Permanence properties for graph algebras

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Graph Algebras: Bridges between graph C^*-algebras and Leavitt path algebras
BIRS
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- for all $\mathfrak{I}_1 \trianglelefteq \mathfrak{I}_2 \trianglelefteq \mathfrak{A}$, the group $K_0(\mathfrak{I}_2/\mathfrak{I}_1)$ is finitely generated and the group $K_1(\mathfrak{I}_2/\mathfrak{I}_1)$ is finitely generated and free, and $\mathrm{rank}(K_0(\mathfrak{I}_2/\mathfrak{I}_1)) = \mathrm{rank}(K_1(\mathfrak{I}_2/\mathfrak{I}_1))$;

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Definition

A C^*-algebra $\mathfrak{A}$ is a *phantom Cuntz-Krieger algebra* if $\mathfrak{A}$ looks like a Cuntz-Krieger algebra but $\mathfrak{A}$ is not isomorphic to a Cuntz-Krieger algebra.

Question of George Elliott (2012)

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Can a phantom Cuntz-Krieger algebra be SME to a Cuntz-Krieger algebra?

$$\mathfrak{A} \otimes \mathbb{K} \cong \mathcal{O}_{\mathsf{A}}^{\mathrm{top}} \otimes \mathbb{K} \quad \text{but} \quad \mathfrak{A} \ncong \mathcal{O}_{\mathsf{A}'}^{\mathrm{top}}$$

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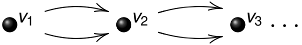
Reformulation

Can a phantom Cuntz-Krieger algebra be isomorphic to a unital full hereditary sub-algebra of a stablized Cuntz-Krieger algebra?

Bad permanence properties

E :



Bad permanence properties

$$E: \quad \bullet^{v_1} \rightrightarrows \bullet^{v_2} \rightrightarrows \bullet^{v_3} \cdots$$

$$p_{v_1}(C^*(E) \otimes \mathbb{K})p_{v_1} \sim_{SME} M_{2^\infty}$$

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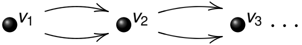

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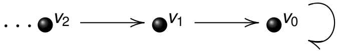

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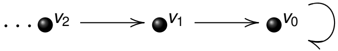

$$p(C^*(E) \otimes \mathbb{K})p \cong \mathrm{M}_{n+1}(C(\mathbb{T})) \cong C^*(F)$$

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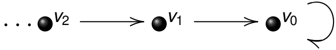

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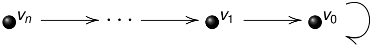

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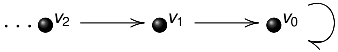

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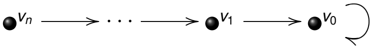

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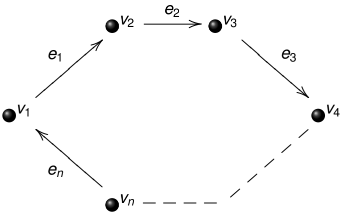
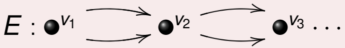
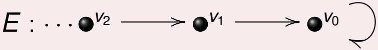




$C^*(E)$ is not SME to a unital graph C^*-algebra



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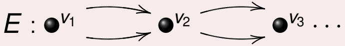

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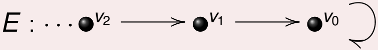

$C^*(E)$ is SME to a unital graph C^*-algebra

Question

Let E be a graph with finitely many vertices.

(1) Is every unital hereditary sub-algebra of $C^*(E) \otimes \mathbb{K}$ isomorphic to a graph C^*-algebra?

(2) Is every hereditary sub-algebra of $C^*(E) \otimes \mathbb{K}$ with an approximate identity consisting of projections isomorphic to a graph C^*-algebra?

Approximate identity consisting of projections is necessary

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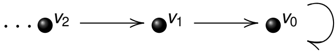

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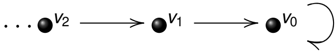

$$\text{Set } \mathfrak{A} = \left\{ f \in C(S^1, \mathrm{M}_2) : f(1) \in \begin{bmatrix} \mathbb{C} & 0 \\ 0 & 0 \end{bmatrix} \right\} \subseteq C(\mathbb{T}) \otimes \mathbb{K} \cong C^*(E).$$

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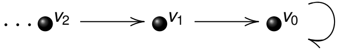

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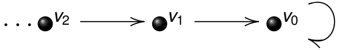

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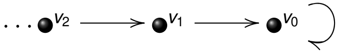

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Therefore, $\mathfrak{A}$ is not isomorphic to a graph C^*-algebra.

