Embeddings of non-orientable surfaces

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Introduction

Basic theme in 3 and 4-manifold theory:

Constraints on genus of embedded orientable surface

$$\Sigma_g \subset M^{3 \text{ or } 4}$$

Thurston norm in dimension 3; adjunction inequalities.

Usually assume M orientable and $[\Sigma] = \alpha \neq 0 \in H_2(M;\mathbb{Z})$.

Define

$$g_M(\alpha) = \min\{g \mid \Sigma_g \subset M,\ [\Sigma_g] = \alpha\}.$$

Embeddings of genus h *non-orientable* surface

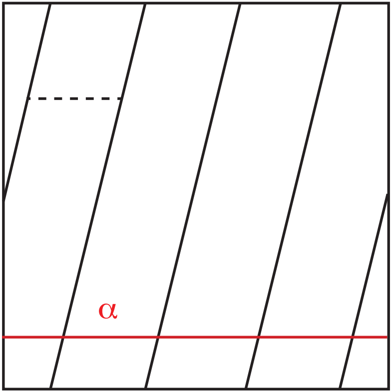
$$F_h = \#_h \mathbb{RP}^2 \subset M.$$

Dimension 3: $[F]$ must be $\neq 0 \in H_2(M;\mathbb{Z}_2)$.

Example: All $L(2k,q)$ contain non-orientable surfaces generating $H_2(L(2k,q);\mathbb{Z}_2) \cong \mathbb{Z}_2$.

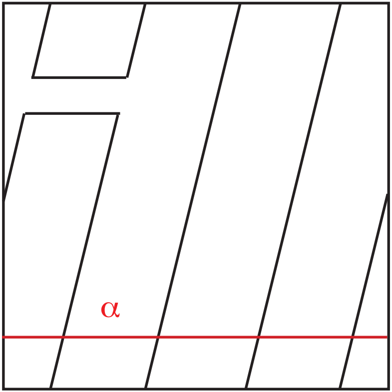
- $\mathbb{RP}^2 \subset \mathbb{RP}^3 = L(2,1)$.
- Klein bottle = $F_2 \subset L(4,1)$

L(4,1)



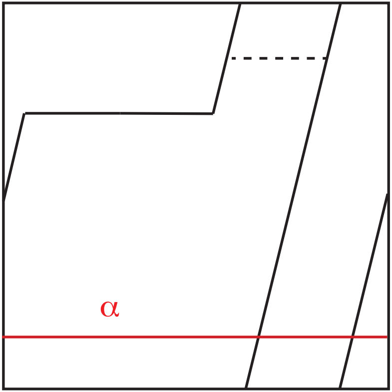
band move

L(4,1)



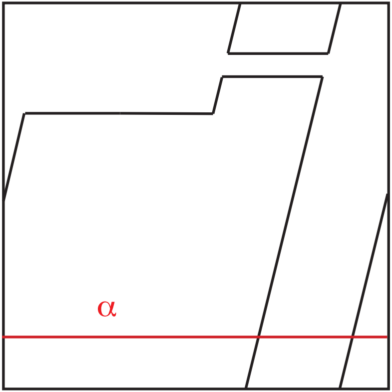
1-handle

L(4,1)



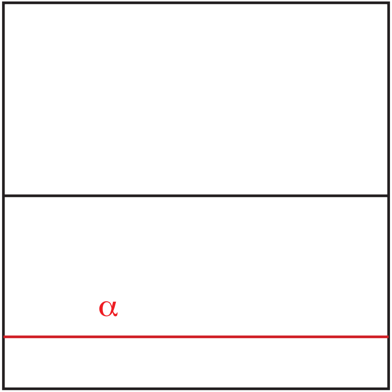
second band move

L(4,1)

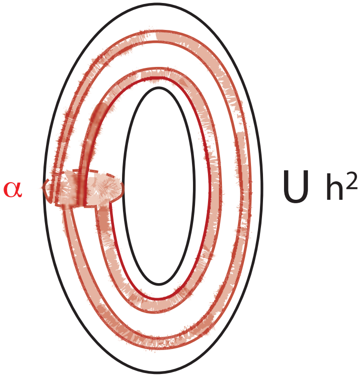


second handle

L(4,1)



fill in over α disk



Klein bottle in L(4,1)

Dimension 4:

$\mathbb{RP}^2 \subset S^4$ with normal Euler number ± 2.

So we'll assume $[F] \neq 0 \in H_2(M; \mathbb{Z}_2)$; say F is ***essential***.

For $F_h \subset M^4$, let n be its normal Euler number $F \cdot F$.

Definition: $h_M(\alpha) = \min\{h \mid F_h \subset M,\ [F_h] = \alpha\}$.

Concentrate on special case: $M = Y^3 \times I$ with $H_2(Y; \mathbb{Z}_2) = \mathbb{Z}_2$, particularly $M = L(2k, q) \times I$.

