Non-planar Periodic Solutions for Spatial Restricted N+1-body Problems

Xiaoxiao Zhao, Fengying Li and Shiqing Zhang

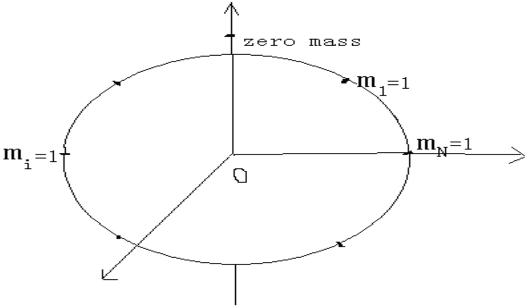
Sichuan University

New Perspectives on the N-body Problem, BIRS , 2013

Construction

In this talk, we use variational minimizing methods and Jacobi's theory for local minimizers to study spatial restricted N+1-body problems with a sufficiently small mass(zero mass) moving on the vertical axis of the moving circular orbit plane for the first N bodies, here the vertical axis passes through the center of masses for the N primaries.

Firstly, for $2 \leq N \leq 10^{10}$, we study the spatial restricted N+1-body problems. N primary bodies with equal masses are located at the vertices of a regular polygon, we prove that the minimizer of the Lagrangian action on the anti-T/2 or odd symmetric loop space must be a non-planar periodic solution.



The orbit $q(t) = (0, 0, z(t)) \in R^3$ for the sufficiently small mass satisfies the following equation

$$\ddot{q} = \sum_{i=1}^{N} \frac{m_i(q_i - q)}{|q_i - q|^3}, \tag{1}$$

and the corresponding functional is

$$\begin{aligned} f(q) &= \int_0^1 \Big[\frac{1}{2}|\dot{q}|^2 + \sum_{i=1}^{N} \frac{1}{|q - q_i|}\Big] dt, \qquad q \in \Lambda_i, i = 1, 2, \\ &= \int_0^1 \Big[\frac{1}{2}|z'|^2 + \frac{N}{\sqrt{r^2 + z^2}}\Big] dt \triangleq f(z), \end{aligned} \tag{2}$$

where

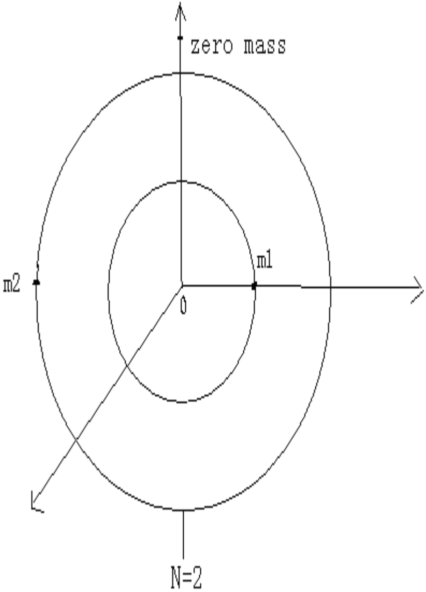
$$W^{1,2}(R/TZ, R) = \left\{ u(t) \middle| u(t), u'(t) \in L^2(R, R),\ u(t+T) = u(t) \right\},$$

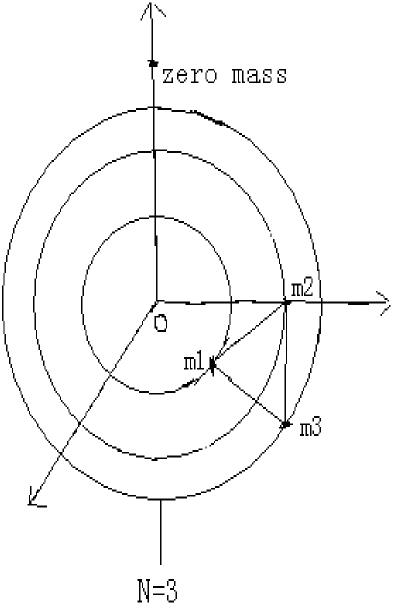
$$\Lambda_1 = \left\{ q(t) = (0, 0, z(t)) \middle| z(t) \in W^{1,2}(R/TZ, R),\ z(t + \frac{T}{2}) = -z(t) \right\},$$

and

$$\Lambda_2 = \left\{ q(t) = (0, 0, z(t)) \middle| z(t) \in W^{1,2}(R/TZ, R),\ z(-t) = -z(t) \right\},$$

Secondly, we prove the existence of non-planar periodic solutions for the following spatial restricted 3-body and 4-body problems: for $N = 2$ or 3, given any positive masses $m_1, \cdots, m_N$ in a central configuration(for $N = 2$, two bodies are in a Euler configuration; for $N = 3$, three bodies are in a Lagrange configuration), and the mass points of $m_1, \cdots, m_N$ move in the plane of their circular obits centered at the center of masses. Using variational minimizing methods, we establish the existence of the minimizer of the Lagrangian action on anti-T/2 or odd symmetric loop space, moreover, we prove the minimizer is non-planar periodic solution by using the Jacobi's Necessary Condition for local minimizers.


