Singular Reduction in the Three- and in the N-Body Problems: Invariant Tori Reconstructed from the Relative Equilibria

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New Perspectives on the N-body Problem

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Goal

Studying the Newtonian three and N body problems in $3D$ from the point of view of averaging and reductions of continuous symmetries:

i) Investigating the dynamics of the most possible reduced problem: existence, stability and bifurcations of the relative equilibria in terms of two parameters.

ii) Reconstructing the flow of the original problem: **KAM tori of various types.**

Method

By introducing a small parameter we average the perturbation over 2 (or $N-1$) angles obtaining a Hamiltonian system on the reduced (orbit) space associated with the symmetries introduced.

We use **singular reduction theory** and have conclusions about the full system.

We obtain new invariant tori of the spatial three body problem (or the spatial N body problem).

The Spatial N Body Problem: Equations of Motion

The Spatial N Body Problem (SNBP) is an IVP:

Given initial values for the positions $\mathbf{q}_j(0)$ and velocities $\dot{\mathbf{q}}_j(0)$ of N particles $j = 0, 1, \ldots, N-1$ with $\mathbf{q}_j(0) \neq \mathbf{q}_k(0)$ for all distinct j and k, find the solution of the second order system whose Hamiltonian is:

$$\mathcal{H} = \frac{1}{2} \sum_{0 \le j \le N-1} \frac{|\mathbf{p}_j|^2}{m_j} - \mathcal{G} \sum_{0 \le j < k \le N-1} \frac{m_j m_k}{|\mathbf{q}_j - \mathbf{q}_k|},$$

where the $\mathbf{p}_j$ are the linear momenta associated to the position vectors $\mathbf{q}_j$.

Counting positions $\mathbf{q}_j \in \mathbb{R}^3$ and momenta $\mathbf{p}_j \in \mathbb{R}^3$ for $j = 0, 1, \ldots, N-1$ one has $6N$ variables.

It is a problem of $3N$ **degrees of freedom**.

The Integrals of the N Body Problem

The N body problem has 10 independent algebraic integrals:

- placing the centre of mass at the origin and fixing the linear momentum reduces the problem to a linear subspace of dimension $6N - 6$,
- fixing the angular momentum vector reduces the problem to a $(6N - 9)$-dimensional space,
- identifying configurations that differ by a rotation about the angular momentum reduces the problem to a space of dimension $6N - 10$.

Conclusion

Thus, it is possible to study the N body problem in a reduced space (symplectic manifold) of dimension $6N - 10$, thus, as a system of $3N - 5$ **degrees of freedom** (respectively $4N - 6$ and $2N - 3$ for the planar N body problem).

For $N = 3$ we study a problem of 4 degrees of freedom

Passing from 9 to 6 Degrees of Freedom #1

(1) The centre of mass moves uniformly with time, then we introduce Jacobi coordinates:

$$\mathbf{x}_0 = \mathbf{q}_0, \quad \mathbf{x}_1 = \mathbf{q}_1 - \mathbf{q}_0, \quad \mathbf{x}_2 = \mathbf{q}_2 - \sigma_0 \mathbf{q}_0 - \sigma_1 \mathbf{q}_1,$$

$$\mathbf{y}_0 = \mathbf{p}_0 + \mathbf{p}_1 + \mathbf{p}_2, \quad \mathbf{y}_1 = \mathbf{p}_1 + \sigma_1 \mathbf{p}_2, \quad \mathbf{y}_2 = \mathbf{p}_2,$$

where

$$1/\sigma_0 = 1 + m_1/m_0, \quad 1/\sigma_1 = 1 + m_0/m_1.$$

Passing from 9 to 6 Degrees of Freedom #2

(2) Attach the reference frame to the centre of mass, i.e. make $\mathbf{y}_0 = 0$, then if $\mathbf{x}_2 \neq 0$ we can write:

$$\mathcal{H} = \mathcal{H}_{\text{Kep}} + \mathcal{H}_{\text{pert}}$$

with

$$\begin{aligned}
\mathcal{H}_{\text{Kep}} &= \frac{|\mathbf{y}_1|^2}{2\mu_1} + \frac{|\mathbf{y}_1|^2}{2\mu_2} - \frac{\mu_1 M_1}{|\mathbf{x}_1|} - \frac{\mu_2 M_2}{|\mathbf{x}_2|}, \\
\mathcal{H}_{\text{pert}} &= -\frac{m_0 m_1 - \mu_1 M_1}{|\mathbf{x}_1|} - \frac{m_1 m_2}{|\mathbf{x}_2 - \sigma_0 \mathbf{x}_1|} - \frac{m_0 m_2}{|\mathbf{x}_2 + \sigma_1 \mathbf{x}_1|} + \frac{\mu_2 M_2}{|\mathbf{x}_2|},
\end{aligned}$$

and

$$\begin{aligned}
\frac{1}{\mu_1} &= \frac{1}{m_0} + \frac{1}{m_1}, \quad \frac{1}{\mu_2} = \frac{1}{m_0 + m_1} + \frac{1}{m_2}, \\
M_1 &= m_0 + m_1, \quad M_2 = m_0 + m_1 + m_2.
\end{aligned}$$

