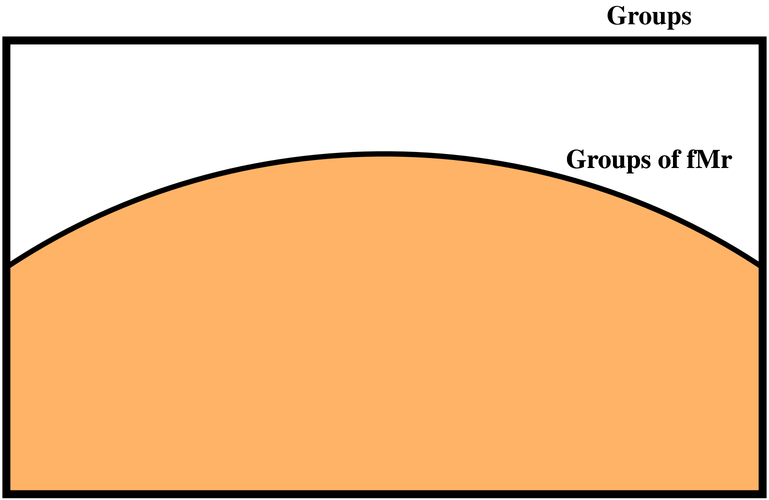
Generically n-transitive permutation groups

Josh Wiscons

Universität Münster

Workshop on Permutation Groups
BIRS - 2013

Groups of finite Morley rank (fMr)



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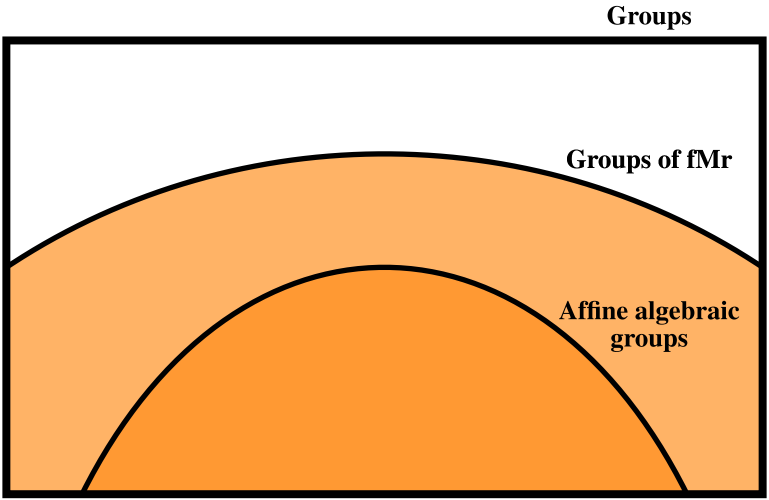

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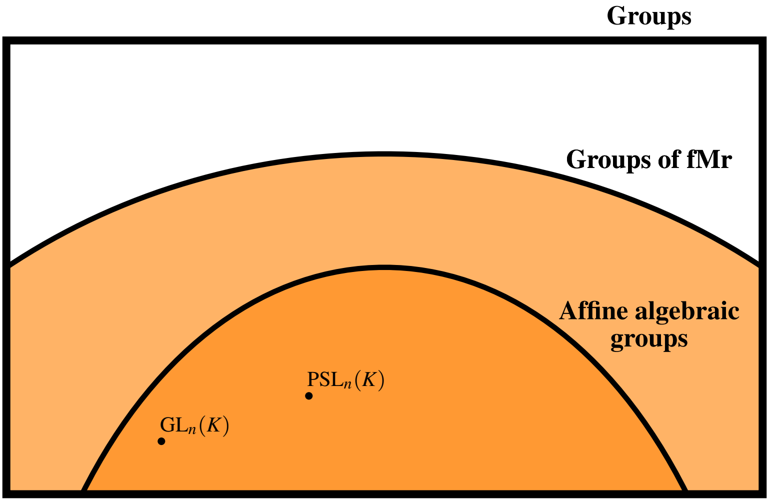

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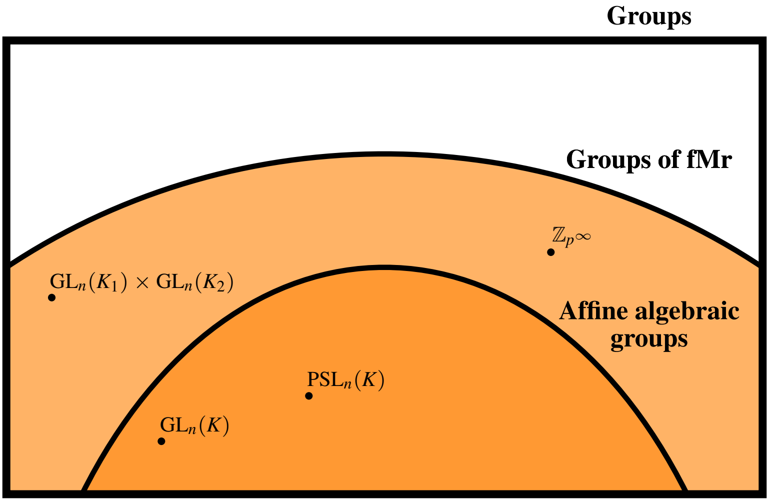

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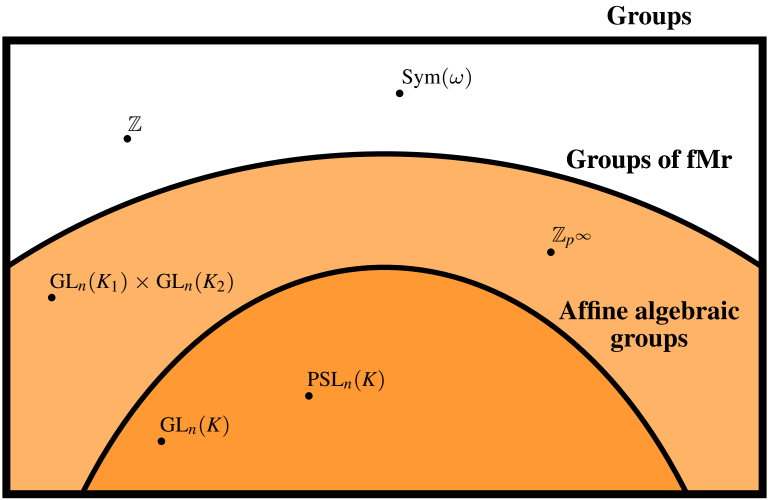

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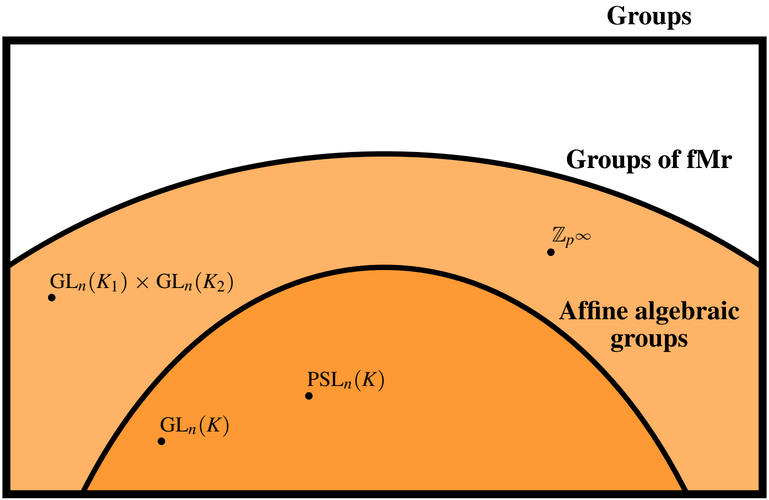

