APPLICATIONS OF THE PRODUCT THEOREM

L. PYBER

RÉNYI INSTITUTE, BUDAPEST

PRODUCT THEOREM

BREUILLARD-GREEN-TAO + PY-SZABÓ

L A FINITE SIMPLE GROUP OF LIE TYPE OF RANK r AND A A GENERATING SUBSET ($\langle A \rangle = L$)

$\Rightarrow A^3 = L$ OR $|A^3| > |A|^{1+\varepsilon}$

FOR SOME $\varepsilon = \varepsilon(r)$

PROVED EARLIER BY $\boxed{\text{HELFGOTT}}$ FOR $PSL(2,p)$, $PSL(3,p)$

DINAI + VARJÚ FOR $PSL(2,q)$

<u>QUESTION</u> IS THIS TRUE WITH $\varepsilon \sim \frac{1}{r}$?

- THERE ARE MANY EXAMPLES WITH $\varepsilon(r) \geq \frac{5}{r}$
- PY-SZABÓ PROOF EFFECTIVE

"MAIN" APPLICATION

TIP OF THE ICEBERG

BOURGAIN-GAMBURD-SARNAK

$\Lambda \leq SL(2,\mathbb{Z})$, Λ NOT VIRTUALLY CYCLIC

$b \in \mathbb{Z}^2$, b NOT A MULTIPLE OF A VECTOR

THEN $\exists\, r$ INFINITE

$\Lambda b \cap P_r^2$ IS ZARISKI DENSE

IN THE PLANE (P_r = PRODUCTS OF $\leq r$ PRIMES), r EFFECTIVE

CONJECTURE TRUE WITH $\boxed{\boxed{r=1}}$

IF THERE ARE NO "OBVIOUS" OBSTRUCTIONS

ICEBERG SALEHY-GOLSEFIDY, SARNAK

AFFINE SIEVE

EXTENSION TO $\Lambda = SL(n,\mathbb{Z})$ AND MUCH MORE

WEISS CONJECTURE (1978)

Γ GRAPH, $G \leq AUT(\Gamma)$ TRANSITIVE AND LOCALLY PRIMITIVE

Γ d-REGULAR $\Rightarrow |G_\alpha| \leq f(d)$

PRAEGER-PY-SPIGA-SZABÓ

TRUE IF THE COMPOSITION FACTORS HAVE BOUNDED RANK

MOST OF THE „REDUCTION TO THE SIMPLE CASE" DUE TO PRAEGER-SPIGA-VERRET

"SIMPLE" IDEA (G SIMPLE, RANK r)

IF g TAKES α TO A NEIGHBOUR

THEN $|G_\alpha \cdot g \cdot G_\alpha| = |G_\alpha| d$

HENCE $|\{G_\alpha, g\}^3| \leq 100 d\, |G_\alpha|$

BUT $|\{G_\alpha, g\}^3| \geq |G_\alpha|^{1+\varepsilon(r)}$

$\Rightarrow |G_\alpha| \leq (100 d)^{1/\varepsilon(r)}$ $\square$

QUESTION PRAEGER - PY - SPIGA - SZABO

Γ CONNECTED, d-REGULAR G-VERTEX TRANSITIVE GRAPH

IS EXPONENT OF $G_\alpha \leq f(d)$?

IF YES + IN THE WEISS CASE THE NUMBER OF GENERATORS OF G_α WOULD BE $\leq f(d)$

THEN WEISS WOULD FOLLOW FROM ZELMANOV'S THEOREM

(OVERKILL)

EASY <u>COROLLARY</u> (OF PRODUCT THEOREM) (5)

L SIMPLE OF RANK r, $\langle A \rangle = L$

$\Rightarrow$ $A^d = L$ FOR $d \leq \left(\frac{\log |L|}{\log |A|}\right)^{C(r)}$

<u>DEF</u> $G \leq \mathrm{Sym}(\Omega)$ TRANSITIVE

$\mathrm{diam}(X, G)$ = LARGEST DIAMETER OF ORBITAL GRAPHS

HIGMAN: G FINITE THEN

$\mathrm{diam}(X, G) < \infty \iff G$ PRIMITIVE

<u>LIEBECK-MACPHERSON-TENT</u>

"DESCRIBE" INFINITE FAMILIES OF FINITE PRIMITIVE PERMUTATION GROUPS FOR WHICH

$\mathrm{diam}(X, G)$ IS BOUNDED

MODEL THEORY

KEY RESULT

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$\mathcal{C}$ A CLASS OF PRIMITIVE ALMOST SIMPLE GROUPS OF BOUNDED „RANK"