Elimination of the Nodes #1

Let the angular momentum vector

$$\sum_{k=1}^{2} \mathbf{G}_k \equiv \sum_{k=1}^{2} \mathbf{x}_k \times \mathbf{y}_k = \mathbf{C} \neq 0.$$

Spatial Delaunay elements are not useful for carrying out the reduction of the nodes as the conservation of the components of $\mathbf{C}$ requires that

$$h_1 - h_2 = \pi, \quad G_1^2 - H_1^2 = G_2^2 - H_2^2, \quad H_1 + H_2 = \mathbf{C} \cdot \mathbf{k},$$

$\mathbf{k}$ is the vertical unit vector of an inertial frame centred at the centre of mass.

Reason

The constrainst given above imply that the transformation is only possible in a submanifold of $\mathbb{R}^{12}$ that has dimension 10

Elimination of the Nodes #2

We use Deprit's coordinates devised by André Deprit in 1983 to deal with the N body problem: [Elimination of the Nodes in Problems of N Bodies, *CM* **30** 181-195 (1983).]

1. Choose an inertial frame $\mathcal{Q} = (\mathbf{i}, \mathbf{j}, \mathbf{k})$: if $\mathbf{C} \neq \mathbf{0}$ then $\mathbf{C} = C\,\mathbf{n}$ with $C > 0$ and $|\mathbf{n}| = 1$.
2. Introduce an angle I such that $\mathbf{k} \cdot \mathbf{n} = \cos I$ with $0 \leq I \leq \pi$: when $I \in (0, \pi)$ there exists a unit vector $\mathbf{l}$ with $\mathbf{k} \times \mathbf{n} = \mathbf{l} \sin I$ and $|\mathbf{l}| = 1$.
3. Define the invariable frame $\mathcal{I} = (\mathbf{n}, \mathbf{l}, \mathbf{m})$.

Elimination of the Nodes #3

1. The angle ν is the longitude of $\mathbf{l}$, i.e. $\mathbf{l} = \mathbf{i}\cos\,\nu + \mathbf{j}\sin\,\nu$ with $0 \leq \nu \leq 2\pi$,

2. If $\mathbf{G}_k = G_k\mathbf{n}_k$ with $|\mathbf{n}_k| = 1$, I_k is the angle between $\mathbf{C}$ and $\mathbf{G}_k$ and $\mathbf{n} \times \mathbf{n}_k = \mathbf{l}_k \sin\,I_k$ with $|\mathbf{l}_k| = 1$, then the angle ν_k is defined such that $\mathbf{l}_k = \mathbf{l}\cos\nu_k + \mathbf{m}\sin\,\nu_k$ with $0 \leq \nu_k \leq 2\pi$,

3. γ_k is the argument of the pericentre in the plane defined by $\mathbf{l}_k$ and $\mathbf{m}_k$,

4. $B = \mathbf{C} \cdot \mathbf{k}$,

5. The coordinates L_k's, G_k's and ℓ_k's are the same as the spatial Delaunay's elements.

We introduce Deprit's coordinates as the set of action-angle variables:

$$(\ell_1, \ell_2, \gamma_1, \gamma_2, \nu_1, \nu, L_1, L_2, G_1, G_2, C, B)$$

Elimination of the Nodes #4

Some considerations:

- For the three body problem the variables C, B and ν are integrals of motion and in particular the nodes ν and ν_1 are not present in the equations.
- Deprit's variables were constructed for N bodies using recursion. See the recent paper [L. Chierchia, G. Pinzari: Deprit's Reduction of the Nodes Revisited, *CMDA* **109** 285-301 (2011).]
- The number of degrees of freedom is then **reduced to** $3N - 5$ (i.e. to 4 if $N = 3$).

Perturbative Region #1

Let a_1, a_2 be the semimajor axes and e_1 and e_2 the corresponding eccentricities of the ellipses 1 and 2 and let $\epsilon_k = \sqrt{1-e_k^2}$.

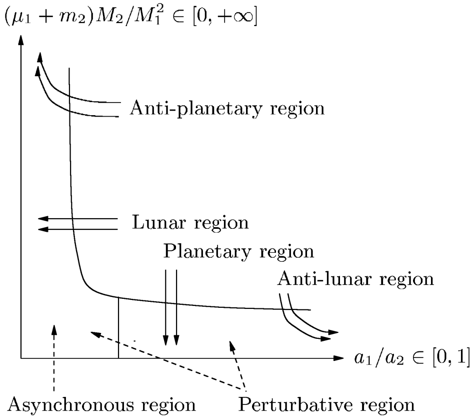
We define

$$\hat{\sigma} = \max\{\sigma_1, \sigma_2\}, \quad \Delta = \hat{\sigma}\frac{a_1(1+e_1)}{a_2(1+e_2)}.$$

For $0 < \varepsilon \ll 1$ and $k \in \mathbb{Z}^+$, **the perturbative region** is