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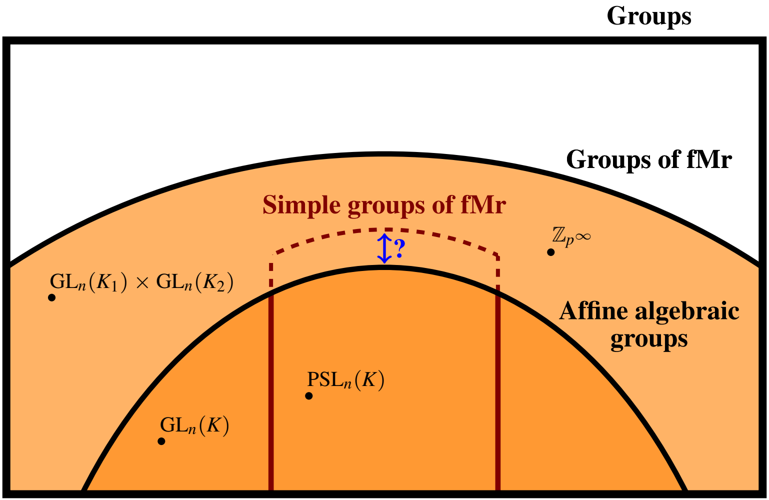

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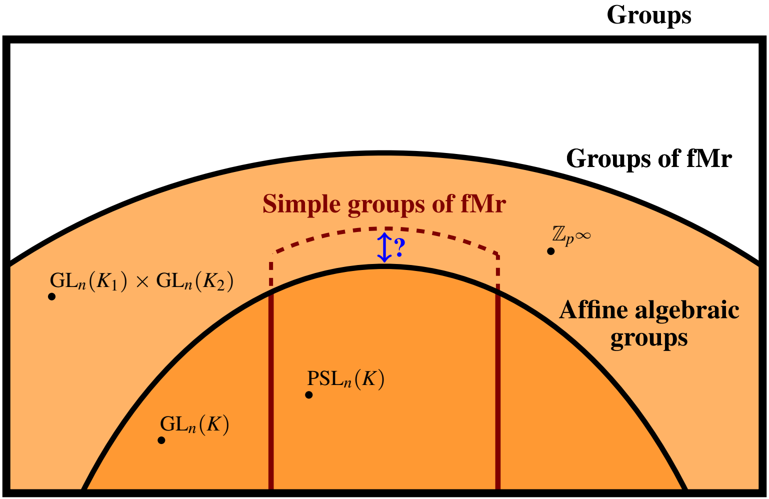

Algebraicity Conjecture:

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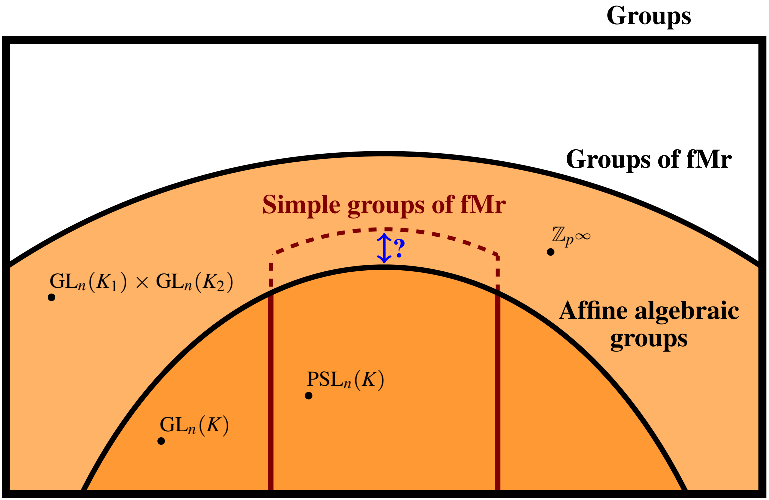

Algebraicity Conjecture: the gap, ↕ , does not exist.

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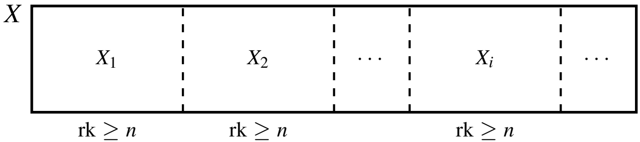

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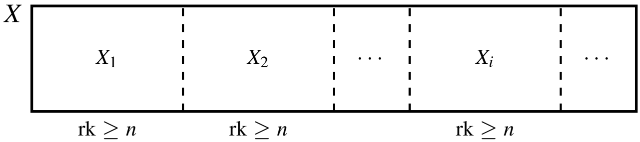

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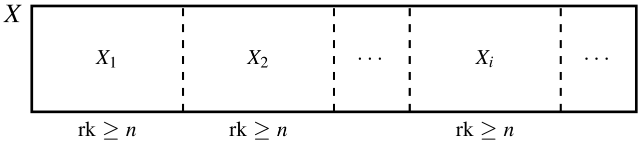

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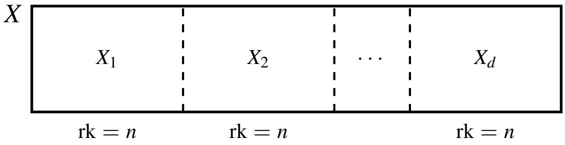

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     - $Y = G/H$ whenever H is definable

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$n+2$	3	3	3	4	3	2	2	2

2. *Let G be an infinite solvable group of fMr. Then $\mathrm{gtd}(G) \leq 2$.*
3. *Let G be an infinite nilpotent group of fMr. Then $\mathrm{gtd}(G) = 1$.*

Generic t-transitivity

Example: $\mathrm{GL}_n(K) \curvearrowright K^n$

- generically sharply n-transitive
- $\mathcal{O}$ is the set of bases of K^n: orbit of $(e_1, \ldots, e_n)$

Example: $\mathrm{PGL}_n(K) \curvearrowright \mathrm{P}^{n-1}(K)$

- gen. sharply $(n+1)$-transitive
- $\mathcal{O}$ is the set bases of $\mathrm{P}^{n-1}(K)$: orbit of $(\langle e_1\rangle, \ldots, \langle e_n\rangle, \langle \sum e_i\rangle)$

Theorem

1. *(Popov '07) Let G be an infinite simple algebraic group over an alg. closed field of characteristic 0. Then $\mathrm{gtd}(G)$ is given by*

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Problem (BC '08)

Show that the above table is valid in arbitrary characteristic.