Cuntz-Krieger algebras

Theorem (Arklint-R)

Let $\mathfrak{B}$ be a unital hereditary sub-algebra of $\mathcal{O}_{\mathsf{A}}^{\mathrm{top}} \otimes \mathbb{K}$. Then $\mathfrak{B} \cong \mathcal{O}_{\mathsf{A}'}^{\mathrm{top}}$.

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Theorem

Let $\mathfrak{B}$ be a hereditary sub-algebra of $\mathcal{O}_{\mathsf{A}}^{\mathrm{top}} \otimes \mathbb{K}$. Then the following are equivalent.

(a) $\mathfrak{B}$ is isomorphic to a graph C^*-algebra.

(b) $\mathfrak{B}$ has an approximate identity consisting of projections.

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An *algebraic Cuntz-Krieger algebra* is $L_K(E)$ arising from a finite graph E with no sinks and sources. $L_K(E) = \mathcal{O}_{\mathsf{A}}^{\mathrm{alg}}$.

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Let S be a hereditary sub-ring of $\mathsf{M}_\infty(\mathcal{O}_{\mathsf{A}}^{\mathrm{alg}})$. Then the following are equivalent.

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(2) $S \cong L_K(F)$.

Moreover, if S is unital, then $S \cong \mathcal{O}_{\mathsf{A}'}^{\mathrm{alg}}$.

Consequences

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C^*-algebras	Rings
(1) Every hereditary sub-algebra of $\mathcal{O}_A^{\text{top}}$ with an approximate identity consisting of projections is a isomorphic to a graph C^*-algebra.	(1) Every hereditary sub-algebra of $\mathcal{O}_A^{\text{alg}}$ with an approximate identity consisting of idempotents is a isomorphic to a Leavitt path algebra.

Consequences

C^*-algebras

(1) Every hereditary sub-algebra of $\mathcal{O}_A^{\mathrm{top}}$ with an approximate identity consisting of projections is a isomorphic to a graph C^*-algebra.

(2) If $\mathfrak{A}$ has an approximate identity consisting of projections and $\mathfrak{A} \sim_{SME} \mathcal{O}_A^{\mathrm{top}}$, then $\mathfrak{A} \cong C^*(E)$.

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(2) If $\mathfrak{A}$ has an approximate identity consisting of projections and $\mathfrak{A} \sim_{SME} \mathcal{O}_{\mathsf{A}}^{\mathrm{top}}$, then $\mathfrak{A} \cong C^*(E)$.

(3) If $\mathfrak{A}$ is unital and $\mathfrak{A} \sim_{SME} \mathcal{O}_{\mathsf{A}}^{\mathrm{top}}$, then $\mathfrak{A} \cong \mathcal{O}_{\mathsf{A}'}^{\mathrm{top}}$.

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Consequences

C^*-algebras	Rings
(4) If $C^*(E)$ is a unital graph C^*-algebra, then $C^*(E) \cong \mathcal{O}_{\mathsf{A}}^{\mathrm{top}}$ if and only if $$\mathrm{rank}(K_0(C^*(E))) = \mathrm{rank}(K_1(C^*(E))).$$	(4) ?

Consequences

C^*-algebras

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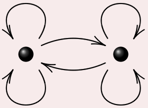
Rings

(4)

?

Example

If

F 

then

$$K_0(L_{\mathbb{Q}}(F)) = \mathbb{Z} \quad \text{and} \quad K_1(L_{\mathbb{Q}}(F)) = \mathbb{Z}_2 \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \cdots .$$

Proof

Definition

A graph E with finitely many vertices is in *standard form* if

(1) every regular vertex of E is a base point of a loop and

(2) for every infinite emitter $v \in E^0$ and $e \in s^{-1}(v)$, we have that $|s^{-1}(v) \cap r^{-1}(r(e))| = \infty$.

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Theorem (Sørensen)

If E is a graph with finitely many vertices, then there exists a graph F in standard form such that $C^*(E) \otimes \mathbb{K} \cong C^*(F) \otimes \mathbb{K}$.

