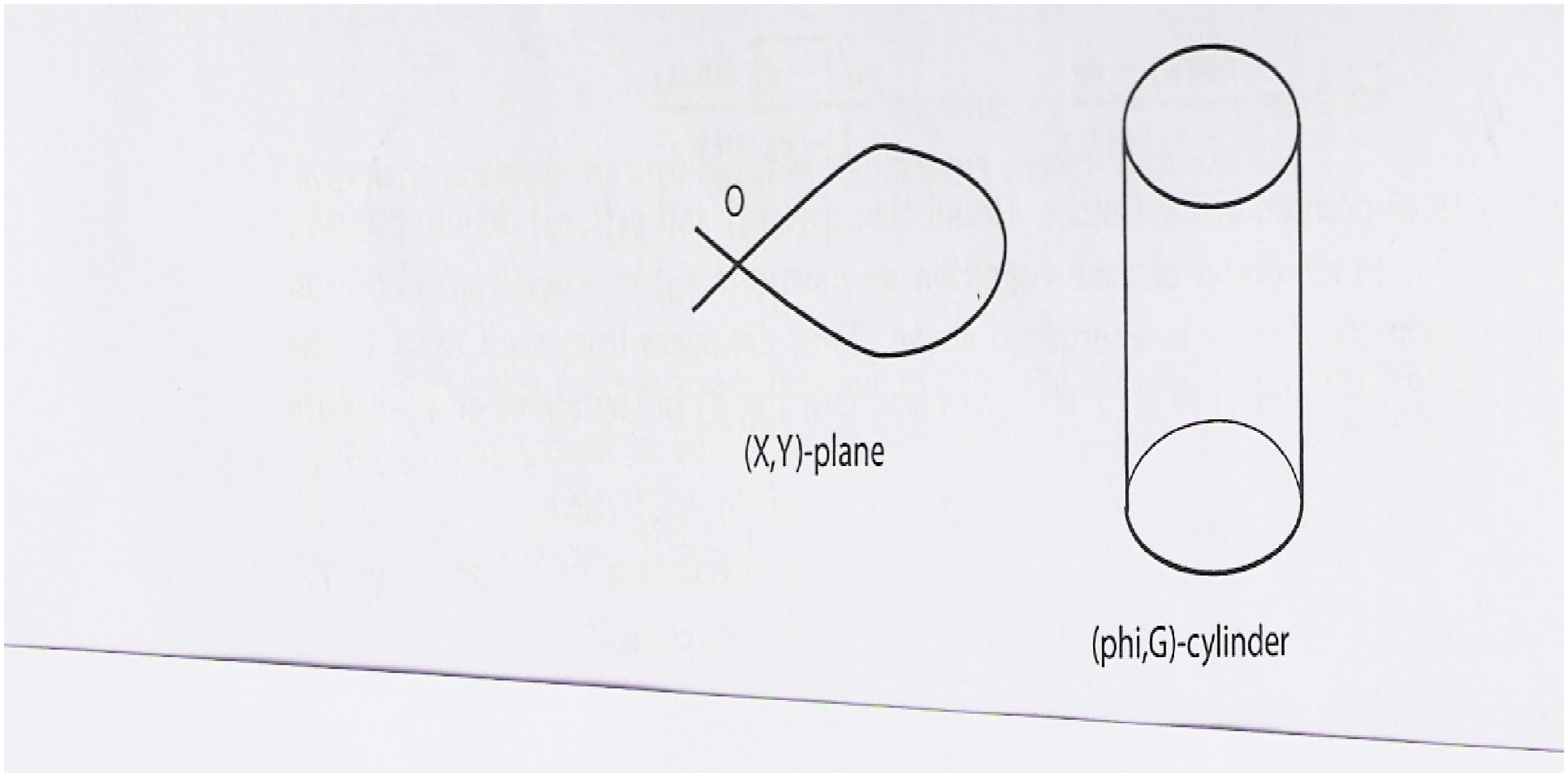
$$\begin{aligned} \tilde{\gamma} &= W_0^s(\tilde{\Lambda}_\infty) = W_0^u(\tilde{\Lambda}_\infty) \\ &= \{\mathcal{H}_0(x, y, G) = 0,\ (\alpha, G, s) \in \mathbb{T} \times \mathbb{R}_+ \times \mathbb{T}\} \end{aligned}$$

that can be seen as a union of parabolic homoclinic orbits to $\tilde{\Lambda}_{\alpha,G}$ (homoclinic manifold).