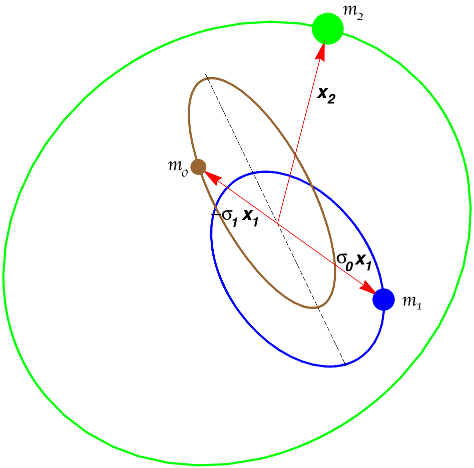
$$\mathcal{P} = \max\left\{\frac{m_2}{M_1}\left(\frac{a_1}{a_2}\right)^{3/2}, \frac{\mu_1\sqrt{M_2}}{M_1^{3/2}}\left(\frac{a_1}{a_2}\right)^2\right\}\frac{1}{\epsilon_2^{3(2+k)}(1-\Delta)^{2k+1}} < \varepsilon.$$

Perturbative Region #2



[J. Féjoz: Quasiperiodic Motions in the Planar Three-Body Problem, *JDE* **183**, 303–341 (2002)]

Inner and Outer Ellipses



Averaging with Respect to the Mean Anomalies

In a region free of resonances among ℓ_1 and ℓ_2 (e.g. the ratio ν_2/ν_1 is not too close to a rational number) we average over the two anomalies:

$$\mathcal{K}_1 = \frac{1}{4\pi^2}\int_0^{2\pi}\!\!\int_0^{2\pi} \mathcal{H}_{\mathrm{pert}}\, d\,\ell_1\; d\,\ell_2,$$

and the generating function satisfies:

$$\nu_1\frac{\partial \mathcal{W}_1}{\partial \ell_1} + \nu_2\frac{\partial \mathcal{W}_1}{\partial \ell_2} = \mathcal{H}_{\mathrm{pert}} - \mathcal{K}_1.$$

Truncating the Legendre expansion at $n = 2$, we get:

$$\begin{aligned}\mathcal{K}_1 = {} & \frac{m_2^7 M_1^7}{64 m_0^3 m_1^3 M_2^3}\,\frac{L_1^2}{L_2^3 G_1^2 G_2^5}\Big(\\ & \big(-3(C^2 - G_1^2)^2 + 2(3C^2 - G_1^2)G_2^2 - 3G_2^4\big)(5L_1^2 - 3G_1^2)\\ & + 15\big((C+G_1)^2 - G_2^2\big)\big((C-G_1)^2 - G_2^2\big)(L_1^2 - G_1^2)\cos 2\gamma_1\Big).\end{aligned}$$

Consequences of Averaging

1. After truncating higher order terms, the resulting system is of two degrees of freedom.

2. However, as γ_2 is not present in the equations (it appears if we truncate at $n = 3$), thus the reduced system is of one degree of freedom.

Regular Reduction #1

Once we truncate higher order terms we construct the orbit space. We apply Meyer (or Marsden-Weinstein) regular reduction theory.

Invariants associated to the symmetries L_1 and L_2:

1. Take the Laplace-Runge-Lenz vectors $\mathbf{A}_k = (\mathbf{y}_k \times \mathbf{G}_k)/\mu_k - \mathbf{x}_k/r_k$, $k = 1, 2$.
2. Introduce $\mathbf{a} = (a_1, a_2, a_3)$, $\mathbf{b} = (b_1, b_2, b_3)$, $\mathbf{c} = (c_1, c_2, c_3)$ and $\mathbf{d} = (d_1, d_2, d_3)$ through

$$\mathbf{a} \;=\; \mathbf{G}_1 + L_1\mathbf{A}_1, \quad \mathbf{b} \;=\; \mathbf{G}_1 - L_1\mathbf{A}_1, \quad \mathbf{c} \;=\; \mathbf{G}_2 + L_2\mathbf{A}_2, \quad \mathbf{d} \;=\; \mathbf{G}_2 - L_2\mathbf{A}_2.$$

3. **a**, **b**, **c** and **d** satisfy

$$|\mathbf{a}|^2 = |\mathbf{b}|^2 \;=\; L_1^2,$$
$$|\mathbf{c}|^2 = |\mathbf{d}|^2 \;=\; L_2^2.$$

Regular Reduction #2

For fixed and strictly positive values of L_1 and L_2 the reduced phase space (i.e. the orbit space) related to the normalisation of ℓ_1 and ℓ_2 and the truncation of the corresponding tail is given by

$$\begin{aligned}\mathcal{A}_{L_1,L_2} &= S^2_{L_1} \times S^2_{L_1} \times S^2_{L_2} \times S^2_{L_2} \\ &= \left\{(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}) \in \mathbb{R}^8 \;\middle|\; |\mathbf{a}|^2 = |\mathbf{b}|^2 = L_1^2,\ |\mathbf{c}|^2 = |\mathbf{d}|^2 = L_2^2,\ C \leq L_1 + L_2\right\}\end{aligned}$$

(i) $\mathcal{A}_{L_1,L_2}$ defines a manifold of dimension eight and is regular.