Unital Case

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Theorem (Ara-Moreno-Pardo)

Let E be a finite graph in standard form such that E has no sinks and sources. If p is a non-zero projection (idempotent) in $C^*(E) \otimes \mathbb{K}$ ($\mathrm{M}_\infty(L_K(E))$), then

$$p \sim \sum_{v \in H} m_v p_v$$

with $m_v > 0$ where H is the hereditary subset of E^0 such that $I_H = \mathrm{Ideal}(p)$.

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$$p(C^*(E) \otimes K)p \cong C^*(F)$$

where F is the graph obtained from $(H, r^{-1}(H), r, s)$ by adding a head of length $m_v - 1$ to each vertex v in H.

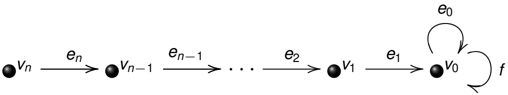
E : 

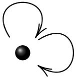
$$p(C^*(E) \otimes \mathbb{K})p$$

E :

$$p(C^*(E) \otimes \mathbb{K})p \cong C^*(F_1)$$

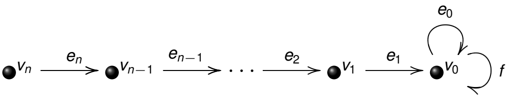
$\underline{F_1}$



E : 

$$p(C^*(E) \otimes \mathbb{K})p \cong C^*(F_1) \cong C^*(F_2)$$

$\underline{F_1}$



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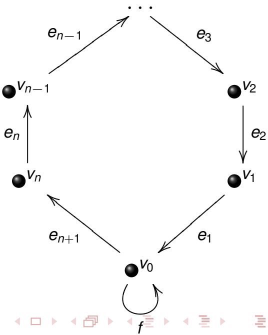

Non-unital case

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Take $\{p_n\}_{n=1}^{\infty}$ be a sequence of non-zero mutually orthogonal projections such that $\{\sum_{k=1}^{n} p_k\}_{n=1}^{\infty}$ is an approximate identity consisting of projections for

$$\mathfrak{B} \cong p(C^*(E) \otimes \mathbb{K})p \subseteq C^*(E) \otimes \mathbb{K},$$

for some projection p in the multiplier algebra $\mathcal{M}(C^*(E) \otimes \mathbb{K})$.

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- Let H be the hereditary subset E^0 such that $I_H = \mathrm{Ideal}\{p_n : n \in \mathbb{N}\}$
- $p_n \sim \sum_{v \in H} m(v,n) p_v$ where $m(v,n) \geq 0$ and

$$\bigcup_{n=1}^{\infty} \{v \in H : m(v,n) > 0\} = H.$$

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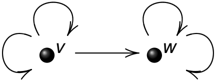
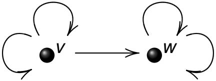
for some projection p in the multiplier algebra $\mathcal{M}(C^*(E) \otimes \mathbb{K})$.

- Let H be the hereditary subset E^0 such that $I_H = \mathrm{Ideal}\{p_n : n \in \mathbb{N}\}$
- $p_n \sim \sum_{v \in H} m(v,n) p_v$ where $m(v,n) \geq 0$ and

$$\bigcup_{n=1}^\infty \{v \in H : m(v,n) > 0\} = H.$$

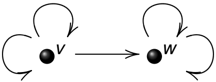
Then $\mathfrak{B} \cong C^*(F)$ where F is obtained from $(H, r^{-1}(H), r, s)$ by adding a head of length $-1 + \sum_{n=1}^\infty m(v,n)$ to each vertex $v \in H$.


v
w



$$p_1 \sim n_1 p_v + m_1 p_w, \quad p_2 \sim n_2 p_v + m_2 p_w, \quad p_3 \sim m_3 p_w, \quad p_4 \sim m_4 p_w,$$

$$\dots$$



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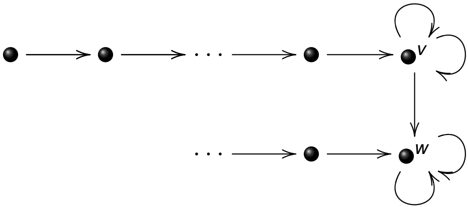

The unitization of a graph algebra

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If $C^*(E)$ is a non-unital C^*-algebra and $C^*(E) \sim_{SME} \mathcal{O}_{\mathsf{A}}^{\text{top}}$, then $C^*(E)^\dagger \cong C^*(F)$.