$$\tilde{\gamma} = \bigcup_{(\alpha,G)} \tilde{\gamma}_{\alpha,G}$$

We can parameterize the 4-dimensional homoclinic manifold as:

$$\tilde{\gamma} = \{\tilde{z}_0 := (x_G(\tau), y_G(\tau), \alpha_G(\tau) + \alpha, G, s), \tau \in \mathbb{R}, G \in \mathbb{R}_+, (\alpha, s) \in \mathbb{T}^2\}$$


O
(X,Y)-plane
(phi,G)-cylinder

One limit: the two body problem: $\mu = 0$

Outer dynamics: the scattering map (Delshams-Llave-S. 2000) in $\tilde{\Lambda}_\infty$.

We can define a map in $\tilde{\Lambda}_\infty$ associated to the homoclinic manifold $\tilde{\gamma}$

$$S_0 : \tilde{\Lambda}_\infty \to \tilde{\Lambda}_\infty$$

by $\tilde{z}_+ = S_0(\tilde{z}_-)$ iff $\exists \tilde{z} \in \tilde{\gamma}$ such that

$$\mathrm{d}(\varphi(t; \tilde{z}), \varphi(t; \tilde{z}_\pm)) \to 0 \text{ as } t \to \pm\infty.$$

The orbit through $\tilde{z}$ is a heteroclinic connection between the orbits through $\tilde{z}_\pm$.

Using the point of $\tilde{z} = \tilde{z}_0 = (x_G(\tau), y_G(\tau), \alpha_G(\tau) + \alpha, G, s)$, one can compute S_0 in coordinates:

$$S_0(\alpha, G, s) = (\alpha, G, s)$$

One limit: the two body problem: $\mu = 0$

Outer dynamics: the *scattering map* in $\tilde{\Lambda}_\infty$.

As $S_0 = Id$, the unperturbed periodic orbits $\tilde{\Lambda}_{\alpha,G}$ only have homoclinic connections.

Main goal:

For $\mu > 0$ (and $e_0 > 0$) we want to see that we can define a scattering map such that the image of one periodic orbit intersects other periodic orbits with larger angular momentum G. Then we will have heteroclinic orbits between periodic orbits

Dynamics of infinity for $e_0 > 0$, $\mu > 0$

In variables (x, y), the Hamiltonian is:

$$H(x, y, \alpha, G, s; e_0, \mu) = \frac{y^2}{2} + \frac{G^2 x^4}{2} - U(x, \alpha, s; e_0, \mu)$$

with $U(x, \alpha, s; e_0) = x^2 \tilde{U}(x, \alpha, s; e_0, \mu)$

Implications:

- $\tilde{\Lambda}_\infty = \{x = y = 0,\ (\alpha, G, s) \in \mathbb{T} \times \mathbb{R}_+ \times \mathbb{T}\}$ is still invariant.
- The periodic orbits $\tilde{\Lambda}_{\alpha,G}$ persist.
- The inner dynamics on $\tilde{\Lambda}_\infty$ is still trivial:

$$(\alpha, G, s) \to (\alpha, G, s + t)$$

The invariant manifolds of $\tilde{\Lambda}_\infty$ for $e_0 > 0$, $\mu > 0$: Melnikov approach

For $\mu > 0$, $e_0 > 0$, the manifolds $W^s_\mu(\tilde{\Lambda}_\infty)$ and $W^u_\mu(\tilde{\Lambda}_\infty)$ intersect transversally along TWO homoclinic manifolds.

This result is based on a Melnikov type computation.

