Deformation theory and finite simple quotients of triangle groups

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A natural question

Fact: Every finite simple group can be generated by two elements.

G_0: a finite (simple) group, $(a, b, c) \in \mathbb{N}^3$.

Question: Can we find $X, Y \in G_0$ s.t. $\langle X, Y \rangle = G_0$ and $X^a = Y^b = (XY)^c = 1$?
In other words, is G_0 an (a, b, c)-group?

i.e. Is there a surjective homomorphism from $T = T_{a,b,c}$ to G_0, where

$$T = \langle x, y, z : x^a = y^b = z^c = xyz = 1 \rangle?$$

If $\mu := 1/a + 1/b + 1/c \geq 1$ then T is either soluble or $T \leq \mathrm{Sym}_5$.

WLOG $\mu < 1$ and $a \leq b \leq c$.

T: hyperbolic triangle group.

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A general problem

<u>Problem:</u> Find the finite simple quotients of a given hyperbolic triangle group T.

Everitt 2000: every hyperbolic triangle group surjects onto all but finitely many alternating groups.

Focus on finding finite simple images of T of Lie type.

<u>Setting:</u>

$T = T_{a,b,c}$: a hyperbolic triangle group.

$G_0 = G(p^r) = G(q)$: a finite quasisimple group of Lie type.

G: corresponding algebraic group over $K = \overline{K}$, $\mathrm{char}\, K = p$.

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Hurwitz generation

- Many results in the literature on $(2,3,7)$-generation of finite groups (Hurwitz generation).
- Motivation: If S is a compact Riemann surface of genus $h \geq 2$ then $|\mathrm{Aut}\ S| \leq 84(h-1)$ and bound is attained iff $\mathrm{Aut}\ S$ is Hurwitz.
- Examples:
  1. Macbeath 1969: $\mathrm{PSL}_2(p^r)$ is Hurwitz $\Leftrightarrow r = 1$ and $p \equiv 0, \pm 1 \mod 7$ or $r = 3$ and $p \equiv \pm 2, \pm 3 \mod 7$.
  2. Lucchini-Tamburini-Wilson 2000: Many classical groups of large rank are Hurwitz.
     e.g. if $n > 267$ then $\mathrm{SL}_n(q)$ is Hurwitz for every q.
  3. Di Martino-Tamburini-Zalesski 2000: If G is classical of low rank, given p there are very few $G(p^r)$ (possibly 0) which are Hurwitz.
- If $(a,b,c) \neq (2,3,7)$: less results in the literature.

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- If $(a,b,c) \neq (2,3,7)$: less results in the literature.

Some groups of low rank

- Fix (a, b, c) with a, b, c primes
- $\underline{G_0 = \mathrm{PSL}_2(p^r)}$

Theorem (M 2009)

Given p, $\exists!$ *r such that* $\mathrm{PSL}_2(p^r)$ *is an* (a, b, c)*-group.*

- $\underline{G_0 = \mathrm{PSL}_3(p^r)}$
  - There is a dichotomy between the case $a = 2$ and $a \neq 2$.

Theorem (M 2013)

1. *Given p, there are at most four r such that* $\mathrm{PSL}_3(p^r)$ *is a* $(2, b, c)$*-group.*
2. *If* $a \neq 2$ *then a randomly chosen homomorphism in* $\mathrm{Hom}(T, G_0)$ *is surjective with probability tending to 1 as* $|G_0| \to \infty$
   i.e. $\exists \infty$ *many r such that* $\mathrm{PSL}_3(p^r)$ *is an* (a, b, c)*-group.*

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Rigidity

- <u>Notation:</u>
  G: a simple algebraic group defined over $K = \overline{K}$, $\mathrm{char}\, K = p$.

  j_a: dimension of the subvariety of G of elements of order dividing a
  i.e. j_a: max. dim. of a conj. class of G of elements of order dividing a.
- e.g. if $G = \mathrm{SL}_2(K)$ where $p \neq 2$, then $j_2 = 0$ and $j_a = 2$ for $a \neq 2$.

Proposition (M 2010)

If $j_a + j_b + j_c < 2 \dim G$ then $G_0 = G(p^r)$ is not an (a,b,c)-group.