$G \curvearrowright X$ is generically t-transitive

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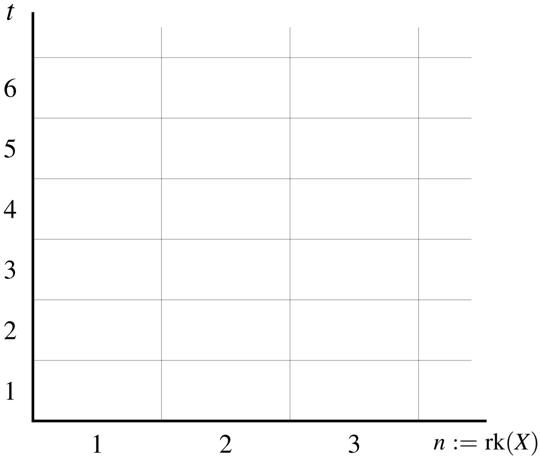

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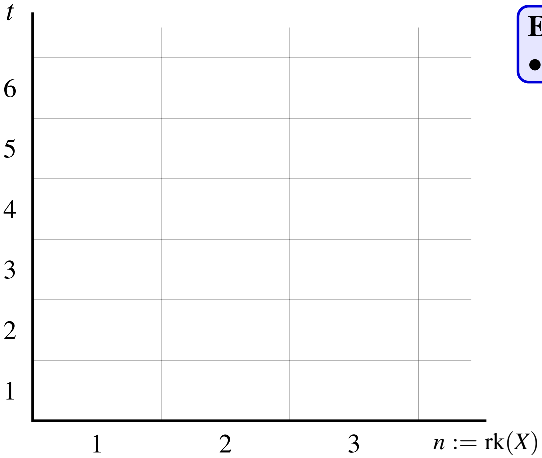

Extra Assumptions

- $G \curvearrowright X$ is transitive

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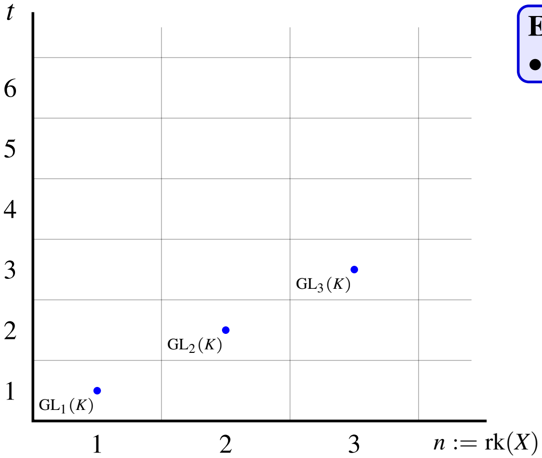

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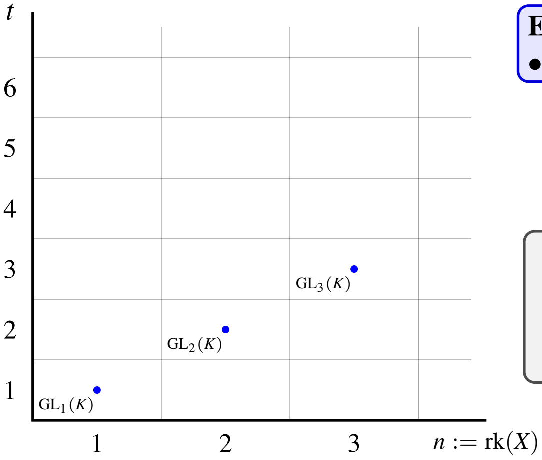

Extra Assumptions

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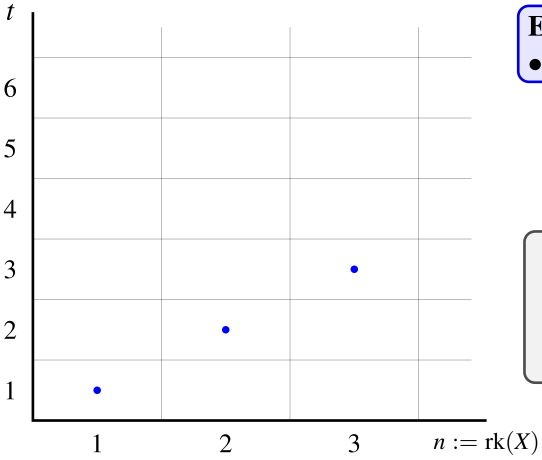

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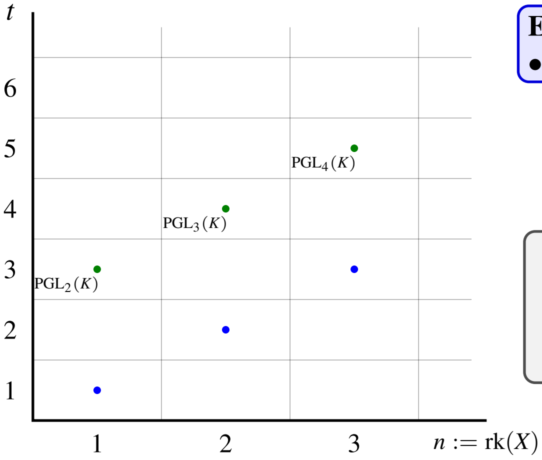

Extra Assumptions

- $G \curvearrowright X$ is transitive

- $\mathrm{PGL}_{n+1}(K) \curvearrowright \mathrm{P}^n(K)$
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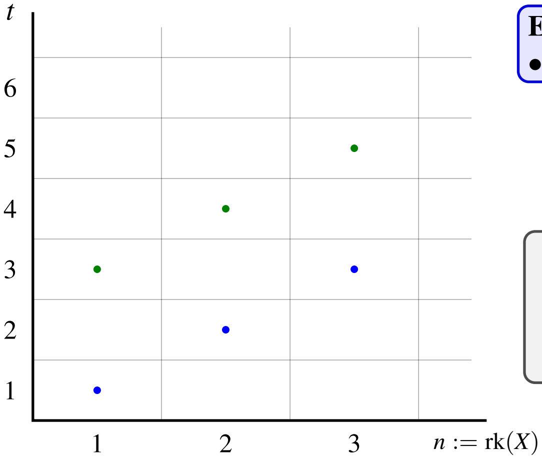

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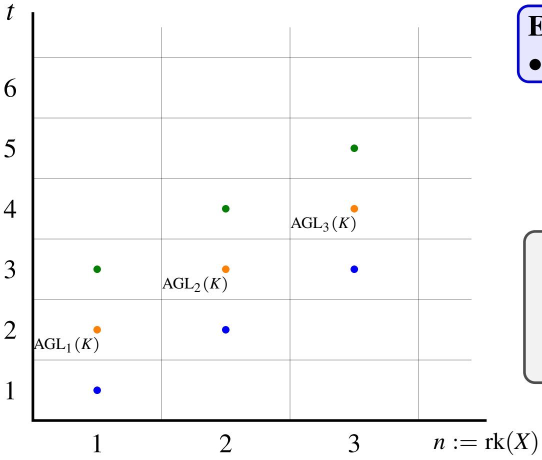