Classical Melnikov potential:

$$\mathcal{L}(\alpha, G, s; e_0) = \int_{\mathbb{R}} \overline{\Delta U}(x_G(t), \alpha_G(t) + \alpha, s + t; e_0)\, dt.$$

where $U(x, \alpha, s; e_0, \mu) = x^2 + \mu\overline{\Delta U}(x, \alpha, s; e_0) + O(\mu^2)$

Intersection property: If the function

$$\tau \mapsto \mathcal{L}(\alpha, G, s - \tau; e_0)$$

has a *non-degenerate critical point* $\tau^*(\alpha, G, s; e_0)$, then there is a transversal intersection between $W^u(\tilde{\Lambda}_\infty)$ and $W^s(\tilde{\Lambda}_\infty)$ close to $\tilde{z}_0 = (x_G(\tau), y_G(\tau), \alpha_G(\tau) + \alpha, G, s)$.

The invariant manifolds of $\tilde{\Lambda}_\infty$ for $e_0 > 0$, $\mu > 0$: the reduced Poincaré function

For any fixed (α, G, e_0), we just need to find a non-degenerate critical point $s^*(\alpha, G; e_0)$ of $s \mapsto \mathcal{L}(\alpha, G, s; e_0)$, that is, a solution $s^*(\alpha, G; e_0)$ of the equation

$$\frac{\partial \mathcal{L}}{\partial s}(\alpha, G, s; e_0) = 0, \quad \frac{\partial^2 \mathcal{L}}{\partial s^2}(\alpha, G, s; e_0) \neq 0$$

and we recover $\tau^*(\alpha, G, s; e_0) = s - s^*(\alpha, G; e_0)$

Once we have $\tau^*(\alpha, G, s; e_0)$ we can consider the *reduced Poincaré function*

$$\mathcal{L}^*(\alpha, G; e_0) = \mathcal{L}(\alpha, G, -\tau^*(\alpha, G, 0; e_0); e_0) = \mathcal{L}(\alpha, G, s^*(\alpha, G; e_0); e_0)$$

The scattering map for $e_0 > 0$, $\mu > 0$

The scattering map S given by the homoclinic intersection associated to the critical point $s^*(\alpha, G; e_0)$ is given as:

$$(\alpha, G, s) \mapsto (\alpha - \mu \frac{\partial \mathcal{L}^*}{\partial G} + O(\mu^2), G + \mu \frac{\partial \mathcal{L}^*}{\partial \alpha} + O(\mu^2), s)$$

S is given, up to first order in μ, as the time $-\mu$ Hamiltonian flow of the autonomous Hamiltonian $\mathcal{L}^*(\alpha, G; e_0)$!

Then, looking at the level curves of $\mathcal{L}^*(\alpha, G; e_0)$ we get the images under the scattering map.

The scattering map for $e_0 > 0$, $\mu > 0$

The inner dynamics in $\tilde{\Lambda}_\infty$ is trivial:

$$(\alpha, G, s) \mapsto (\alpha, G, s + t)$$

The classical geometric mechanism to obtain diffusion does not work: there is no possibility of combining the inner and the outer dynamics to obtain large changes of G.

The time 2π-Poincaré map $P(\alpha, G, s) = (\alpha, G, s)$, therefore $S \circ P = S$

Only with one scattering map we cannot get large changes in G.

Combining two scattering maps for $e_0 > 0$, $\mu > 0$

The function $\mathcal{L}(\alpha, G, s; e_0)$ has two non-degenerate critical points $s^*_+(\alpha, G; e_0)$, $s^*_-(\alpha, G; e_0)$ which give rise to two different perturbed scattering maps S_+, S_-.

The foliations of their level curves are transversal.

We can construct heteroclinic chains of periodic orbits with increasing angular momentum choosing the right scattering map any time

Computation of the Melnikov potential $\mathcal{L}$ for $e_0 G \ll 1$ and big G

Computation of the Melnikov potential is delicate.

We have rigourous computations and bounds of the errors for $e_0 G \ll 1$.