(ii) The trajectories of the inner ellipses can be rectilinear, i.e. $e_2 = 1$ as we regularise the inner collisions.

(iii) Circular and/or coplanar trajectories are also studied properly in $\mathcal{A}_{L_1,L_2}$.

Singular Reductions #1

Reduction by the symmetry related with C and B:

Arms, Cushman and Gotay: As the reduction process has non-trivial isotropy groups **the reduction is singular.**

Central question

How do we get the invariants associated to the symmetries

The actions C and B in terms of $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ are:

$$
\begin{aligned}
C &= \tfrac{1}{2}\sqrt{(a_1+b_1+c_1+d_1)^2+(a_2+b_2+c_2+d_2)^2+(a_3+b_3+c_3+d_3)^2},\\
B &= \tfrac{1}{2}(a_3+b_3+c_3+d_3).
\end{aligned}
$$

Invariants: we look for polynomials in a_k's, b_k's, c_k's and d_k's such that $\{p, C^2\} = \{p, B\} = 0$.

Singular Reductions #2

We proceed constructively, starting by polynomials of degree one, then polynomials of degree two and so on, all with arbitrary coefficients that we have to determine.

The result yields one valid combination:

$$\pi_1 = a_3 + b_3 + c_3 + d_3,$$

$$\pi_2 = a_1 b_1 + a_2 b_2 + a_3 b_3, \ \pi_3 = a_1 c_1 + a_2 c_2 + a_3 c_3, \ \pi_4 = a_1 d_1 + a_2 d_2 + a_3 d_3,$$

$$\pi_5 = b_1 c_1 + b_2 c_2 + b_3 c_3, \ \pi_6 = b_1 d_1 + b_2 d_2 + b_3 d_3, \ \pi_7 = c_1 d_1 + c_2 d_2 + c_3 d_3,$$

$$\pi_8 = (a_1 + b_1 + c_1 + d_1)^2 + (a_2 + b_2 + c_2 + d_2)^2,$$

$$\pi_9 = -(a_1 + b_1 + c_1)(a_1 + b_1 + c_1 + 2d_1) - (a_2 + b_2 + c_2)(a_2 + b_2 + c_2 + 2d_2) + d_3^2.$$

Singular Reductions #3

Questions:

- Are they independent invariants?
- Where do we have to stop?
- Do we have to compute invariants of degree three?

It is a topic of Computer Algebra: we need to find out a Hilbert basis (fundamental set of invariants).

- If $\{\pi_1, \pi_2, \ldots, \pi_n\}$ is the Hilbert basis, the reduced Hamiltonian has to be expressible in terms of the basis, and the basis together with the constraints involved has to define the singular space.
- If we build a Gröbner basis from the Hilbert basis, say, $\{\bar{\pi}_1, \bar{\pi}_2, \ldots, \bar{\pi}_n\}$, pick any invariant and apply the multivariate division algorithm with respect to it, then the remainder of the division must be zero.

To compute the Gröbner basis is a formidable task.
However we can use the relationships between **a**, **b**, **c** and **d** and Deprit's elements (we have obtained all of them). We easily obtain that

$$\pi_2 = 2G_1^2 - L_1^2, \quad \pi_7 = 2G_2^2 - L_2^2,$$

$$\pi_3 + \pi_4 - \pi_5 - \pi_6 = \frac{2}{G_1}\sqrt{\left((C+G_2)^2 - G_1^2\right)\left(G_1^2 - (C-G_2)^2\right)}\sqrt{L_1^2 - G_1^2}\,\sin\gamma_1,$$

$$\pi_3 - \pi_4 + \pi_5 - \pi_6 = \frac{2}{G_2}\sqrt{\left((C+G_2)^2 - G_1^2\right)\left(G_1^2 - (C-G_2)^2\right)}\sqrt{L_2^2 - G_2^2}\,\sin\gamma_2.$$

Thus, we define:

$$\sigma_1 = \pi_2, \qquad \sigma_2 = \pi_7,$$

$$\sigma_3 = \tfrac{1}{2}(\pi_3 + \pi_4 - \pi_5 - \pi_6), \qquad \sigma_4 = \tfrac{1}{2}(\pi_3 - \pi_4 + \pi_5 - \pi_6).$$

Singular Reductions #5

However this is not enough:
$\{\sigma_1, \sigma_2, \sigma_3, \sigma_4\}$ does not verify the multivariate division algorithm, i.e., they cannot form a Hilbert basis.
Going to degree three, we arrive at the following invariants:

$$\begin{aligned}\sigma_5 &= \tfrac{1}{2}\Big(a_1\big(b_3(c_2+d_2)-b_2(c_3+d_3)\big)+a_2\big(-b_3(c_1+d_1)+b_1(c_3+d_3)\big)\\ &\quad +a_3\big(b_2(c_1+d_1)-b_1(c_2+d_2)\big)\Big),\\ \sigma_6 &= \tfrac{1}{2}\Big(c_1\big(-d_2(a_3+b_3)+d_3(a_2+b_2)\big)+c_2\big(d_1(a_3+b_3)-d_3(a_1+b_1)\big)\\ &\quad +c_3\big(-d_1(a_2+b_2)+d_2(a_1+b_1)\big)\Big).\end{aligned}$$