- e.g. Let $G_0 = \mathrm{Sp}_6(p^r)$ with p odd, and $(a,b,c) = (2,5,5)$. Then $j_2 = 8$, $j_5 = 16$ but $\dim G = 21$. Hence G_0 is not a $(2,5,5)$-group.

A conjecture

Definition

1. If $j_a + j_b + j_c < 2\dim G$ then (a, b, c) is reducible for G.
2. If $j_a + j_b + j_c = 2\dim G$ then (a, b, c) is rigid for G.
3. If $j_a + j_b + j_c > 2\dim G$ then (a, b, c) is nonrigid for G.

e.g. (a, b, c) is always rigid for $\mathrm{PSL}_2(K)$.
e.g. for $\mathrm{PSL}_3(K)$, (a, b, c) is rigid $\Leftrightarrow a = 2$. It is nonrigid otherwise.

Conjecture (M 2010)

Fix p. If (a, b, c) is rigid for G, a, b, c primes, then $\exists$ only finitely r such that $G(p^r)$ is an (a, b, c)-group.

The conjecture holds for $\mathrm{PSL}_2(p^r)$ (exactly one r) and $\mathrm{PSL}_3(p^r)$ (at most four r).
It agrees with the known results in the literature on Hurwitz groups.

The conjecture holds in many cases

- (M 2010) The conjecture holds in many cases.
- Proof by a case by case study:
  - First classify rigid triples of primes for simple algebraic groups.
  - Use the concept of linear rigidity.

Remark

The converse of the conjecture is false. e.g. $(2, 3, 7)$ is nonrigid for $\mathrm{SL}_7(K)$ but $\mathrm{SL}_7(q)$ is never a Hurwitz group.

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Reducible triples when a, b, c primes

G	p	(a, b, c)
$\mathrm{SL}_2(K)$	$p \neq 2$	$a = 2$
$\mathrm{Sp}_4(K)$	$p \neq 2$	$a = 2$, $b = 3$
$\mathrm{Sp}_6(K)$	$p \neq 2$	$a = 2$, $b = 3$
		or $a = 2$, $b = c = 5$
$\mathrm{Sp}_{2m}(K)$, $m \in \{4, 5, 6, 7, 8, 9, 11\}$	$p \neq 2$	$a = 2$, $b = 3$, $c = 7$

Lemma

Table gives quasisimple groups of Lie type that are not (a, b, c)-groups.

Rigid triples when a, b, c primes

$s.c$	p	(a,b,c)
$\mathrm{SL}_2(K)$	2	any
	$\neq 2$	$a > 2$
$\mathrm{SL}_3(K)$	any	$a = 2$
$\mathrm{SL}_4(K)$	any	$a = 2$, $b = 3$
$\mathrm{SL}_5(K)$	any	$a = 2$, $b = 3$
$\mathrm{SL}_6(K)$	$\neq 2$	$a = 2$, $b = 3$
$\mathrm{SL}_{10}(K)$	$\neq 2$	$(2,3,7)$
$\mathrm{Sp}_4(K)$	2	$b = 3$
	$\neq 2$	$a = b = 3$
		or $a = 2$, $b > 3$
$\mathrm{Sp}_6(K)$	$\neq 2$	$a = 2$, $b = 5$ $c \geq 7$
$\mathrm{Sp}_8(K)$	$\neq 2$	$a = 2$, $b = 3$, $c > 7$
		or $(2,5,5)$
$\mathrm{Sp}_{10}(K)$	$\neq 2$	$a = 2$, $b = 3$, $c > 7$

$s.c$	p	(a,b,c)
$\mathrm{Sp}_{2m}(K)$	$\neq 2$	$(2,3,7)$
$m = 10, 12, 13$		
$\mathrm{Spin}_{11}(K)$	$\neq 2$	$(2,3,7)$
$\mathrm{Spin}_{12}(K)$	$\neq 2$	$(2,3,7)$