zero mass
m1
m2
0
N=2


zero mass
m2
O
m1
m3
N=3

The orbit $q(t) = (0, 0, z(t)) \in R^3$ for the sufficiently small mass satisfies the following equation

$$\ddot{q} = \sum_{i=1}^{N} \frac{m_i(q_i - q)}{|q_i - q|^3}, \;\; N = 2 \text{ or } 3, \tag{3}$$

and the corresponding functional is

$$\begin{aligned} f(q) &= \int_0^T \Big[\frac{1}{2}|\dot{q}|^2 + \sum_{i=1}^{N} \frac{m_i}{|q - q_i|}\Big] dt, \;\; q \in \Lambda_j, j = 1, 2, \\ &= \int_0^T \Big[\frac{1}{2}|z'|^2 + \sum_{i=1}^{N} \frac{m_i}{\sqrt{r_i^2 + z^2}}\Big] dt, \\ &\triangleq f(z). \end{aligned} \tag{4}$$

Variational Methods

In 1975 and 1977, Gordon firstly used variational methods to study the periodic solutions of two body problems. Later, many people used variational methods to study N ($N \geq 3$) body problems.

In 2004, Souissi used variational minimax methods and approximations to study the following restricted 3-body problem:

$$\ddot{z}(t) + \alpha \frac{z(t)}{(|z(t)|^2 + |r|^2)^{\alpha/2+1}} = 0. \tag{5}$$

He got

Theorem

Theorem A(Chouhaïd Souissi, Math. Comp. Sci. Ser., 2004)
For $0 < \alpha \le 1$, problem (5) has at least one parabolic orbit.

Recently, by using variational minimizing methods, Zhang ShiQing considered the restricted 3-body problem (5) with weak forces. He got the following

Theorem

Theorem B (Zhang Shiqing, Sci. China. Math., 2012)
For problem (5) with $0 < \alpha < 2$*, there exists one odd parabolic or hyperbolic orbit which minimizes the corresponding variational functional.*

Our Works

Theorem

Theorem 1 *The minimizer of* $f(q)$ *on the closure* $\overline{\Lambda}_i$ *of* Λ_i*(i=1,2) is a non-constant periodic solution for* $2 \leq N \leq 10^{10}$*,hence the zero mass must oscillate,so that it can't be always in the same plane with the other bodies.*

Theorem

Theorem 2 *For $N = 2$, let mass points of m_1, m_2 be in a Euler configuration, then the minimizer of $f(q)$ on the closure $\overline{\Lambda}_i$ of $\Lambda_i (i = 1, 2)$ is a nonplanar periodic solution.*

Theorem

Theorem 3 *For $N = 3$, let mass points of m_1, m_2, m_3 be in a Lagrange configuration, then the minimizer of $f(q)$ on the closure $\overline{\Lambda}_i$ of $\Lambda_i (i = 1, 2)$ is a nonplanar periodic solution.*

Theorem

Theorem 2 *For $N = 2$, let mass points of m_1, m_2 be in a Euler configuration, then the minimizer of $f(q)$ on the closure $\overline{\Lambda}_i$ of $\Lambda_i (i = 1, 2)$ is a nonplanar periodic solution.*

Theorem

Theorem 3 *For $N = 3$, let mass points of m_1, m_2, m_3 be in a Lagrange configuration, then the minimizer of $f(q)$ on the closure $\overline{\Lambda}_i$ of $\Lambda_i (i = 1, 2)$ is a nonplanar periodic solution.*

Here i list some famous theorems we needed.

Theorem(Palais 1979) *Let σ be an orthogonal representation of a finite or compact group G in the real Hilbert space H such that for any $\sigma \in G$,*

$$f(\sigma \cdot x) = f(x),$$

where $f \in C^1(H, R^1)$. Let $S = \{x \in H | \sigma x = x, \forall \sigma \in G\}$, then the critical point of f in S is also a critical point of f in H.

Theorem(Tonelli [1]) *Let X be a reflexive Banach space, S be a weakly closed subset of X, $f : S \to R \cup \{+\infty\}$. If $f \not\equiv +\infty$ is weakly lower semi-continuous and coercive($f(x) \to +\infty$ as $\|x\| \to +\infty$), then f attains its infimum on S.*

Jacobi's Necessary Condition[2] *Let* $F \in C^3([a,b] \times R \times R, R)$. *If the critical point* $y = \tilde{y}(x)$ *corresponds to a minimum of the functional* $\int_a^b F(x, y(x), y'(x))dx$ *on* $M = \{y \in W^{1,2}([a,b], R) | y(a) = A, y(b) = B\}$ *and if* $F_{y'y'} > 0$ *along this critical point, then the open interval* (a,b) *contains no points conjugate to a, that is, for* $\forall c \in (a,b)$, *the following problem:*

$$\begin{cases} -\frac{d}{dx}(Ph') + Qh = 0, \\ h(a) = 0, \quad h(c) = 0, \end{cases}$$

has only the trivial solution $h(x) \equiv 0, \ \forall x \in (a,c)$, *where*

$$P = \frac{1}{2} F_{y'y'}|_{y=\tilde{y}},$$

$$Q = \frac{1}{2}\Big(F_{yy} - \frac{d}{dx}F_{yy'}\Big)\Big|_{y=\tilde{y}}.$$

- **Remark 1** Cleary, the Jacobi's Necessary Condition is suitable for the fixed end problem. But we consider the periodic solutions of the move equations on $\overline{\Lambda}_i = \Lambda_i (i = 1, 2)$, hence we need to establish a similar conclusion as Jacobi's Necessary Condition for the periodic boundary problem.
- Therefore we get

Lemma 1 Let $F \in C^3(R \times R \times R, R)$. Assume that $u = \tilde{u}(t)$ is a critical point of the functional $\int_0^T F(t, u(t), u'(t))dt$ on $W^{1,2}(R/TZ, R)$ and $F_{u'u'}|_{u=\tilde{u}} > 0$. If the open interval $(0, T)$ contains a point c conjugate to 0, then $u = \tilde{u}(t)$ is not a minimizer of the functional $\int_0^T F(t, u(t), u'(t))dt$.