The expressions of σ_5 and σ_6 in terms of Deprit's elements are:

$$\begin{aligned}\sigma_5 &= \sqrt{\big((C+G_2)^2-G_1^2\big)\big(G_1^2-(C-G_2)^2\big)}\sqrt{L_1^2-G_1^2}\cos\gamma_1,\\ \sigma_6 &= \sqrt{\big((C+G_2)^2-G_1^2\big)\big(G_1^2-(C-G_2)^2\big)}\sqrt{L_2^2-G_2^2}\cos\gamma_2.\end{aligned}$$

Singular Reductions #6

$\{\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6\}$ satisfies all the requirements and is a fundamental set of invariants.
The reduced space is:

$$\mathcal{S}_{L_1,L_2,C} = \Big\{(\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6) \in \mathbb{R}^6 \;\Big|\; \text{the } \sigma_i\text{'s satisfy } (1)\Big\},$$

with the constraints

$$(1)\quad \begin{aligned} (\sigma_1 - L_1^2)\Big((\sigma_2 - \sigma_1 + L_2^2 - L_1^2 + C^2)^2 - 8C^2(\sigma_2 + L_2^2)\Big) &= \\ &4(\sigma_1 + L_1^2)\sigma_3^2 + 8\sigma_5^2, \\ (\sigma_2 - L_2^2)\Big((\sigma_1 - \sigma_2 + L_1^2 - L_2^2 + 2C^2)^2 - 8C^2(\sigma_1 + L_1^2)\Big) &= \\ &4(\sigma_2 + L_2^2)\sigma_4^2 + 8\sigma_6^2. \end{aligned}$$

$\mathcal{S}_{L_1,L_2,C}$ is a four-dimensional symplectic orbifold as it has singularites.

Singular Reductions #6

Reduction by the symmetry related with G_2:

$$\mathcal{T}_{L_1,C,G_2} = \Big\{(\tau_1,\tau_2,\tau_3) \in \mathbb{R}^3 \;\Big|\; \text{the invariants } \tau_i\text{'s satisfy } (2)\Big\},$$

where

$$(\tau_1 - L_1^2)\big((\tau_1 + L_1^2 - 2C^2 - 2G_2^2)^2 - 16C^2G_2^2\big) = 4(\tau_1 + L_1^2)\tau_2^2 + 8\tau_3^2. \quad (2)$$

It defines an orbifold of dimension two.

- The invariants τ_1, τ_2 and τ_3 generate the fully-reduced space.
- The rest of invariants of different degrees belong to the ideal defined by the selected invariants (i.e. the Hilbert basis) using the multivariate division algorithm.

Fully-Reduced Phase Space: $\mathcal{T}_{L_1,C,G_2}$

τ_1, τ_2 and τ_3 are represented in terms of Deprit's coordinates by

$$\begin{aligned}
\tau_1 &= 2G_1^2 - L_1^2, \\
\tau_2 &= \frac{1}{G_1}\sqrt{\left((C+G_2)^2 - G_1^2\right)\left(G_1^2 - (C-G_2)^2\right)}\sqrt{L_1^2 - G_1^2}\,\sin\,\gamma_1, \\
\tau_3 &= \sqrt{\left((C+G_2)^2 - G_1^2\right)\left(G_1^2 - (C-G_2)^2\right)}\sqrt{L_1^2 - G_1^2}\,\cos\,\gamma_1.
\end{aligned}$$

- Coordinates: G_1 and γ_1.
- Parameters: L_1, C and G_2.
- $\gamma_1 \in [0, 2\pi)$, $G_1 \in [0, L_1]$.
- If $L_1 = |C - G_2|$: the space gets reduced to a unique point.

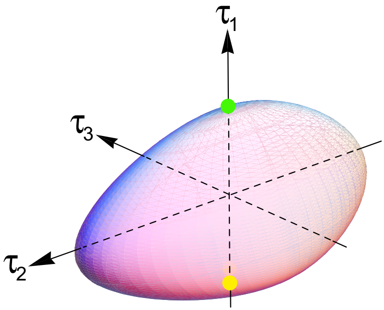
Fully-Reduced Phase Space: $\mathcal{T}_{L_1,C,G_2}$

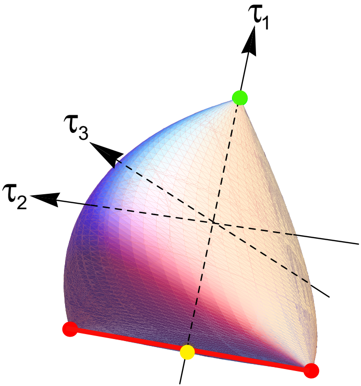
Special motions concerning the inner bodies which represent points where Deprit's coordinates are singular:

- Circular motions: $(L_1^2, 0, 0)$
- Coplanar motions: $(2(C - G_2)^2 - L_1^2, 0, 0)$
- Circular coplanar motions: $((C + G_2)^2, 0, 0)$
- Set of rectilinear solutions:

$$\left\{(-L_1^2\,,\ \tau_2\,,\ 0)\ \mid\ \tau_2 \in [-2L_1G_2,\ 2L_1G_2]\right\}.$$

Fully-Reduced Phase Space: $\mathcal{T}_{L_1,C,G_2}$

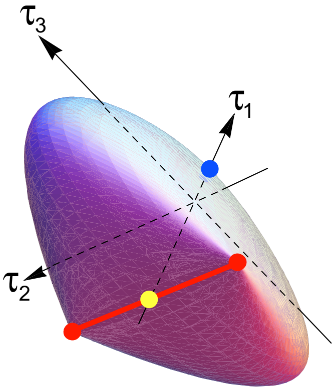


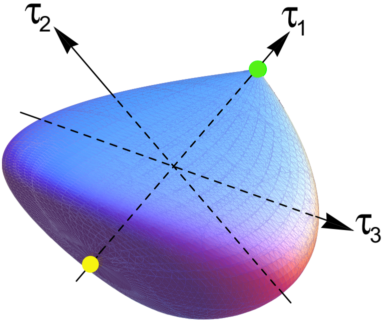


- Green points: circular motions.
- Yellow points: coplanar solutions.
- Red segment: all possible rectilinear motions.

There can be 0, 1, 2 or 3 singular points in $\mathcal{T}_{L_1,C,G_2}$

Two More Pictures of $\mathcal{T}_{L_1,C,G_2}$





Application to the STBP: The Reduced Hamiltonian and The Equations of Motion

After dropping constant terms, The fully-reduced Hamiltonian is:

$$\mathcal{K}_1 = 2(-L_1^2 + 2C^2 + 6G_2^2)\tau_1 - \tau_1^2 + 20\tau_2^2.$$

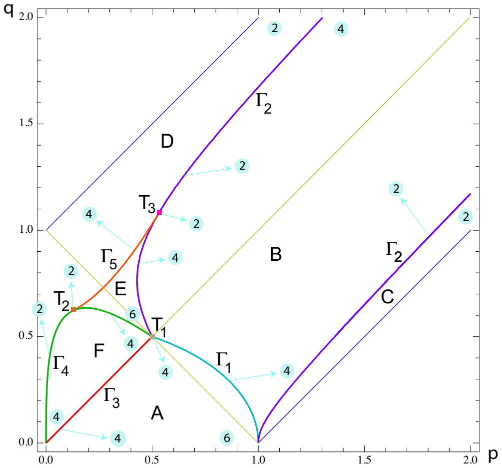
The vector field associated to $\mathcal{K}_1$ is:

$$\begin{aligned}
\dot{\bar{\tau}}_1 &= -160\bar{\tau}_2\bar{\tau}_3, \\
\dot{\bar{\tau}}_2 &= -8(\bar{\tau}_1 - 2p^2 - 6q^2 + 1)\bar{\tau}_3, \\
\dot{\bar{\tau}}_3 &= 2\bar{\tau}_2\Big((\bar{\tau}_1 + 1)(-13\bar{\tau}_1 + 7) + 20\bar{\tau}_2^2 + 4p^2(9\bar{\tau}_1 - 1) \\
&\qquad + 4q^2(7\bar{\tau}_1 + 10p^2 - 3) - 20(p^4 + q^4)\Big),
\end{aligned}$$

where $p = C/L_1$, $q = G_2/L_1$, $\bar{\tau}_1 = \tau_1/L_1^2$, $\bar{\tau}_2 = \tau_2/L_1^2$ and $\bar{\tau}_3 = \tau_3/L_1^3$.

Circular and coplanar type of trajectories are always equilibria.

Application to the STBP: Plane of Parametric Bifurcations



- Γ_1, Γ_2, Γ_3, Γ_4: Hamiltonian Pitchfork
- Γ_5: Double Centre-Saddle
- T_2 and T_3: Reversible Elliptic Umbilic

Some Remarks

1. This approach for the STBP follows works initiated by Jefferys and Moser (1966), Harrington (1968, 1969), Lidov and Ziglin (1976), Ferrer and Osácar (1994), Farago and Laskar (2010).

   Our study is **global in phase space and makes use of singular reduction**, avoiding degeneracy in the analysis of bifurcations (using regular reduction there are some artificial distortions).

2. We have found relative equilibria of **rectilinear type** (for the inner ellipses), specifically vertical solutions and coplanar solutions.

Reconstruction of the Dynamics

We can apply KAM theory to get families of invariant tori of the system in four degrees of freedom.