Main idea:

- $\mathcal{L}$ is periodic in s and α.
- The k-th Fourier coefficient in the angle s is of order $O(e^{-k\frac{G^3}{3}})$.
  This is difficult to prove.
- One needs to compute the asymptotic of the first Fourier coefficients and bound the rest.

Computation of the Melnikov potential $\mathcal{L}$ for $e_0 G \ll 1$ and big G

Fourier expanding in the angle s (and α), we get

$$\begin{aligned}\mathcal{L}(\alpha, G, s; e_0) &= \mathcal{L}_0(\alpha, G; e_0) + \mathcal{L}_1(\alpha, G, s; e_0) \\ &\quad + F(\alpha, G; e_0) + E(\alpha, G, s; e_0)\end{aligned}$$

$$\begin{aligned}\mathcal{L}_0(\alpha, G; e_0) &= -\frac{\pi}{G^3} - \frac{15\pi e_0}{8G^5}\cos\alpha, \\ \mathcal{L}_1(\alpha, G, s; e_0) &= \sqrt{\frac{\pi}{8}}\frac{e^{-G^3/3}}{G^{1/2}}\left(\cos(s-\alpha) + p\cos(s-2\alpha)\right),\end{aligned}$$

where $p = 10ee_0G^2$, F is small: $F = O\left(e_0^2 G^{-7}\right)$, and E is exponentially small: $E = e^{-G^3/3} O(G^{-3/2}, e_0 G^{1/2}, e_0^2 G^{5/2})$.

- $\mathcal{L}_0$ contains the main term of the zero harmonic in s.
- $\mathcal{L}_0$ contains the main terms of the two first order harmonics in s.
- $e_0 G \leq 1$ needed for the convergence of the expansions.

Computation of the term $\mathcal{L}_1$ for $e_0 G \ll 1$

The form of $\mathcal{L}_1$ ensures the existence of TWO scattering maps.

$s \mapsto \mathcal{L}_1(\alpha, G, s; e_0)$ is indeed a cosine function:

$$\mathcal{L}_1(\alpha, G, s; e_0) = \sqrt{\frac{\pi}{8}} \frac{e^{-G^3/3}}{G^{1/2}} \sqrt{1 + 2p\cos\alpha + p^2} \cos(s - \alpha - \alpha*),$$

where $\alpha^* = \alpha^*(p, \alpha) = 2 \arctan \frac{p \sin\alpha}{1 + p\cos\alpha}$ $(p = 10 e e_0 G^2)$, with a unique non-degenerate maximum for $s = \alpha + \alpha^*$ and a unique non-degenerate minimum for $s = \alpha + \alpha^* + \pi$, where $\mathcal{L}_1$ takes the values

$$\pm \mathcal{L}_1^*(\alpha, G; e_0) = \pm \sqrt{\frac{\pi}{8}} \frac{e^{-G^3/3}}{G^{1/2}} \sqrt{1 + 2p\cos\alpha + p^2}.$$

Computation of the reduced Poincaré functions $\mathcal{L}^*$

Since $\left|\frac{\partial E}{\partial s}\right| \ll \left|\frac{\partial \mathcal{L}_1}{\partial s}\right|$, the function $s \mapsto \mathcal{L}(\alpha, G, s; e_0)$ is a "cosine-like" function, with unique non-degenerate maximum and minimum at $s^*_\pm$. We can define two *reduced Poincaré functions*

$$\mathcal{L}^*_\pm(\alpha, G; e_0) = \mathcal{L}(\alpha, G, s^*_\pm; e_0) = \mathcal{L}_0 \pm \mathcal{L}^*_1 + F + E^*_\pm$$