- **Remark 1** Cleary, the Jacobi's Necessary Condition is suitable for the fixed end problem. But we consider the periodic solutions of the move equations on $\overline{\Lambda}_i = \Lambda_i (i = 1, 2)$, hence we need to establish a similar conclusion as Jacobi's Necessary Condition for the periodic boundary problem.
- Therefore we get

  Lemma 1 Let $F \in C^3(R \times R \times R, R)$. Assume that $u = \tilde{u}(t)$ is a critical point of the functional $\int_0^T F(t, u(t), u'(t))dt$ on $W^{1,2}(R/TZ, R)$ and $F_{u'u'}|_{u=\tilde{u}} > 0$. If the open interval $(0, T)$ contains a point c conjugate to 0, then $u = \tilde{u}(t)$ is not a minimizer of the functional $\int_0^T F(t, u(t), u'(t))dt$.

Proof. Suppose $u = \tilde{u}(t)$ is a minimizer of the functional $\int_0^T F(t, u(t), u'(t))dt$. The second variation of $\int_0^T F(t, u(t), u'(t))dt$ is

$$\int_0^T (Ph'^2 + Qh^2)dt, \tag{6}$$

where

$$P = \frac{1}{2} F_{u'u'}|_{u=\tilde{u}},$$

$$Q = \frac{1}{2}\left(F_{uu} - \frac{d}{dt}F_{uu'}\right)\Big|_{u=\tilde{u}}.$$

Set

$$Q_{\tilde{u}}(h) = \int_0^T (Ph'^2 + Qh^2)dt. \tag{7}$$

For $\forall h \in C_0^1([0, T], R)$, it is easy to see that $Q_{\tilde{u}}(h) \geq 0$. Then by $Q_{\tilde{u}}(\theta) = 0$, θ is a minimizer of $Q_{\tilde{u}}(h)$.

The Euler-Lagrange equation which is called the Jacobi equation of (7) is

$$-\frac{d}{dt}(Ph') + Qh = 0. \tag{8}$$

Since the interval $(0, T)$ contains a point c conjugate to 0, there exists a nonzero Jacobi field $h_0 \in C^2([0, T], R)$ satisfying

$$\begin{cases} -\frac{d}{dt}(Ph_0') + Qh_0 = 0, \\ h_0(0) = 0, \quad h_0(c) = 0. \end{cases} \tag{9}$$

Let

$$\hat{h}(t) = \begin{cases} h_0(t) & t \in [0, c], \\ 0 & t \in (c, T], \end{cases}$$

we have $\hat{h} \in C^2([0, T] \backslash \{c\}, R)$, $\hat{h}(0) = \hat{h}(c) = \hat{h}(T) = 0$ and

$$Q_{\tilde{u}}(\hat{h}) = \int_0^T (P\hat{h}'^2 + Q\hat{h}^2)dt = \int_0^c (Ph_0'^2 + Qh_0^2)dt = 0. \tag{10}$$

Notice that we can extend $\hat{h}$ periodically when we take T as the period, so $\hat{h} \in W_0^{1,2}(R/TZ, R)$. For $\forall h \in C_0^1([0, T], R)$, it is easy to check that $Q_{\tilde{u}}(h) \geq 0$. Then by (10), one has $\hat{h} \in C^2([0, T]\backslash\{c\}, R) \cap W_0^{1,2}(R/TZ, R)$ is a minimizer of $Q_{\tilde{u}}(h)$. Hence we get

$$-\frac{d}{dt}(P\hat{h}') + Q\hat{h} = 0. \tag{11}$$

Combine with $\hat{h}(0) = \hat{h}(c) = 0$, by the uniqueness of initial value problems for second order differential equation, we have $\hat{h}(t) \equiv 0$ on [0,c], which contradicts the definition of $\hat{h}$. The proof is completed. □

Restricted N+1-body problems with N equal masses

The orbit $q(t) = (0, 0, z(t)) \in R^3$ for the sufficiently small mass satisfies the following equation

$$\ddot{q} = \sum_{i=1}^{N} \frac{m_i(q_i - q)}{|q_i - q|^3}, \tag{12}$$

and the corresponding functional is

$$\begin{aligned} f(q) &= \int_0^1 [\frac{1}{2}|\dot{q}|^2 + \sum_{i=1}^{N} \frac{1}{|q - q_i|}] dt, \qquad q \in \Lambda_i, i = 1, 2, \\ &= \int_0^1 [\frac{1}{2}|z'|^2 + \frac{N}{\sqrt{r^2 + z^2}}] dt \triangleq f(z). \end{aligned} \tag{13}$$

- Clearly, $q(t) = (0, 0, 0)$ is a critical point of $f(q)$ on $\bar{\Lambda}_i = \Lambda_i (i = 1, 2)$.