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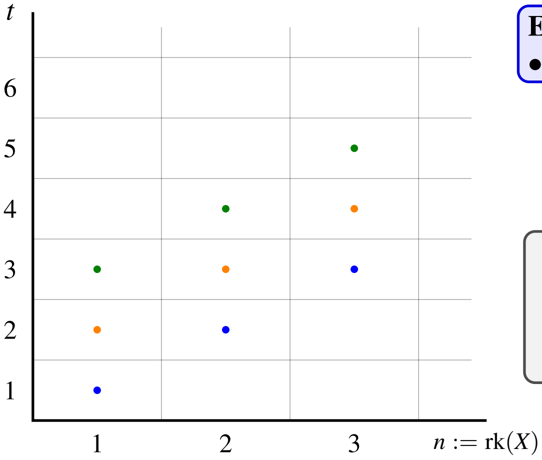

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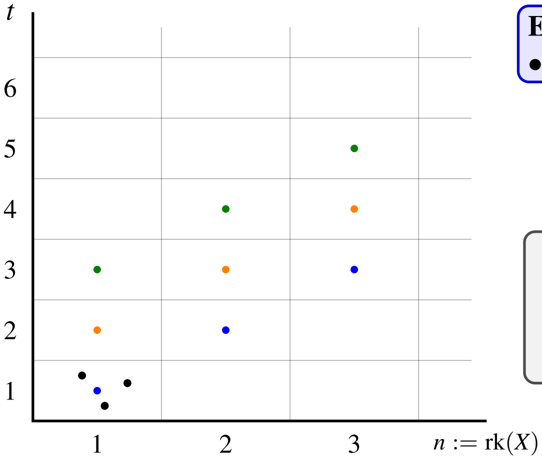

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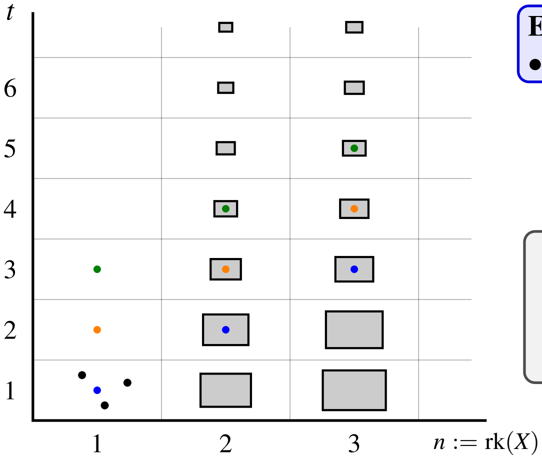

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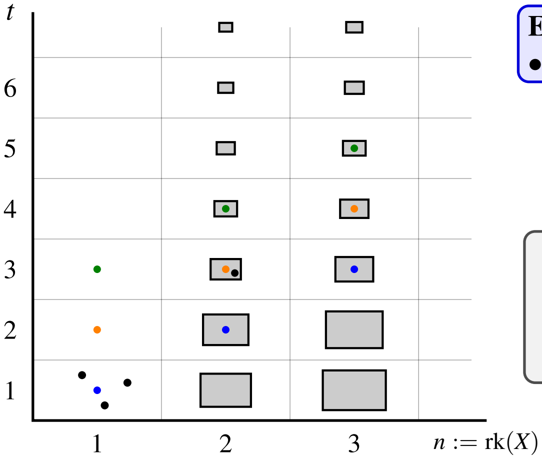

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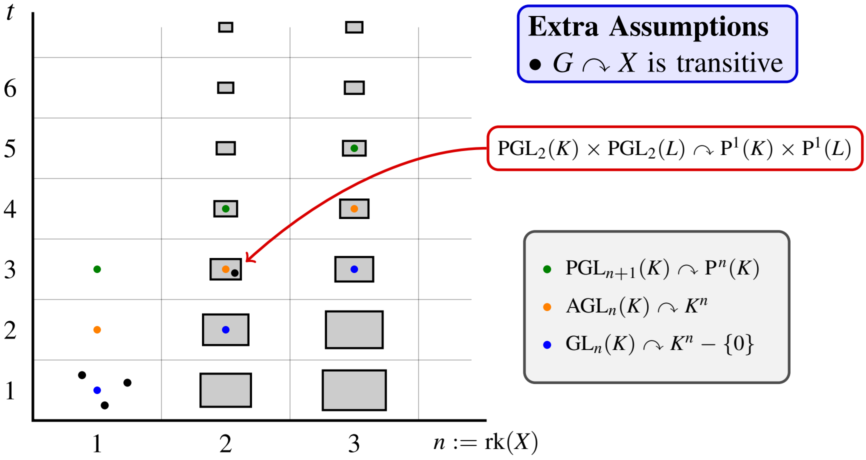

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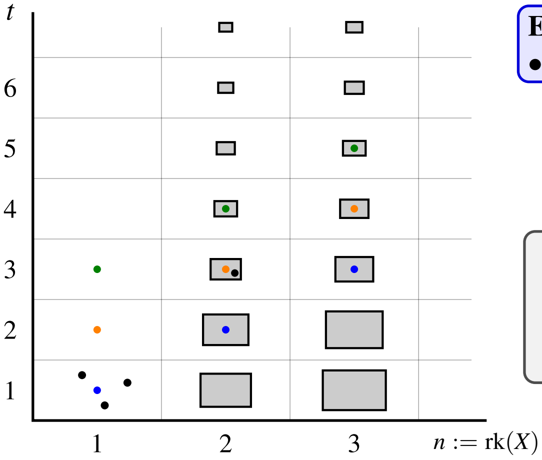

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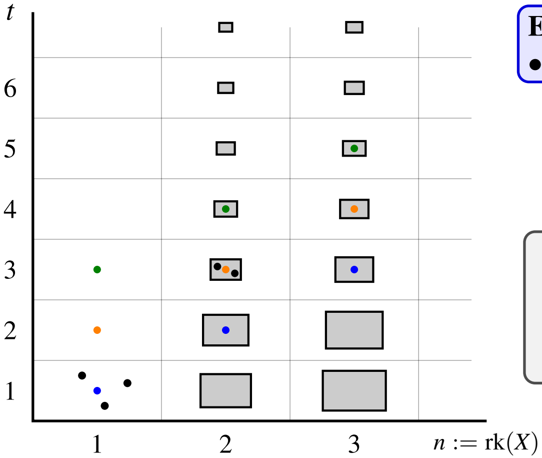

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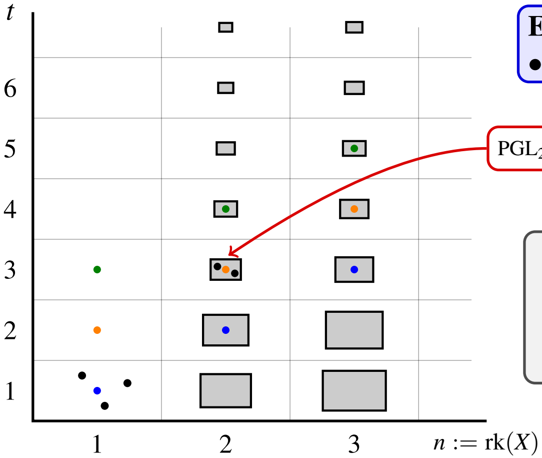

Extra Assumptions

- $G \curvearrowright X$ is transitive

$\mathrm{PGL}_2(L) \curvearrowright \mathrm{P}^1(L)$ with $\mathrm{rk}(L) = 2$!