Other related issues:

- Do the relative equilibria analysed so far correspond with invariant 2-tori of the 4 DOF Hamiltonian?
- Can we obtain periodic solutions related to the KAM tori?
- Can we study the bifurcations of the invariant tori following the guides provided by the reduced system and the bifurcation of the relative equilibria?
- Can we establish the existence of lower-dimensional tori?

Different Cases of Invariant Tori

All possible relative equilibria that are elliptic

Space	Dimension	Cases (Inner / Outer Ellipses)
$\mathcal{T}_{L_1,C,G_2}$	2	No Circular/No Circular - No Coplanar Circular / No Circular - No Coplanar Rectilinear / No Circular
$\mathcal{S}_{L_1,L_2,C}$	4	No Circular / Circular - No Coplanar Circular/Circular - No Coplanar No Circular / No Circular - Coplanar Circular / No Circular - Coplanar No Circular / Circular - Coplanar
$\mathcal{R}_{L_1,L_2,B}$	6	Circular / Circular - Coplanar - $C \neq \lvert B \rvert$
$\mathcal{U}_{L_1,B,G_2}$	6	Rectilinear / Circular - $C \neq \lvert B \rvert$
$\mathcal{A}_{L_1,L_2}$	8	Circular / Circular - Coplanar - $C = \lvert B \rvert$ Rectilinear /Circular - $C = \lvert B \rvert$

Circular / Circular - Coplanar: $G_1 = C + G_2$, $C = |N|$

$\mathcal{H} = \mathcal{H}_{\mathrm{Kep}} + \varepsilon \mathcal{K}_1 + \mathcal{O}(\varepsilon^2)$, where $\mathcal{H}_{\mathrm{Kep}} \equiv \mathcal{H}_{\mathrm{Kep}}(L_1, L_2)$ is the sum of two Keplerian Hamiltonians.

We introduce local coordinates of $\mathcal{A}_{L_1,L_2}$ through:

$$
\begin{aligned}
x_1 &= \sqrt{2(L_1 - G_1)}\cos(g_1 \pm \nu + \nu_1) & y_1 &= \sqrt{2(L_1 - G_1)}\sin(g_1 \pm \nu + \nu_1)\\
x_2 &= \sqrt{2(L_2 - G_2)}\cos(g_2 \mp \nu - \nu_1) & y_2 &= \sqrt{2(L_2 - G_2)}\sin(g_2 \mp \nu - \nu_1)\\
x_3 &= \mp\sqrt{2(C + G_2 - G_1)}\cos(\nu \pm \nu_1) & y_3 &= \sqrt{2(C + G_2 - G_1)}\sin(\nu \pm \nu_1)\\
x_4 &= \mp\sqrt{2(C \mp N)}\sin\nu & y_4 &= \sqrt{2(C \mp N)}\cos\nu
\end{aligned}
$$

$$\bar{\mathcal{H}} \equiv \mathcal{H}(L_1, L_2, x_1, x_2, x_3, -, y_1, y_2, y_3, -).$$

Invariant Tori in Hamiltonian Systems with High Order Proper Degeneracy

$$h(I,\varphi,\varepsilon) = h_0(I^{n_0}) + \varepsilon^{m_1} h_1(I^{n_1}) + \ldots + \varepsilon^{m_a} h_a(I^{n_a}) + \varepsilon^{m_a+1} p(I,\varphi,\varepsilon), \qquad (1)$$

- $(I,\varphi) \in \mathbb{R}^n \times \mathbb{T}^n$ are action-angle variables,
- $\varepsilon > 0$ is a sufficiently small parameter,
- h is real analytic and is considered in a closed region $Z \times \mathbb{T}^n \subset \mathbb{R}^n \times \mathbb{T}^n$,
- a, m_j, n_i $(j = 0, 1, \ldots, a)$ and $(i = 0, 1, \ldots, a)$ are positive integers,
- $n_0 \leq n_1 \leq \ldots \leq n_a = n$, $m_1 \leq m_2 \leq \ldots \leq m_a = m$,
- $I^{n_i} = (I_1, \ldots, I_{n_i})$, $i = 1, 2, \ldots, a$,
- p depends on ε smoothly,
- the intermediate Hamiltonian $\tilde{h}(I,\varphi,\varepsilon) = h_0(I^{n_0}) + \varepsilon^{m_1} h_1(I^{n_1}) + \ldots + \varepsilon^{m_a} h_a(I^{n_a})$ admits a family of invariant n-tori $T^{\varepsilon}_{\zeta} = \{\zeta\} \times \mathbb{T}^n$.

Invariant Tori in Hamiltonian Systems with High Order Proper Degeneracy

Let

- $\bar{I}^{n_i} = (I_{n_{i-1}+1}, \ldots, I_{n_i})$, with $n_{-1} = 0$ and $\bar{I}^{n_0} = I^{n_0}$.
- $\Omega = \Big(\nabla_{\bar{I}^{n_0}} h_0(I^{n_0}), \ldots, \nabla_{\bar{I}^{n_a}} h_{n_a}(I^{n_a})\Big)$, $i = 0, 1, \ldots, a$.
- Condition (A): Rank $\left\{\partial_I^\alpha \Omega(I) \;:\; 0 \leq |\alpha| \leq s\right\} = n, \quad \forall\, I \in Z$.