- It is easy to get that

Lemma 2 $f(q)$ attains its infimum on $\bar{\Lambda}_1 = \Lambda_1$ or $\bar{\Lambda}_2 = \Lambda_2$.

- Clearly, $q(t) = (0, 0, 0)$ is a critical point of $f(q)$ on $\bar{\Lambda}_i = \Lambda_i (i = 1, 2)$.

- It is easy to get that

  Lemma 2 $f(q)$ attains its infimum on $\bar{\Lambda}_1 = \Lambda_1$ or $\bar{\Lambda}_2 = \Lambda_2$.

- The radius r for the moving orbit of N equal masses is

$$r = \left(\frac{1}{4\pi}\right)^{\frac{2}{3}} \Big[\sum_{1 \le j \le N-1} csc(\frac{\pi}{N} j) \Big]^{\frac{1}{3}}.$$

- The Euler equation of the second variation of (13) is called the Jacobi equation of the original functional (13), which is

$$h'' + \frac{N}{r^3} h = 0. \tag{14}$$

- The radius r for the moving orbit of N equal masses is

$$r = \left(\frac{1}{4\pi}\right)^{\frac{2}{3}} \Big[\sum_{1 \le j \le N-1} csc(\frac{\pi}{N} j) \Big]^{\frac{1}{3}}.$$

- The Euler equation of the second variation of (13) is called the Jacobi equation of the original functional (13), which is

$$h'' + \frac{N}{r^3} h = 0. \tag{14}$$

- Next, we study the solution of (14) with initial values $h(0) = 0, \ h'(0) = 1$.
- It is easy to get

$$h(t) = \sqrt{\frac{r^3}{N}} \cdot sin\sqrt{\frac{N}{r^3}}t, \tag{15}$$

which is not identically zero on $[0, 1]$, but we can prove $h(c) = 0$ for some $c \in (0, 1)$.

- Next, we study the solution of (14) with initial values $h(0) = 0, \; h'(0) = 1$.
- It is easy to get

$$h(t) = \sqrt{\frac{r^3}{N}} \cdot sin\sqrt{\frac{N}{r^3}}t, \tag{15}$$

which is not identically zero on $[0, 1]$, but we can prove $h(c) = 0$ for some $c \in (0, 1)$.

In order to find some a $c \in (0,1)$ satisfying $h(c) = 0$, we need the following Lemma

Lemma 3(Moeckel R.,Simo C.,SIAM J. MATH. ANAL.,1995)
Let $A(N) = \sum_{j=1}^{N-1} csc(\frac{\pi}{N}j)$. Then, $A(N)$ has the following asymptotic expansion for N large:

$$A(N) \sim \frac{2N}{\pi}(\gamma + \log \frac{2N}{\pi}) + \sum_{k \geq 1} \frac{(-1)^k (2^{2k-1} - 1) B_{2k}^2 \pi^{2k-1}}{(2k)(2k)!} \frac{1}{N^{2k-1}},$$

where γ stands for the Euler-Mascheroni constant and B_{2k}^2 stands for the Bernoulli numbers.

By Lemma 3, we can choose $c = \left[\frac{\sum_{j=1}^{N-1} csc(\frac{\pi}{N}j)}{16N}\right]^{1/2} < \frac{1}{2}$ for $2 \leq N \leq 10^{10}$.

Case 1: Minimizing $f(q)$ on $\bar{\Lambda}_1 = \Lambda_1$.

- Let

$$\tilde{h}(t) = \begin{cases} h(t) & t \in [0, c], \\ 0 & t \in (c, \frac{1}{2}], \\ -h(t - \frac{1}{2}) & t \in (\frac{1}{2}, \frac{1}{2} + c], \\ 0 & t \in (\frac{1}{2} + c, 1]. \end{cases} \tag{16}$$

  It is easy to check that $\tilde{h}(t) \in C^2([0,1] \backslash \{c, \frac{1}{2}, \frac{1}{2} + c\}, R) \cap W^{1,2}(R, R)$, $\tilde{h}(t + \frac{1}{2}) = -\tilde{h}(t)$, $\tilde{h}(0) = h(0) = 0$, $\tilde{h}(c) = h(c) = 0$ and $\tilde{h}$ is a nonzero solution of (14).

- Notice that we can extend $\tilde{h}$ periodically when we take 1 as the period, so $\tilde{h} \in \Lambda_1$.

By Lemma 3, we can choose $c = \left[\frac{\sum_{j=1}^{N-1} csc(\frac{\pi}{N}j)}{16N}\right]^{1/2} < \frac{1}{2}$ for $2 \leq N \leq 10^{10}$.

Case 1: Minimizing $f(q)$ on $\bar{\Lambda}_1 = \Lambda_1$.

- Let
$$\tilde{h}(t) = \begin{cases} h(t) & t \in [0, c], \\ 0 & t \in (c, \frac{1}{2}], \\ -h(t - \frac{1}{2}) & t \in (\frac{1}{2}, \frac{1}{2} + c], \\ 0 & t \in (\frac{1}{2} + c, 1]. \end{cases} \tag{16}$$
  It is easy to check that $\tilde{h}(t) \in C^2([0, 1] \backslash \{c, \frac{1}{2}, \frac{1}{2} + c\}, R) \cap W^{1,2}(R, R)$, $\tilde{h}(t + \frac{1}{2}) = -\tilde{h}(t)$, $\tilde{h}(0) = h(0) = 0$, $\tilde{h}(c) = h(c) = 0$ and $\tilde{h}$ is a nonzero solution of (14).
- Notice that we can extend $\tilde{h}$ periodically when we take 1 as the period, so $\tilde{h} \in \Lambda_1$.