- $\mathrm{PGL}_{n+1}(K) \curvearrowright \mathrm{P}^n(K)$
- $\mathrm{AGL}_n(K) \curvearrowright K^n$
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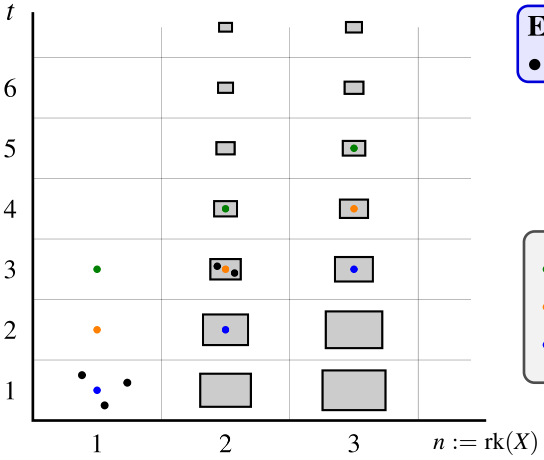
$G \curvearrowright X$ is generically t-transitive



Extra Assumptions

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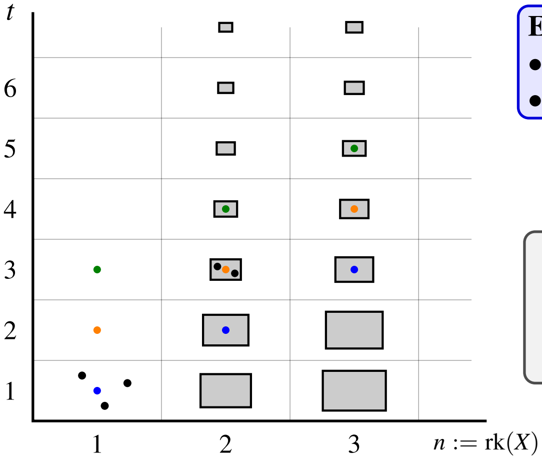
The Problem (BC '08)

$G \curvearrowright X$ is generically t-transitive



Extra Assumptions

- $G \curvearrowright X$ is transitive
- G is connected

- $\mathrm{PGL}_{n+1}(K) \curvearrowright \mathrm{P}^n(K)$
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The Problem (BC ’08)

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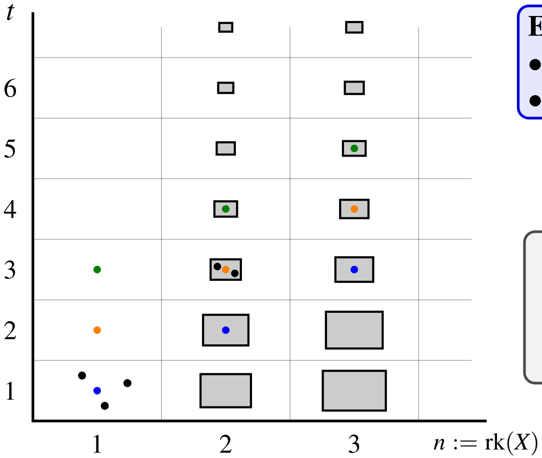

Extra Assumptions

- $G \curvearrowright X$ is transitive
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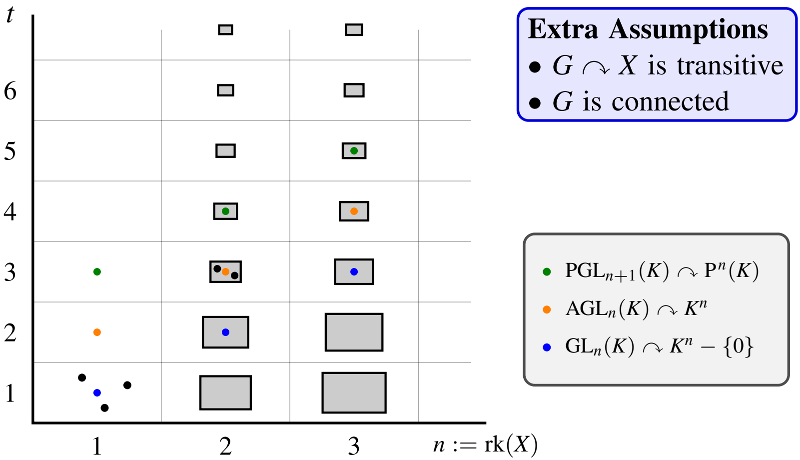
- (green) $\mathrm{PGL}_{n+1}(K) \curvearrowright \mathrm{P}^n(K)$
- (orange) $\mathrm{AGL}_n(K) \curvearrowright K^n$
- (blue) $\mathrm{GL}_n(K) \curvearrowright K^n - \{0\}$

The Problem (BC '08)

Show that $t \geq n+2 \implies$

$G \curvearrowright X$ is generically t-transitive



Extra Assumptions
- $G \curvearrowright X$ is transitive
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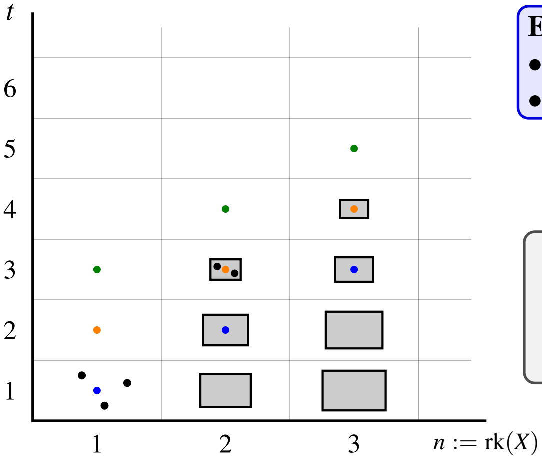
The Problem (BC '08)

Show that $t \geq n+2 \implies G \curvearrowright X \cong \mathrm{PGL}_{n+1}(K) \curvearrowright \mathrm{P}^n(K)$

$G \curvearrowright X$ is generically t-transitive



Extra Assumptions
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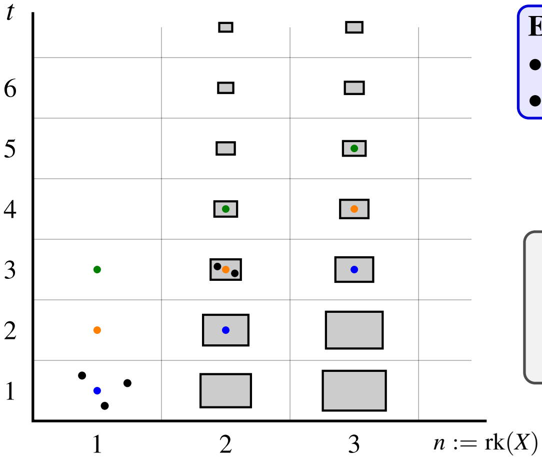

Extra Assumptions

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The Problem (BC ’08)

Let $G = G^\circ$. Suppose $G \curvearrowright X$ is transitive and generically $(n+2)$-transitive with $\mathrm{rk}(X) = n$. Show that $G \curvearrowright X \cong \mathrm{PGL}_{n+1}(K) \curvearrowright \mathrm{P}^n(K)$.