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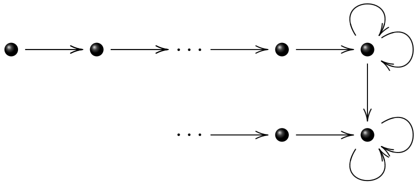

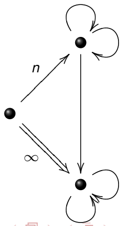

Unital graph algebras

Theorem

Let E be a graph with finitely many vertices.

(1) Every unital hereditary sub-algebra of $C^*(E) \otimes \mathbb{K}$ is isomorphic to a graph C^*-algebra.

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Let E be a graph and p be a projection (idempotent) in $C^*(E) \otimes \mathbb{K}$ ($\mathsf{M}_\infty(L_K(E))$). Then

$$p \sim \sum_{v \in S} m_v \left(p_v - \sum_{e \in T_v} s_e s_e^* \right)$$

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Change graph

$$C^*(E) \otimes \mathbb{K} \cong C^*(F) \otimes \mathbb{K}$$

$$L_K(E) \otimes \mathbb{K} \cong L_K(F) \otimes \mathbb{K}$$

$$p \mapsto q \sim \sum_{v \in S} m_v q_v$$

Theorem (work in progress)

Let E be a graph with finitely many vertices, $\mathfrak{A} \subseteq_{\text{her}} C^*(E) \otimes \mathbb{K}$, and $A \subseteq_{\text{her}} \mathsf{M}_\infty(L_K(E))$.

(1) $\mathfrak{A}$ has an approximate identity consisting of projections if and only if $\mathfrak{A}$ is isomorphic to a graph C^*-algebra.

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Theorem (work in progress)

Let E be an infinite graph.

(1) $C^*(E)^\dagger$ is isomorphic to a graph C^*-algebra if and only if $C^*(E)$ is SME to a unital graph C^*-algebra.

(2) $L_K(E)^\dagger$ is isomorphic to a Leavitt path algebra if and only if $L_K(E)$ is ME to a unital Leavitt path algebra.

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(3) If $\mathfrak{J}_2/\mathfrak{J}_1$ is "gauge simple" and $K_0(\mathfrak{J}_2/\mathfrak{J}_1)_+ \neq K_0(\mathfrak{J}_2/\mathfrak{J}_1)$, then $K_0(\mathfrak{J}_2/\mathfrak{J}_1) \cong \mathbb{Z}$ and $K_0(\mathfrak{J}_2/\mathfrak{J}_1)_+ \cong \mathbb{Z}_{\geq 0}$.

Theorem (Arklint-Bentmann-Katsura)

Let $C^*(E)$ purely infinite graph C^*-algebra with finitely many ideals. If

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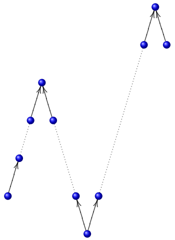
If X is an accordion space, then

$$C^*(E) \otimes \mathbb{K} \cong C^*(F) \otimes \mathbb{K}.$$



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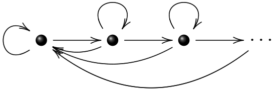
Reformulation

Suppose $C^*(E)$ is a non-unital graph C^*-algebra with finitely many ideals and "K-theory" as a Cuntz-Krieger algebra (unital graph C^*-algebra). Is every unital hereditary sub-algebra of $C^*(E)$ isomorphic to a Cuntz-Krieger algebra (unital graph C^*-algebra).



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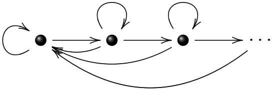

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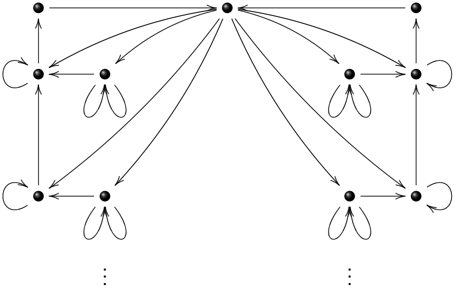

$$K_0(C^*(E)) = 0 \quad K_1(C^*(E)) = \mathbb{Z}$$