Theorem (Han, Li and Yi):
Assume the condition (A) and let δ with $0 < \delta < 1/5$ be given. Then there exists an $\varepsilon_0 > 0$ and a family of Cantor sets $Z_\varepsilon \subset Z$, $0 < \varepsilon < \varepsilon_0$, with $|Z \setminus Z_\varepsilon| = O(\varepsilon^{\delta/s})$, such that each $\zeta \in Z_\varepsilon$ corresponds to a real analytic, invariant, quasi-periodic n-torus $\bar{T}_\zeta^\varepsilon$ of the Hamiltonian (1) which is slightly deformed from the intermediate n-torus T_ζ^ε. Moreover, the family $\{\bar{T}_\zeta^\varepsilon : \zeta \in Z_\varepsilon,\; 0 < \varepsilon < \varepsilon_0\}$ varies Whitney smoothly.

We linearize $\bar{\mathcal{H}}$ around the origin.

This point represents motions in $\mathcal{A}_{L_1,L_2}$ of the type circular / circular and coplanar:

$$x_i = \nu^{1/8}\bar{x}_i \qquad y_i = \nu^{1/8}\bar{y}_i$$

The change is symplectic with multiplier $\nu^{-1/4}$.

The next step is the passage to action-angle coordinates:

$$\bar{x}_i = \sqrt{2I_i}\sin\varphi_i \qquad \bar{y}_i = \sqrt{2I_i}\cos\varphi_i$$

After averaging over φ_i we end up with:

$$\bar{\mathcal{H}}_\varepsilon(L_1, L_2, I_1, I_2, I_3) = h_0(L_1, L_2) + \varepsilon^4 h_1(L_1, L_2) + \varepsilon^5 h_2(L_1, L_2, I_1, I_2, I_3) + \varepsilon^6 h_3(L_1, L_2, I_1, I_2, I_3) + \mathcal{O}(\varepsilon^7),$$

where $\mathcal{H}_{\text{Kep}} = h_0$ and $\nu = \varepsilon^4$.

$$\Omega \equiv (\Omega_1,\ \Omega_2,\ \Omega_3,\ \Omega_4,\ \Omega_5,\ \Omega_6,\ \Omega_7,\ \Omega_8) = \left(\frac{\partial h_0}{\partial L_1}, \frac{\partial h_0}{\partial L_2}, \frac{\partial h_2}{\partial I_1}, \frac{\partial h_2}{\partial I_2}, \frac{\partial h_2}{\partial I_3}, \frac{\partial h_3}{\partial I_1}, \frac{\partial h_3}{\partial I_2}, \frac{\partial h_3}{\partial I_3}\right).$$

$$\text{Rank}\left\{\partial^\alpha_{L,I}\,\Omega(L,I)\ :\ 0 \le |\alpha| \le s\right\} = 5.$$

There are families of KAM 5-tori around relative equilibria of type circular / circular - coplanar.

- These invariant tori are a particular case of the tori computed by Chierchia and Pinzari for the N body problem: [The Planetary N-Body Problem: Symplectic Foliation, Reductions and Invariant Tori, *Invent. Math.* **186** 1-77 (2011)] and other papers by them; see also the papers by J. Féjoz.
- We conclude with **the persistence of different types of invariant tori**, not only in a circular/circular and coplanar regime, enlarging the known results by Moser and Jefferys (1966), Robutel (1993), Chierchia and Pinzari (2011), etc. for $N = 3$.
- There are families of KAM 5-tori around each **elliptic equilibrium**, even for the equilibria of **rectilinear type**.

Invariant Tori for the N Body Problem

We plan to apply a similar scheme to the N–body problem with the aim of finding families KAM tori apart from the ones of circular coplanar type.
After averaging over $\ell_1, \ldots, \ell_{n_1}$, the reduced space is

$$\begin{aligned}\mathcal{A}_{L_1,\ldots,L_{N-1}} &= S^2_{L_1} \times S^2_{L_1} \times \ldots \times S^2_{L_{N-1}} \times S^2_{L_{N-1}} \\ &= \Big\{(\mathbf{a}_1, \mathbf{b}_1, \ldots, \mathbf{a}_{N-1}, \mathbf{d}_{N-1}) \in \mathbb{R}^{4\,N-1} \;\mid\; |\mathbf{a}_k|^2 = |\mathbf{b}_k|^2 = L_k^2, \\ &\qquad\qquad C \leq L_1 + \ldots + L_{N-1}\Big\}.\end{aligned}$$

Next step is the construction of the singular reduced space after reducing the nodes.
Ingredients:

- The Hamiltonian has to be averaged w.r.t $\ell_1, \ldots, \ell_{N_1}$ using Deprit's variables.
- One needs to apply an inductive process to get the invariants σ_k's and the reduced space $\mathcal{S}_{L_1,\ldots,L_{N-1},C}$