The Rank Two Problem

Rank Two Problem

Let $G = G^\circ$*. Suppose* $G \curvearrowright X$ *is transitive and generically* 4*-transitive with* $\mathrm{rk}(X) = 2$*. Show* $G \curvearrowright X \cong \mathrm{PGL}_3(K) \curvearrowright \mathrm{P}^2(K)$.

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Let $G = G^\circ$. *Suppose* $G \curvearrowright X$ *is transitive and generically* 4-*transitive with* $\mathrm{rk}(X) = 2$. *Show* $G \curvearrowright X \cong \mathrm{PGL}_3(K) \curvearrowright \mathrm{P}^2(K)$.

Rank Two Problem (Sharp version)

Let $G = G^\circ$. *Suppose* $G \curvearrowright X$ *is transitive and generically sharply* t-*transitive with* $\mathrm{rk}(X) = 2$. *Show that* $t \geq 4$ *implies* $G \curvearrowright X \cong \mathrm{PGL}_3(K) \curvearrowright \mathrm{P}^2(K)$.

The Rank Two Problem

Rank Two Problem

Let $G = G^\circ$. Suppose $G \curvearrowright X$ is transitive and generically 4-transitive with $\mathrm{rk}(X) = 2$. Show $G \curvearrowright X \cong \mathrm{PGL}_3(K) \curvearrowright \mathrm{P}^2(K)$.

Rank Two Problem (Sharp version)

Let $G = G^\circ$. Suppose $G \curvearrowright X$ is transitive and generically sharply t-transitive with $\mathrm{rk}(X) = 2$. Show that $t \geq 4$ implies $G \curvearrowright X \cong \mathrm{PGL}_3(K) \curvearrowright \mathrm{P}^2(K)$.

Really, we have two things to show.

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Let $G = G^\circ$. Suppose $G \curvearrowright X$ is transitive and generically 4-transitive with $\mathrm{rk}(X) = 2$. Show $G \curvearrowright X \cong \mathrm{PGL}_3(K) \curvearrowright \mathrm{P}^2(K)$.

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Let $G = G^\circ$. Suppose $G \curvearrowright X$ is transitive and generically sharply t-transitive with $\mathrm{rk}(X) = 2$. Show that $t \geq 4$ implies $G \curvearrowright X \cong \mathrm{PGL}_3(K) \curvearrowright \mathrm{P}^2(K)$.

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1. $t \geq 4 \implies t = 4$

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Really, we have two things to show.

1. $t \geq 4 \implies t = 4$
2. $t = 4 \implies \mathrm{PGL}_3(K) \curvearrowright \mathrm{P}^2(K)$

The Rank Two Problem

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Let $G = G^\circ$. Suppose $G \curvearrowright X$ is transitive and generically 4-transitive with $\mathrm{rk}(X) = 2$. Show $G \curvearrowright X \cong \mathrm{PGL}_3(K) \curvearrowright \mathrm{P}^2(K)$.

Rank Two Problem (Sharp version)

Let $G = G^\circ$. Suppose $G \curvearrowright X$ is transitive and generically sharply t-transitive with $\mathrm{rk}(X) = 2$. Show that $t \geq 4$ implies $G \curvearrowright X \cong \mathrm{PGL}_3(K) \curvearrowright \mathrm{P}^2(K)$.

Really, we have two things to show.

√ 1. $t \geq 4 \implies t = 4$
2. $t = 4 \implies \mathrm{PGL}_3(K) \curvearrowright \mathrm{P}^2(K)$

The Rank Two Problem: building $(\mathcal{P}, \mathcal{L})$

Rank Two Problem (Sharp version)

Let $G = G^\circ$. *Suppose* $G \curvearrowright X$ *is transitive and generically sharply* 4*-transitive with* $\mathrm{rk}(X) = 2$. *Show that* $G \curvearrowright X \cong \mathrm{PGL}_3(K) \curvearrowright \mathrm{P}^2(K)$.

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Want to build a projective plane.

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Want to build a projective plane. Set $\mathcal{P} := X$.

The Rank Two Problem: building $(\mathcal{P}, \mathcal{L})$

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G_x

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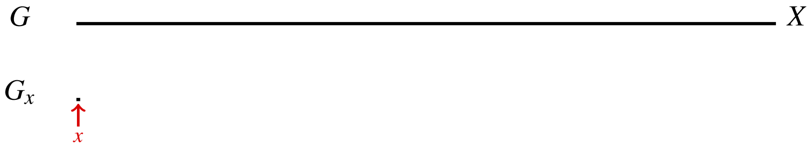

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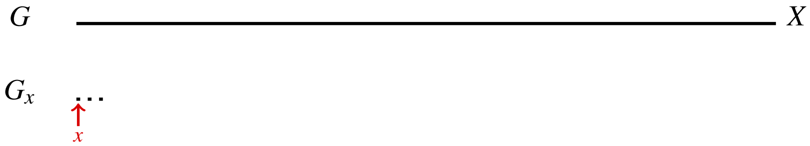

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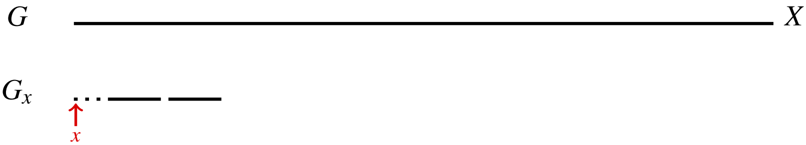

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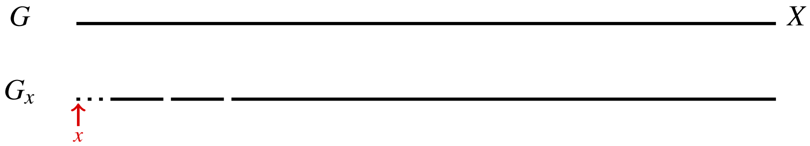

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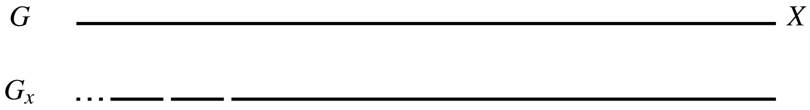

Rank Two Problem (Sharp version)

Let $G = G^\circ$. Suppose $G \curvearrowright X$ is transitive and generically sharply 4-transitive with $\operatorname{rk}(X) = 2$. Show that $G \curvearrowright X \cong \mathrm{PGL}_3(K) \curvearrowright \mathrm{P}^2(K)$.