so that the associated *scattering maps* $S_\pm$ are given by

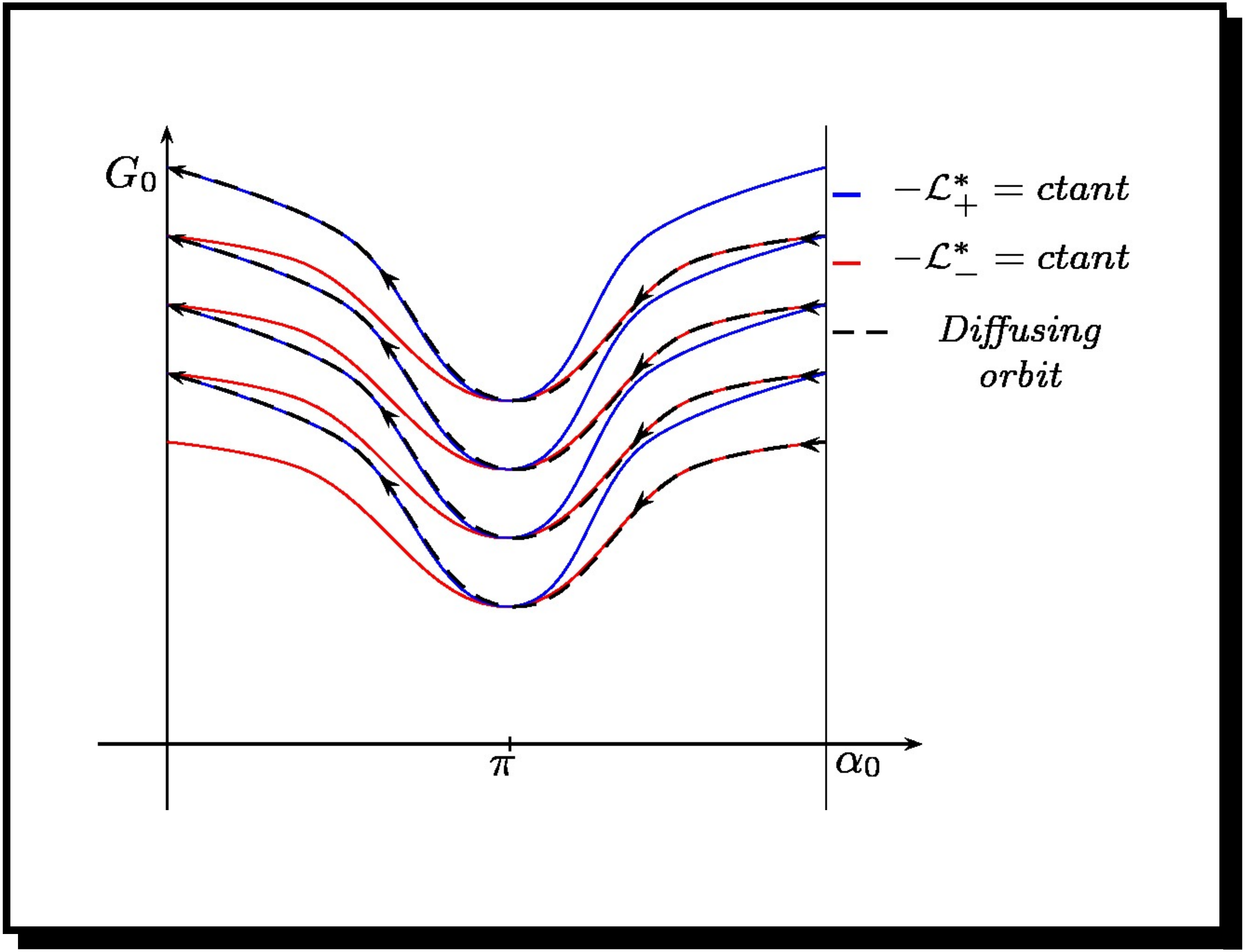
$$(\alpha, G, s) \mapsto \left(\alpha - \mu \frac{\partial \mathcal{L}^*_\pm}{\partial G} + O(\mu^2), G + \mu \frac{\partial \mathcal{L}^*_\pm}{\partial \alpha} + O(\mu^2), s\right).$$

Functionally independent Scattering maps $S_\pm$

The *scattering maps* $S_\pm$ are given by

$$(\alpha, G) \mapsto \left(\alpha - \mu \frac{\partial \mathcal{L}^*_\pm}{\partial G} + O(\mu^2), G + \mu \frac{\partial \mathcal{L}^*_\pm}{\partial \alpha} + O(\mu^2)\right).$$