Case 2: Minimizing $f(q)$ on $\bar{\Lambda}_2 = \Lambda_2$.

- Let
$$\bar{h}(t) = \begin{cases} h(t) & t \in [0, c], \\ 0 & t \in (c, 1-c], \\ -h(1-t) & t \in (1-c, 1]. \end{cases} \tag{17}$$
It is easy to check that $\bar{h}(t) \in C^2([0,1]\backslash\{c, 1-c\}, R) \cap W^{1,2}(R, R)$, $\bar{h}(-t) = -\bar{h}(t)$, $\bar{h}(0) = h(0) = 0$, $\bar{h}(c) = h(c) = 0$ and $\bar{h}$ is a nonzero solution of (14).
- Notice that we can extend $\bar{h}$ periodically when we take 1 as the period, so $\bar{h} \in \Lambda_2$.

Case 2: Minimizing $f(q)$ on $\bar{\Lambda}_2 = \Lambda_2$.

- Let

$$\bar{h}(t) = \begin{cases} h(t) & t \in [0, c], \\ 0 & t \in (c, 1-c], \\ -h(1-t) & t \in (1-c, 1]. \end{cases} \tag{17}$$

  It is easy to check that $\bar{h}(t) \in C^2([0,1] \backslash \{c, 1-c\}, R) \cap W^{1,2}(R, R)$, $\bar{h}(-t) = -\bar{h}(t)$, $\bar{h}(0) = h(0) = 0$, $\bar{h}(c) = h(c) = 0$ and $\bar{h}$ is a nonzero solution of (14).
- Notice that we can extend $\bar{h}$ periodically when we take 1 as the period, so $\bar{h} \in \Lambda_2$.

Restricted 3-Body and 4-Body Problems

The orbit $q(t) = (0, 0, z(t)) \in R^3$ for the sufficiently small mass satisfies the following equation

$$\ddot{q} = \sum_{i=1}^{N} \frac{m_i(q_i - q)}{|q_i - q|^3}, \;\; N = 2 \text{ or } 3, \tag{18}$$

and the corresponding functional is

$$\begin{aligned} f(q) &= \int_0^T \left[\frac{1}{2}|\dot{q}|^2 + \sum_{i=1}^{N} \frac{m_i}{|q - q_i|}\right] dt, \qquad q \in \Lambda_j, j = 1, 2, \\ &= \int_0^T \left[\frac{1}{2}|z'|^2 + \sum_{i=1}^{N} \frac{m_i}{\sqrt{r_i^2 + z^2}}\right] dt, \\ &\triangleq f(z). \end{aligned} \tag{19}$$

- Clearly, $q(t) = (0, 0, 0)$ is a critical point of $f(q)$ on $\bar{\Lambda}_i = \Lambda_i (i = 1, 2)$.
- It is easy to check Lemma 2 also hold for the circular restricted 3-body and 4-body problems.

- Clearly, $q(t) = (0, 0, 0)$ is a critical point of $f(q)$ on $\bar{\Lambda}_i = \Lambda_i (i = 1, 2)$.
- It is easy to check Lemma 2 also hold for the circular restricted 3-body and 4-body problems.

N=2

- The radius r_1, r_2 of the planar circular orbits for the masses m_1, m_2 are

$$r_1 = \left(\frac{T}{2\pi(m_1+m_2)}\right)^{\frac{2}{3}} m_2, \qquad r_2 = \left(\frac{T}{2\pi(m_1+m_2)}\right)^{\frac{2}{3}} m_1.$$

- The Euler equation of the second variation of (19) is called the Jacobi equation of the original functional (19), which is

$$h'' + \left(\frac{m_1}{r_1^3} + \frac{m_2}{r_2^3}\right) h = 0. \tag{20}$$

N=2

- The radius r_1, r_2 of the planar circular orbits for the masses m_1, m_2 are

$$r_1 = \left(\frac{T}{2\pi(m_1+m_2)}\right)^{\frac{2}{3}} m_2, \qquad r_2 = \left(\frac{T}{2\pi(m_1+m_2)}\right)^{\frac{2}{3}} m_1.$$

- The Euler equation of the second variation of (19) is called the Jacobi equation of the original functional (19), which is

$$h'' + \left(\frac{m_1}{r_1^3} + \frac{m_2}{r_2^3}\right) h = 0. \qquad (20)$$

- Next, we study the solution of (20) with initial values $h(0) = 0,\ h'(0) = 1$.
- It is easy to get

$$h(t) = \frac{\sqrt{m_1^3 m_2^3}T}{2\pi\sqrt{m_1^4 + m_2^4(m_1 + m_2)}} \cdot sin\left(\sqrt{\frac{m_1^4 + m_2^4}{m_1^3 m_2^3}}(m_1+m_2)\cdot\frac{2\pi}{T}t\right), \tag{21}$$

which is not identically zero on $[0, \frac{\sqrt{m_1^3 m_2^3}T}{\sqrt{m_1^4+m_2^4(m_1+m_2)}}]$, but we can obtain $h(c) = 0$ for some $c \in (0, T)$.