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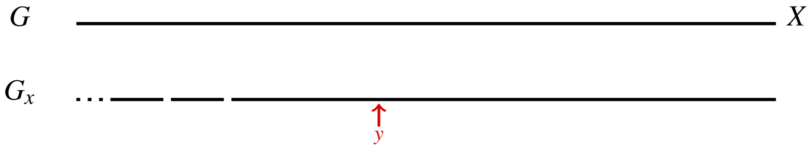

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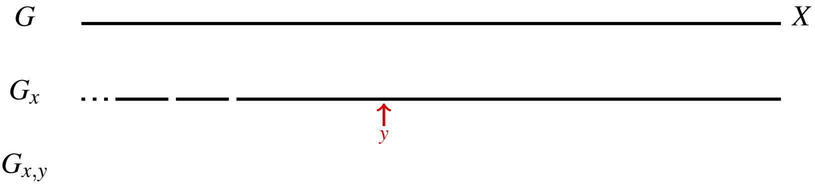

The Rank Two Problem: building $(\mathcal{P}, \mathcal{L})$

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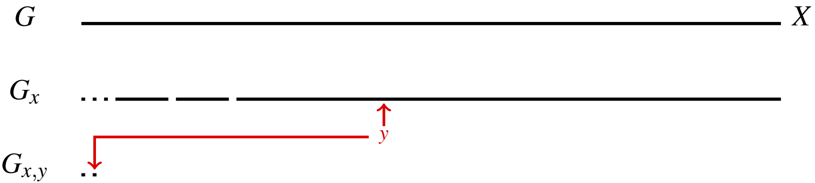

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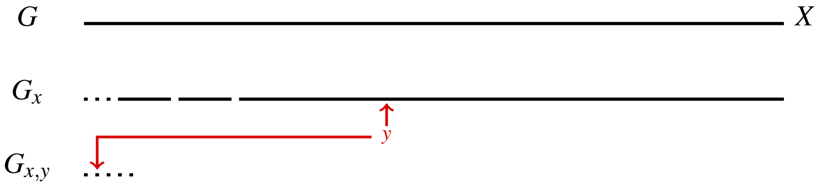

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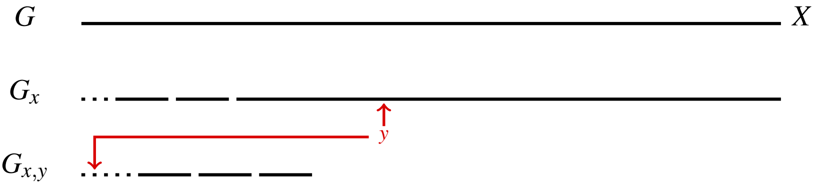

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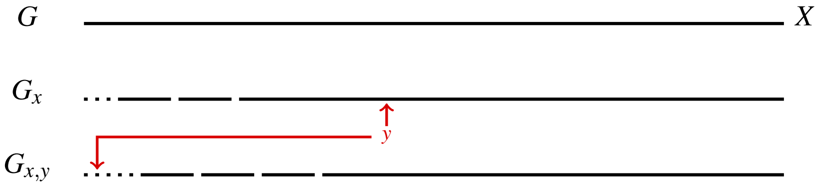

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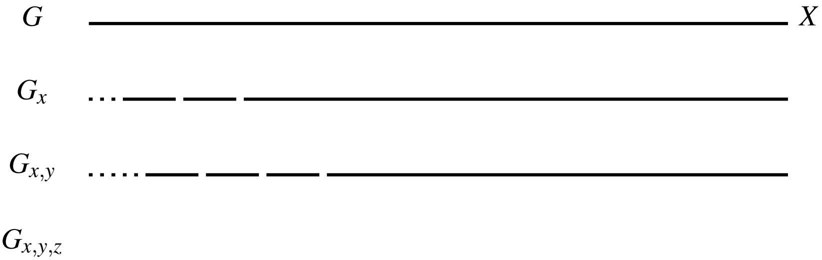

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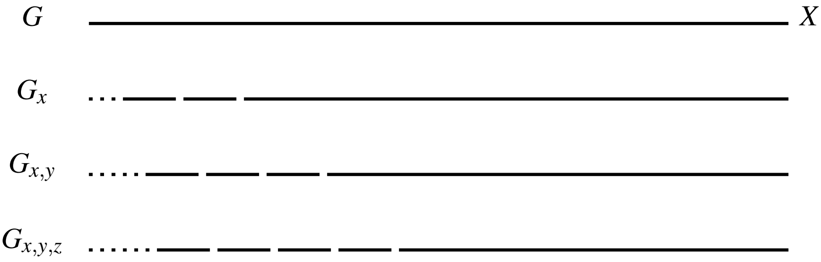

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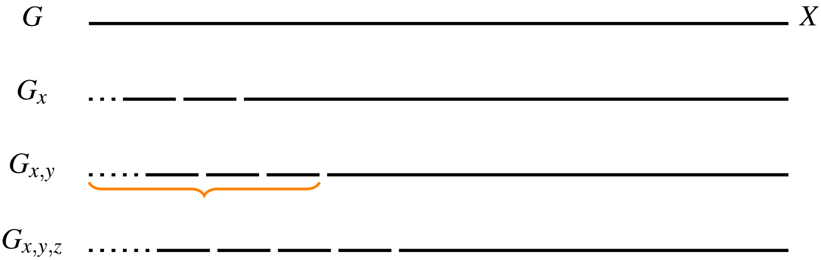

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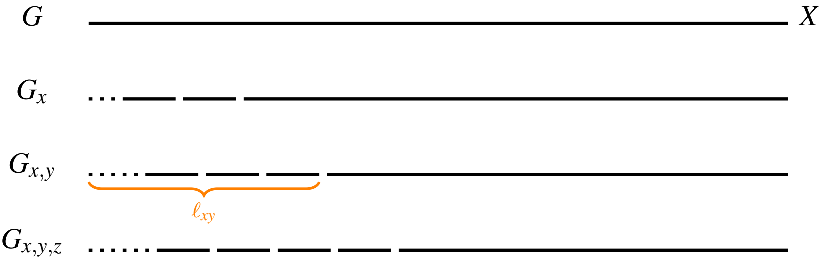

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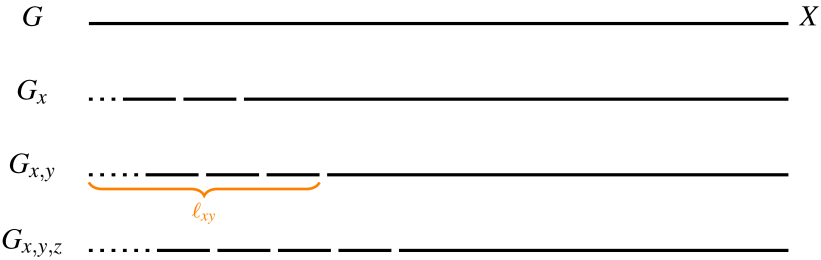
- $\ell_{xy} := \{a : \mathrm{rk}(G_{x,y}a) < 2\}$



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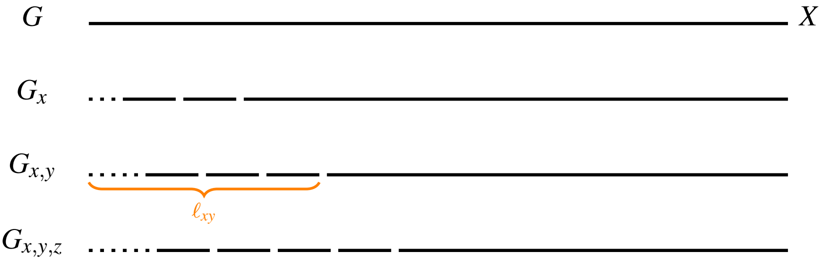