- $S_\pm$ are given, except for $O(\mu^2)$, as the time μ Hamiltonian flow of the autonomous Hamiltonians $-\mathcal{L}^*_\pm(\alpha, G)$.
- The iterates under $S_\pm$ follow the level curves of $\mathcal{L}^*_\pm$.
- Since $\{\mathcal{L}^*_+, \mathcal{L}^*_-\} = -2\{\mathcal{L}_0, \mathcal{L}^*_1\} + \cdots$ only vanishes on $\alpha = 0, \pi$, we can choose alternatively $S_\pm$ to get diffusing pseudo-orbits and get diffusion along $1 \ll G \leq 1/e_0$.



G_0
π
α_0
$-\mathcal{L}^*_+ = ctant$
$-\mathcal{L}^*_- = ctant$
Diffusing orbit

Arnold diffusion: $e_0 > 0$**,** $\mu > 0$

- We have rigourous results for the existence of heteroclinic orbits with increasing angular momentum if $e_0 G \leq 1$ and $\mu e^{\frac{G^3}{3}} <\ll 1$
- A rigourous λ-lemma is needed to get true orbits.

How can improve the range of the parameters with the same results?

- A priori chaotic: In a recent work (Guardia-Martin-S) we have proved that $W^u(\tilde{\Lambda}_\infty)$ and $W^s(\tilde{\Lambda}_\infty)$ intersect transversally for $e_0 = 0$. Then, the circular restricted theree body problem becomes a priori chaotic for any value of μ, and we get results for $|e_0 e^{\frac{G^3}{3}}| << 1$

Arnold diffusion: $e_0 > 0$, any $\mu > 0$

One can see that this problem is a perturbation of the two body problem without assuming μ small, nor e_0 small.

Take ε small and perform the following changes of variables

$$r = \frac{1}{\varepsilon^2}\widetilde{r}, \quad y = \varepsilon\widetilde{y}, \quad \alpha = \widetilde{\alpha} \quad \text{and} \quad G = \frac{1}{\varepsilon}\widetilde{G}$$

and we rescale time as

$$t = \frac{1}{\varepsilon^3}s.$$

The rescaled system is Hamiltonian with respect

$$\widetilde{H}(\widetilde{r}, \widetilde{y}, \alpha, \widetilde{G}, \frac{s}{\varepsilon^3}; \mu, e_0) = \frac{\widetilde{y}^2}{2} + \frac{\widetilde{G}^2}{2\widetilde{r}^2} - \widetilde{V}(\widetilde{r}, \alpha, \frac{s}{\varepsilon^3}; \varepsilon, e_0, \mu),$$

The equations in scaled variables for small ε where

$$\begin{aligned}\widetilde{V}(\widetilde{r}, \alpha, \tfrac{s}{\varepsilon^3}; \varepsilon, e_0, \mu) &= \frac{1-\mu}{(\widetilde{r}^2 - 2(\mu\varepsilon^2)\widetilde{r}\cos\phi + (\mu\varepsilon^2)^2)^{1/2}} \\ &+ \frac{\mu}{(\widetilde{r}^2 + 2((1-\mu)\varepsilon^2)\widetilde{r}\cos\phi + ((1-\mu)\varepsilon^2)^2)^{1/2}}.\end{aligned}$$

where $\phi = \alpha + f(t_0 + \frac{s}{\varepsilon^3}; e_0)$.

Note that, for any μ, and e_0, $\widetilde{V} = \frac{1}{\widetilde{r}} + O(\varepsilon^2)$ and its dependence on time is through $\phi = \alpha + f(t_0 + \frac{s}{\varepsilon^3}; e_0)$,

In this way one can see:

- The exponentially small splitting comes from the fact that the restricted three body problem is a small and fast perturbation of the two body problem for ε small and any e_0 and μ.
- One can expect the diffusion phenomenon if we are able to deal with these exponentially small phenomena
- The first step will be the case $e_0 G$ small without assumptions in μ