Remark: For $M = Y^3 \times I$, the Euler number n is even.

Remark:

In orientable case, Gabai showed for $\alpha \neq 0 \in H_2(Y;\mathbb{Z})$

$$g_Y(\alpha) = \min\{g \mid f : \Sigma_g \to Y,\ f_*[\Sigma] = \alpha\}$$

So $g_Y(\alpha) = g_{Y\times I}(\alpha)$.

Proof uses taut foliations; doesn't work in non-orientable case:

for any k and q there's an essential map $f : \mathbb{RP}^2 \to L(2k,q)$.

Nevertheless, we conjecture (a precise version of)

$$h_{L(2k,q)\times I} = h_{L(2k,q)}.$$

Non-orientable genus bound

Lemma 1 (Cf. B.-H. Li, M. Mahowald)

For essential $F_h \subset L(2k, q) \times I$, we have the congruence $n \equiv 2k - 2h + 2 \pmod 4$.

Remark: Connect sum with $\mathbb{RP}^2 \subset S^4$ gives $F_{h+1} \subset M$ in same homology class with Euler number $= n \pm 2$.

Theorem 2 (Levine-R.-Strle 2013)

Let $h \leq 5$. If $F_h \subset L(2k, q) \times I$ is an essential embedding with normal Euler number n, then there is an i, $(1 \leq i \leq h)$ with $|n| \leq 2h - 2i$ and an embedding $F_i \subset L(2k, q)$.

Conjecture: Theorem 2 holds for all h.

What does this mean? Let's see for small h.

$\underline{h=1}$. If $\mathbb{RP}^2 \subset L(2k,q) \times I$ then $i=1$ only choice. So $n=0$ and there's an embedding of $\mathbb{RP}^2$ in $L(2k,q)$.

Easy to see this means $L(2k,q) = L(2,1) \cong \mathbb{RP}^3$.

$\underline{h=2}$. If Klein bottle $\subset L(2k,q) \times I$ then either

- $i=2$ and $n=0$, and F_2 embeds in $L(2k,q)$. (Bredon-Wood: $\Leftrightarrow k$ even, $q=k\pm 1$)
- $i=1$ and $n=\pm 2$ and F_1 embeds in $L(2k,q)$. So $L(2k,q)=L(2,1)$.

Remark: Theorem 2 for h implies same statement for $h-1$. So it suffices to prove Theorem 2 for h odd (assume from now on).

Surfaces in lens spaces

Work of Bredon-Wood (1969) calculates $h_{L(2k,q)} := N(2k, q)$ defined recursively for $1 \leq q < k$:

- $N(2,1) = 1$
- $N(2k,q) = N(2(k-q),q') + 1$ where $1 \leq q' < k - q$ and $q' = \pm q \pmod{2(k-q)}$.

Realizing lower bound done by technique for $L(4,1)$ from earlier.

Embedding obstructions from d-invariants

If Y is a $\mathbb{Q}HS^3$, d-invariants for $\mathfrak{s} \in \mathrm{Spin}^c(Y)$ defined by

$$\min\{\mathrm{gr}(x) \mid 0 \neq x \in \mathrm{Image}(U^m),\ \forall m \geq 0\}$$

where U acts on the Heegaard-Floer homology $\mathrm{HF}^+(Y, \mathfrak{s})$.

Useful fact: (Ni-Wu; Gessel) For $k \in H_1(L(2k, q))$ order 2:

$$N(2k, q) = 2 \max_{\mathfrak{s} \in \mathrm{Spin}^c(L(2k,q))} \{d(L(2k, q), \mathfrak{s} + k) - d(L(2k, q), \mathfrak{s})\}$$

For torsion Spin^c structure $\mathfrak{s}$ on Y non-$\mathbb{Q}HS^3$ with standard $\mathrm{HF}^\infty(Y,\mathfrak{s})$, there are *two* d-invariants $d_{\mathrm{bot}}(Y,\mathfrak{s})$ and $d_{\mathrm{top}}(Y,\mathfrak{s})$ corresponding to the kernel and cokernel of the action of $H_1(Y)$.

We're interested in $Q_{h,n} =$ the non-orientable S^1 bundle of Euler class n over F_h.

- Recall n even
- $H_1(Q_{h,n}) \cong \mathbb{Z}^{h-1} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ so not a $\mathbb{Q}HS^3$ for $h > 1$.
- Two torsion Spin^c structures extend over D^2 bundle.
- Two torsion Spin^c structures don't extend over D^2 bundle.
- These are the *twisted* Spin^c structures.

The invariants d_{bot} and d_{top} yield bounds on $h_{L\times I}$ for L oriented with $H_1(L) = \mathbb{Z}_{2k}$.