- Next, we study the solution of (20) with initial values $h(0) = 0,\ h'(0) = 1.$
- It is easy to get

$$h(t) = \frac{\sqrt{m_1^3 m_2^3}\,T}{2\pi\sqrt{m_1^4 + m_2^4}(m_1 + m_2)} \cdot sin\left(\sqrt{\frac{m_1^4 + m_2^4}{m_1^3 m_2^3}}(m_1+m_2)\cdot\frac{2\pi}{T}t\right), \tag{21}$$

which is not identically zero on $[0, \frac{\sqrt{m_1^3 m_2^3}\,T}{\sqrt{m_1^4+m_2^4}(m_1+m_2)}]$, but we can obtain $h(c) = 0$ for some $c \in (0, T)$.

In fact, we can choose $0 < c = \frac{\sqrt{m_1^3 m_2^3}T}{2\sqrt{m_1^4+m_2^4}(m_1+m_2)} < \frac{T}{2}$. Hence

$$h(c) = \frac{\sqrt{m_1^3 m_2^3}T}{2\pi\sqrt{m_1^4 + m_2^4}(m_1 + m_2)} \cdot sin\pi = 0.$$

Case 1: Minimizing $f(q)$ on $\bar{\Lambda}_1 = \Lambda_1$.

- Let

$$\tilde{h}(t) = \begin{cases} h(t) & t \in [0, c], \\ 0 & t \in (c, \frac{T}{2}], \\ -h(t - \frac{T}{2}) & t \in (\frac{T}{2}, \frac{T}{2} + c], \\ 0 & t \in (\frac{T}{2} + c, T]. \end{cases} \quad (22)$$

  It is easy to check that
  $\tilde{h}(t) \in C^2([0, T] \backslash \{c, \frac{T}{2}, \frac{T}{2} + c\}, R) \cap W^{1,2}(R, R)$,
  $\tilde{h}(t + \frac{T}{2}) = -\tilde{h}(t)$, $\tilde{h}(0) = h(0) = 0$, $\tilde{h}(c) = h(c) = 0$ and $\tilde{h}$ is a nonzero solution of (20).

- Notice that we can extend $\tilde{h}$ periodically when we take T as the period, so $\tilde{h} \in \Lambda_1$.

Case 1: Minimizing $f(q)$ on $\bar{\Lambda}_1 = \Lambda_1$.

- Let
$$\tilde{h}(t) = \begin{cases} h(t) & t \in [0, c], \\ 0 & t \in (c, \frac{T}{2}], \\ -h(t - \frac{T}{2}) & t \in (\frac{T}{2}, \frac{T}{2} + c], \\ 0 & t \in (\frac{T}{2} + c, T]. \end{cases} \tag{22}$$
It is easy to check that $\tilde{h}(t) \in C^2([0, T] \backslash \{c, \frac{T}{2}, \frac{T}{2} + c\}, R) \cap W^{1,2}(R, R)$, $\tilde{h}(t + \frac{T}{2}) = -\tilde{h}(t)$, $\tilde{h}(0) = h(0) = 0$, $\tilde{h}(c) = h(c) = 0$ and $\tilde{h}$ is a nonzero solution of (20).
- Notice that we can extend $\tilde{h}$ periodically when we take T as the period, so $\tilde{h} \in \Lambda_1$.

Case 2: Minimizing $f(q)$ on $\bar{\Lambda}_2 = \Lambda_2$.

- Let

$$\bar{h}(t) = \begin{cases} h(t) & t \in [0, c], \\ 0 & t \in (c, T - c], \\ -h(T - t) & t \in (T - c, T]. \end{cases} \tag{23}$$

It is easy to check that $\bar{h}(t) \in C^2([0, T]\backslash\{c, T - c\}, R) \cap W^{1,2}(R, R)$, $\bar{h}(-t) = -\bar{h}(t)$, $\bar{h}(0) = h(0) = 0$, $\bar{h}(c) = h(c) = 0$ and $\bar{h}$ is a nonzero solution of (20).

- Notice that we can extend $\bar{h}$ periodically when we take T as the period, so $\bar{h} \in \Lambda_2$.

Case 2: Minimizing $f(q)$ on $\bar{\Lambda}_2 = \Lambda_2$.

- Let

$$\bar{h}(t) = \begin{cases} h(t) & t \in [0, c], \\ 0 & t \in (c, T - c], \\ -h(T - t) & t \in (T - c, T]. \end{cases} \quad (23)$$

  It is easy to check that $\bar{h}(t) \in C^2([0, T] \backslash \{c, T - c\}, R) \cap W^{1,2}(R, R)$, $\bar{h}(-t) = -\bar{h}(t)$, $\bar{h}(0) = h(0) = 0$, $\bar{h}(c) = h(c) = 0$ and $\bar{h}$ is a nonzero solution of (20).

- Notice that we can extend $\bar{h}$ periodically when we take T as the period, so $\bar{h} \in \Lambda_2$.