(i) $|G_\alpha|$ $(G \in \mathcal{C})$ UNBOUNDED

(ii) IF $soc(G) = G(q)$ AND G_α IS A SUBFIELD SUBGROUP $G(q_0)$ $\Rightarrow$ $|F_q : F_{q_0}|$ BOUNDED

THEN $diam(X, G)$ IS BOUNDED

FOLLOWS FROM „EASY COROLLARY"

+ LEMMA (LIEBECK, NIKOLOV, SHALEV)

L LIE TYPE OVER $\mathbb{F}_q$, rank r

IF H IS MAXIMAL IN L

THEN (i) $|H| \le f(r)$

OR (ii) H IS A SUBFIELD SUBGROUP

OR (iii) $|H| \geq q-1$

HENCE IF X, G IS AS IN "KEY RESULT" $\Rightarrow$

$$\frac{\log|G|}{\log|G_\alpha|} \lesssim r^2, \quad A = \{G_\alpha, g, g^{-1}\}$$

$\Rightarrow$ $A^d = L$ FOR $d = (r^2)^{c(r)}$

HENCE $\operatorname{diam}(X, G) \leq (r^2)^{c(r)}$ $\square$

MORE GENERALLY

$$\frac{\log|G|}{\log|M|} \leq \operatorname{diam}(X, G) \leq \left(\frac{\log|G|}{\log|M|}\right)^{c(r)}$$

WHERE $M = G_\alpha$ IS ANY MAXIMAL SUBGROUP

<u>QUESTION</u> (a' la Babai)

$$\operatorname{diam}(X, G) \leq \left(\frac{\log|G|}{\log|M|}\right)^{C}$$

ABSOLUTE ?

ANSWER NO (NOT QUITE)

EXAMPLE (IMPLICIT INBANNAI + KANTOR)

$G \leq \mathrm{Sym}(X)$, G HAS A CONJUGACY CLASS $|\mathcal{K}| \leq t$

$\Rightarrow G_\alpha$ HAS AN ORBIT $\leq t$

HENCE IF G_α IS MAXIMAL IN $G = SL(n, q)$, $|G_\alpha| \sim \sqrt{|G|}$

THEN $\mathrm{diam}(X, G) \gtrsim n$

(AND IS NOT ABSOLUTELY BOUNDED) (THIS FOLLOWS FROM LIEBECK-MACPHERSON-TENT)

QUESTION (P4)

IS $\mathrm{diam}(X, G) \leq \dfrac{\log|G|}{\log|G_\alpha|} \cdot C(r)$

FOR G PRIMITIVE ?

BABAI'S CONJECTURE

G SIMPLE $\Rightarrow$ THE DIAMETER OF ANY CONNECTED CAYLEY GRAPH OF G IS $\leq (\log|G|)^c$

FOLLOWS FROM "EASY COROLLARY" IF G WAS BOUNDED LIE RANK E.G. EXCEPTIONAL (BGT + P_y S_z)

HELFGOTT-SERESS (ANNALS to appear)

THE DIAMETER OF ANY CAYLEY GRAPH OF Alt(n) IS $\leq n^{c(\log n)^4}$ ("ALMOST" $\log|G|^c$)

$\boxed{\text{ONE}}$ NEW INGREDIENT: STRATEGY FROM THE ELEMENTARY PROOF OF

Py (1990) (BASED ON BABAI (1980))

G 2-TRANSITIVE $G \leq S_n$, $G \not\geq A_n$ $\Rightarrow$ $|G| \leq n^{c(\log n)^2}$

WHAT ABOUT $SL(n, \mathbb{Z})$?
ANY ELEMENTARY RESULTS?

CONJECTURE SERESS (2006)

L SIMPLE OF LIE TYPE OVER F_q, LIE RANK $= r$
$\Rightarrow$ THE DIAMETER OF CAYLEY GRAPHS IS $\leq (q^r)^c$

THIS WOULD HAVE APPLICATION TO THE CONJECTURE

$G \leq S_n$ TRANSITIVE $\Rightarrow$
DIAM OF CAYLEY GRAPHS $\leq n^c$

ALSO PROVED WITH
DIAM $\leq n^{(\log n)^5}$ BY HARALD HELFGOTT - SERESS ÁKOS