Lemma 3

Suppose $F_h \subset L \times I$ with normal Euler number n, and exterior $V = L \times I - \nu(F_h)$. For any $\mathfrak{s} \in \mathrm{Spin}^c(L)$, there is a unique Spin^c structure $\tilde{\mathfrak{s}}$ on V that restricts to $\mathfrak{s}$ on L_0 and does not extend over $L \times I$.

Let $\mathfrak{t}_{\mathfrak{s}} \in \mathrm{Spin}^c(Q_{h,n})$ be the restriction of $\tilde{\mathfrak{s}}$ to $Q_{h,n}$; this is one of the twisted Spin^c structures. The restriction of $\tilde{\mathfrak{s}}$ to L_1 is $\mathfrak{s} + k$.

Main result

Theorem 4 (Levine-R.-Strle 2013)

Suppose $F_h \subset L \times I$ with normal Euler number n. For each $\mathfrak{s} \in \mathrm{Spin}^c(L)$, we have

$$d_{\mathrm{top}}(Q_{h,n}, \mathfrak{t}_{\mathfrak{s}}) - \frac{h-1}{2} \leq d(L, \mathfrak{s}+k) - d(L, \mathfrak{s}) \leq d_{\mathrm{bot}}(Q_{h,n}, \mathfrak{t}_{\mathfrak{s}}) + \frac{h-1}{2}.$$

Get the strongest results by varying $\mathfrak{s} \in \mathrm{Spin}^c(L)$ to maximize or minimize $d(L, \mathfrak{s}+k) - d(L, \mathfrak{s})$.

Computing $d(Q_{h,n}, \mathfrak{t})$

We've verified the following conjecture for $h = 1, 3, 5$ by one method, $h = 2$ by another.

Conjecture 5

For odd genus h there are two twisted spin structures $\mathfrak{t}_1$ *and* $\mathfrak{t}_2$ *such that*

$$d_{bot}(Q_{h,n}, \mathfrak{t}_1) = d_{top}(Q_{h,n}, \mathfrak{t}_1) = \frac{n+2}{4}$$

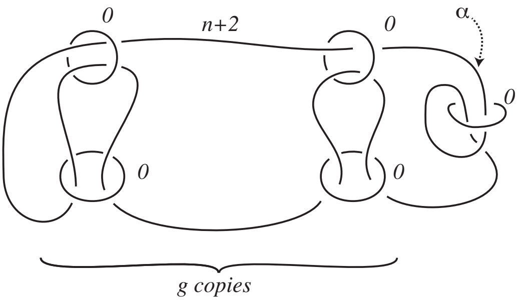
and

$$d_{bot}(Q_{h,n}, \mathfrak{t}_2) = d_{top}(Q_{h,n}, \mathfrak{t}_2) = \frac{n-2}{4}.$$

Similar statement for even genus.

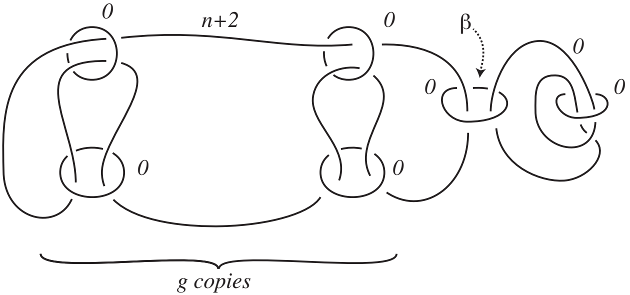
The d-invariant seems to depend only on the Euler class n (i.e., is independent of h).

Surgery picture for $Q_{h,n}$



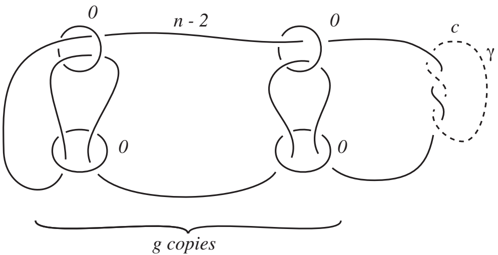
Can't apply surgery formula to surgery on α since it is of infinite order in $H_1(\#^{2g+1}S^1 \times S^2)$.

Better idea: integer surgery formula, based on the following surgery diagram for $Q_{h,n}$.



$Q_{2g+1,n}$ as surgery on rationally null-homologous knot β in $M_{g,n+2} \# Q_{1,-2}$.

Second idea: surgery exact sequence, relating $Q_{2g+1,n}$ to the orientable circle bundles $M_{g,n\pm 2}$.



For $n \neq 2$, γ rationally null-homologous; surgery produces

- $M_{g,n-2}$ for coefficient $c = \infty$;
- $Q_{2g+1,n}$ for $c = 0$;
- $M_{g,n+2}$ for $c = -1$.