N=3

The radius r_1, r_2, r_3 of the planar circular orbits for the masses m_1, m_2, m_3 are

$$r_1 = (\frac{T}{2\pi})^{1/3} M^{-2/3} \sqrt{m_2^2 + m_2 m_3 + m_3^2},$$

$$r_2 = (\frac{T}{2\pi})^{1/3} M^{-2/3} \sqrt{m_1^2 + m_1 m_3 + m_3^2},$$

$$r_3 = (\frac{T}{2\pi})^{1/3} M^{-2/3} \sqrt{m_1^2 + m_1 m_2 + m_2^2}.$$

- The Euler equation of the second variation of (19) is called the Jacobi equation of the original functional (19), which is

$$h'' + \left(\frac{m_1}{r_1^3} + \frac{m_2}{r_2^3} + \frac{m_3}{r_3^3}\right)h = 0. \quad (24)$$

- Next, we study the solution of (24) with initial values $h(0) = 0, \ h'(0) = 1.$
- It is easy to get

$$h(t) = \sqrt{\frac{r_1^3 r_2^3 r_3^3}{m_3 r_1^3 r_2^3 + m_2 r_1^3 r_3^3 + m_1 r_2^3 r_3^3}} \cdot sin\sqrt{\frac{m_1}{r_1^3} + \frac{m_2}{r_2^3} + \frac{m_3}{r_3^3}}t.$$

- The Euler equation of the second variation of (19) is called the Jacobi equation of the original functional (19), which is

$$h'' + \left(\frac{m_1}{r_1^3} + \frac{m_2}{r_2^3} + \frac{m_3}{r_3^3}\right)h = 0. \qquad (24)$$

- Next, we study the solution of (24) with initial values $h(0) = 0, \ h'(0) = 1$.
- It is easy to get

$$h(t) = \sqrt{\frac{r_1^3 r_2^3 r_3^3}{m_3 r_1^3 r_2^3 + m_2 r_1^3 r_3^3 + m_1 r_2^3 r_3^3}} \cdot sin\sqrt{\frac{m_1}{r_1^3} + \frac{m_2}{r_2^3} + \frac{m_3}{r_3^3}}t.$$

- The Euler equation of the second variation of (19) is called the Jacobi equation of the original functional (19), which is

$$h'' + \left(\frac{m_1}{r_1^3} + \frac{m_2}{r_2^3} + \frac{m_3}{r_3^3}\right) h = 0. \tag{24}$$

- Next, we study the solution of (24) with initial values $h(0) = 0,\ h'(0) = 1.$
- It is easy to get

$$h(t) = \sqrt{\frac{r_1^3 r_2^3 r_3^3}{m_3 r_1^3 r_2^3 + m_2 r_1^3 r_3^3 + m_1 r_2^3 r_3^3}} \cdot sin \sqrt{\frac{m_1}{r_1^3} + \frac{m_2}{r_2^3} + \frac{m_3}{r_3^3}}\, t.$$

Let

$$A = \frac{\sqrt{m_2^2 + m_2 m_3 + m_3^2}}{M},$$

$$B = \frac{\sqrt{m_1^2 + m_1 m_3 + m_3^2}}{M},$$

$$C = \frac{\sqrt{m_1^2 + m_1 m_2 + m_2^2}}{M}.$$

Then

$$h(t) = \frac{\sqrt{M}T}{2\pi\sqrt{\frac{m_1}{A^3} + \frac{m_2}{B^3} + \frac{m_3}{C^3}}} \cdot sin\left(\sqrt{\frac{m_1}{A^3} + \frac{m_2}{B^3} + \frac{m_3}{C^3}} \cdot \frac{2\pi}{\sqrt{M}T} t\right), \tag{25}$$

which is not identically zero on $[0, \frac{\sqrt{M}T}{\sqrt{\frac{m_1}{A^3} + \frac{m_2}{B^3} + \frac{m_3}{C^3}}}]$, but we can obtain $h(c) = 0$ for some $c \in (0, T)$.

In fact, we can choose $0 < c = \frac{\sqrt{M}T}{2\sqrt{\frac{m_1}{A^3}+\frac{m_2}{B^3}+\frac{m_3}{C^3}}} < \frac{T}{2}$. Hence

$$h(c) = \frac{\sqrt{M}T}{2\pi\sqrt{\frac{m_1}{A^3}+\frac{m_2}{B^3}+\frac{m_3}{C^3}}} \cdot sin\pi = 0.$$

Case 1: Minimizing $f(q)$ on $\bar{\Lambda}_1 = \Lambda_1$.

- Let
$$\tilde{h}(t) = \begin{cases} h(t) & t \in [0, c], \\ 0 & t \in (c, \frac{T}{2}], \\ -h(t - \frac{T}{2}) & t \in (\frac{T}{2}, \frac{T}{2} + c], \\ 0 & t \in (\frac{T}{2} + c, T]. \end{cases} \quad (26)$$
It is easy to check that
$\tilde{h}(t) \in C^2([0, T] \backslash \{c, \frac{T}{2}, \frac{T}{2} + c\}, R) \cap W^{1,2}(R, R)$,
$\tilde{h}(t + \frac{T}{2}) = -\tilde{h}(t)$, $\tilde{h}(0) = h(0) = 0$, $\tilde{h}(c) = h(c) = 0$ and $\tilde{h}$ is a nonzero solution of (24).
- Notice that we can extend $\tilde{h}$ periodically when we take T as the period, so $\tilde{h} \in \Lambda_1$.