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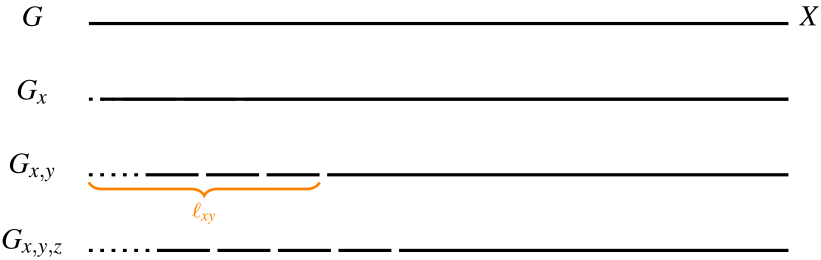

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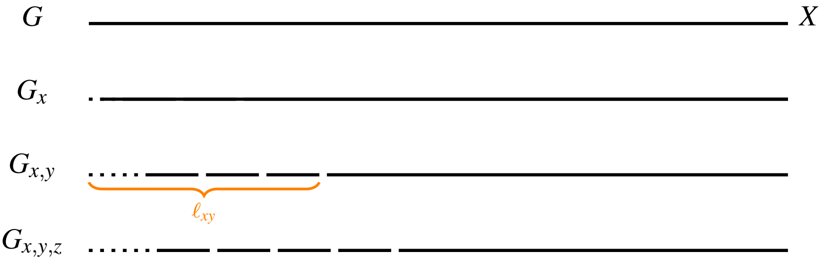

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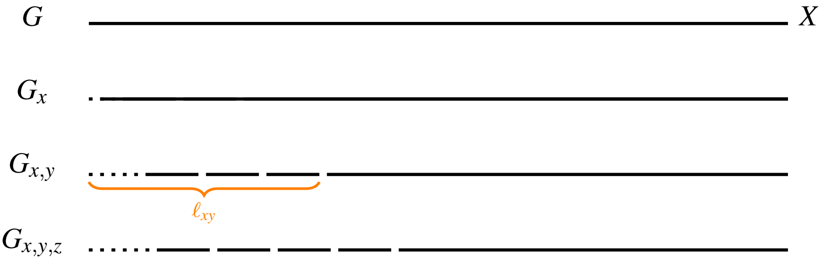

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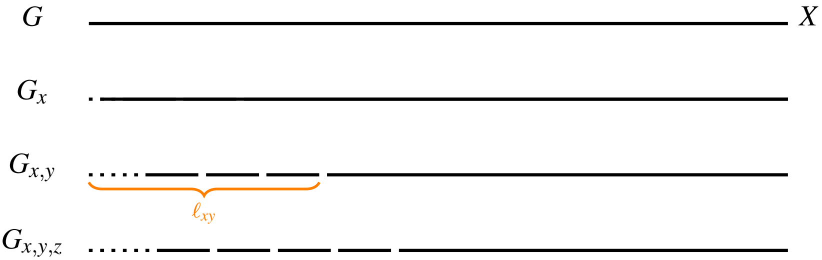

The Rank Two Problem: properties of $(\mathcal{P}, \mathcal{L})$

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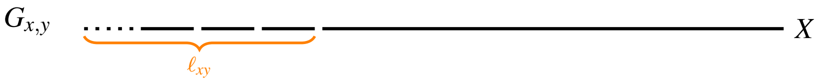

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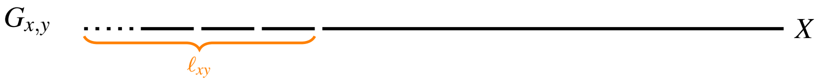

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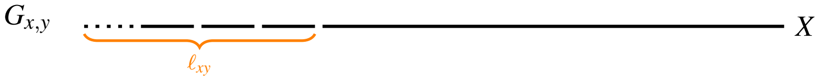
- Every 2 points lie on a line
- There are 4 points no 3 of which are collinear



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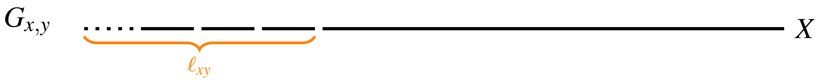

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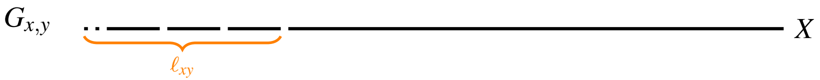

The Rank Two Problem: properties of $(\mathcal{P}, \mathcal{L})$

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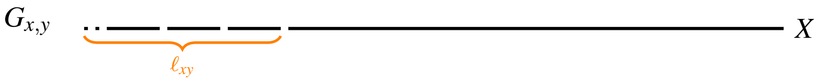

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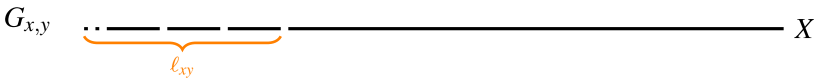

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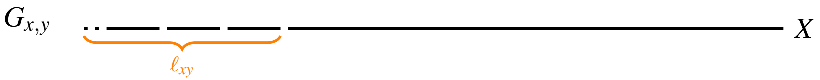

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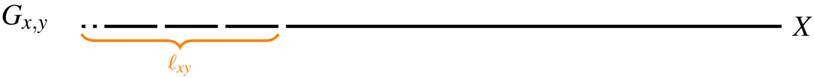

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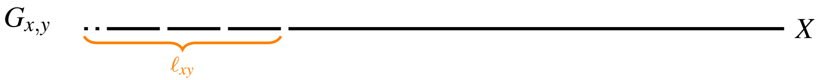
Also,

- G is generically transitive on 4-gons



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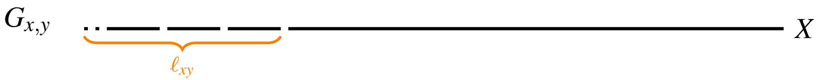
Problem is solved*



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Rank Two Problem (Sharp version)

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Also assume: ~~*2-transitivity;* $\approx$ *NOT 3-transitivity;*~~ $\mathrm{Fix}(G_{x,y,z}) = \{x, y, z\}$

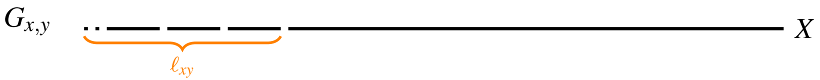
The geometry: $\mathcal{P} := X$ and $\mathcal{L} := \{\ell_{xy} : x \neq y\}$

- Every 2 points lie on a unique line
- Every 2 lines intersect in at most one point and generically lines intersect
- There are 4 points no 3 of which are collinear

Also,

- G is generically transitive on 4-gons

Problem is solved*



Summary

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The Problem (BC '08)

Let $G = G^\circ$. *Suppose* $G \curvearrowright X$ *is transitive and generically* $(n+2)$*-transitive with* $\mathrm{rk}(X) = n$. *Show that* $G \curvearrowright X \cong \mathrm{PGL}_{n+1}(K) \curvearrowright \mathrm{P}^n(K)$.

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1. Remove the fixed-point criterion
2. Try to recognize higher dimensional projective spaces in a similar way, with perhaps an analogous fixed-point criterion.
3. Deal with the non-sharp case.

Thank You