Case 1: Minimizing $f(q)$ on $\bar{\Lambda}_1 = \Lambda_1$.

- Let
$$\tilde{h}(t) = \begin{cases} h(t) & t \in [0, c], \\ 0 & t \in (c, \frac{T}{2}], \\ -h(t - \frac{T}{2}) & t \in (\frac{T}{2}, \frac{T}{2} + c], \\ 0 & t \in (\frac{T}{2} + c, T]. \end{cases} \quad (26)$$
It is easy to check that
$\tilde{h}(t) \in C^2([0, T] \backslash \{c, \frac{T}{2}, \frac{T}{2} + c\}, R) \cap W^{1,2}(R, R)$,
$\tilde{h}(t + \frac{T}{2}) = -\tilde{h}(t)$, $\tilde{h}(0) = h(0) = 0$, $\tilde{h}(c) = h(c) = 0$ and $\tilde{h}$ is a nonzero solution of (24).
- Notice that we can extend $\tilde{h}$ periodically when we take T as the period, so $\tilde{h} \in \Lambda_1$.

Case 2: Minimizing $f(q)$ on $\bar{\Lambda}_2 = \Lambda_2$.

- Let
$$\bar{h}(t) = \begin{cases} h(t) & t \in [0, c], \\ 0 & t \in (c, T-c], \\ -h(T-t) & t \in (T-c, T]. \end{cases} \tag{27}$$
It is easy to check that
$\bar{h}(t) \in C^2([0, T] \backslash \{c, T-c\}, R) \cap W^{1,2}(R, R)$,
$\bar{h}(-t) = -\bar{h}(t)$, $\bar{h}(0) = h(0) = 0$, $\bar{h}(c) = h(c) = 0$ and $\bar{h}$ is a nonzero solution of (24).
- Notice that we can extend $\tilde{h}$ periodically when we take T as the period, so $\tilde{h} \in \Lambda_2$.

Case 2: Minimizing $f(q)$ on $\bar{\Lambda}_2 = \Lambda_2$.

- Let

$$\bar{h}(t) = \begin{cases} h(t) & t \in [0, c], \\ 0 & t \in (c, T-c], \\ -h(T-t) & t \in (T-c, T]. \end{cases} \tag{27}$$

  It is easy to check that $\bar{h}(t) \in C^2([0,T]\backslash\{c, T-c\}, R) \cap W^{1,2}(R, R)$, $\bar{h}(-t) = -\bar{h}(t)$, $\bar{h}(0) = h(0) = 0$, $\bar{h}(c) = h(c) = 0$ and $\bar{h}$ is a nonzero solution of (24).

- Notice that we can extend $\tilde{h}$ periodically when we take T as the period, so $\tilde{h} \in \Lambda_2$.

Open Problems

- N primary bodies with equal masses are located at the vertices of a regular polygon, and the sufficiently mass(zero mass) move in the space. How to get the non-collision solution and whether the solution is non-planar?
- Given any positive masses $m_1, \cdots, m_N$ in a central configuration(for $N = 2$, two bodies are in a Euler configuration; for $N = 3$, three bodies are in a Lagrange configuration), and the sufficiently mass(zero mass) move in the space. How to get the non-collision solution and whether the solution is non-planar?

Open Problems

- N primary bodies with equal masses are located at the vertices of a regular polygon, and the sufficiently mass(zero mass) move in the space. How to get the non-collision solution and whether the solution is non-planar?
- Given any positive masses $m_1, \cdots, m_N$ in a central configuration(for $N = 2$, two bodies are in a Euler configuration; for $N = 3$, three bodies are in a Lagrange configuration), and the sufficiently mass(zero mass) move in the space. How to get the non-collision solution and whether the solution is non-planar?

References

[1] Ambrosetti. A, Coti Zelati. V, Periodic Solutions of Singular Lagrangian Systems, Birkhäuser, Basel, 1993.
[2] Gelfand. I, Formin. S, Calculus of Variations, Nauka, Moscow, English edition, Prentice-Hall, Englewood Cliffs, NJ, 1965.
[3] Mathlouthi. S, Periodic oribits of the restricted three-body problem, Trans. Amer. Math. Soc, 350(1998) 2265-2276.
[4] Mawhin. J, Willem. M, Critical Point Theory and Hamiltonian Systems, Applied Mathematical Sciences, 74, Springer-Verlag, New York, 1989.
[5] McGehee. R, Parabolic Orbits of the Restricted Three-body Problem, Academic Press, New York, London, 1973.
[6] Moeckel R., Simo C., Bifurcation of spatial central configurations from planar ones, SIAM J. Math Anal., 26(4)(1995), 978-998.

[7] Palais. R, The principle of symmetric criticality, Comm. Math. Phys, 69(1)(1979) 19-30.
[8] Sitnikov, K, Existence of oscillating motions for the three-body problem, Dokl. Akad. Nauk, USSR, 133(2)(1960) 303-306.
[9] Souissi. C, Existence of parabolic orbits for the restricted three-body problem, Annals of University of Craiova, Math. Comp. Sci. Ser, 31(2004) 85-93.
[10] Zhang. S. Q, Variational minimizing parabolic and hyperbolic orbits for the restricted 3-body problems, Sci. China. Math, 55(4) 721-725.