$ad.$	p	(a,b,c)
$\mathrm{PSL}_2(K)$	any	any
$\mathrm{PSL}_3(K)$	any	$a = 2$
$\mathrm{PSL}_4(K)$	any	$a = 2$, $b = 3$
$\mathrm{PSL}_5(K)$	any	$a = 2$, $b = 3$
$\mathrm{PSp}_4(K)$	any	$b = 3$
$G_2(K)$	any	$(2,5,5)$

Another approach through deformation theory

- Let $\rho \in \mathrm{Hom}(T, G)$ and $s = \mathrm{Ad} \circ \rho$ so that T acts on $V = L(G)$ through s.
- $\phi : T \to V$ is a 1-cocycle if $\phi(t_1 t_2) = \phi(t_1) + s(t_1)\phi(t_2)\ \forall\ t_1, t_2 \in T$.
- $\phi : T \to V$ is a 1-coboundary if $\exists v \in V$ s.t. $\forall\ t \in T$: $\phi(t) = v - s(t)v$.
- Set $Z^1(T, s)$: space of cocycles, $B^1(T, s)$: space of coboundaries and

$$H^1(T, s) = Z^1(T, s)/B^1(T, s).$$

Theorem (Weil 1964)

- *$Z^1(T, s)$ is the tangent space to $\mathrm{Hom}(T, G)$ at ρ.*
- *If $p \nmid abc$ then*

$$\begin{aligned} \dim H^1(T, s) &= \dim V + i + i^* - \dim V^x - \dim V^y - \dim V^z \\ &= -2 \dim V + i + i^* + \operatorname{codim} V^x + \operatorname{codim} V^y + \operatorname{codim} V^z \\ &\leq -2 \dim G + i + i^* + j_a + j_b + j_c. \end{aligned}$$

where i, i^* are the dimensions of the space of invariants of s and s^*.

- *If $\dim H^1(T, s) = 0$ then ρ is locally rigid. i.e. $\exists$ a neighborhood of ρ in which every element of it is obtained from ρ by conjugation by an element of G.*

Proof of the conjecture

Here we no longer assume a, b, c primes.
Let d be the determinant of the Cartan matrix of $L(G)$.

Theorem (Larsen - Lubotzky - M 2013)

If $p \nmid abcd$ and (a, b, c) is rigid for G, then there are only finitely many r such that $G(p^r)$ is an (a, b, c)-group.

Proof (by contradiction).

- Let $\rho : T \to G(p^r)$ be an epimorphism and consider ρ as an element of $\mathrm{Hom}(T, G)$.
- Since $p \nmid abc$ and (a, b, c) is rigid for G, get $\dim H^1(T, \mathrm{Ad} \circ \rho) \leq i + i^*$. Because $p \nmid d$, a result of Hiss implies $i = i^* = 0$ for $r >> 0$.
- So WLOG ρ is locally rigid.
- The orbit of ρ under the action of G by conjugation is open
- As in a variety one can have only finitely many open orbits, the result follows.

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Positive results

- X: simple Dynkin diagram
- $\underline{G} = X(\mathbb{C})$: simple Chevalley group of type X over $\mathbb{C}$
- $X(p^r)$: finite untwisted simple group of Lie type X.

Definition

T is saturated with finite quotients of type X if: $\exists\ p_0, e \in \mathbb{N}$ s. t. $\forall$ prime $p > p_0$, $X(p^{e\ell})$ is a quotient of T for all $\ell \in \mathbb{N}$.

- Main point: T is satured with finite quotients of type X $\Leftrightarrow \exists\ \rho \in \mathrm{Hom}(T, X(\mathbb{C}))$ with Zariski dense image and $\dim H^1(T, \mathrm{Ad} \circ \rho) > 0$.
- From now on - no finite fields or finite groups!

Criterion for Zariski-dense representation

$\underline{G} = X(\mathbb{C})$

Theorem (Larsen-Lubotzky 2012)

Let $\rho_0 : T \to \underline{G}$ *and* $\underline{H}$*: Zariski closure of* $\rho_0(T)$*. Assume:*

1. $\underline{H}$ *is semisimple and connected.*
2. $\underline{H}$ *is a maximal subgroup of* $\underline{G}$*.*
3. $\dim \mathrm{Epi}(T, \underline{H}) - \dim \underline{H} < \dim H^1(T, \mathrm{Ad}\,|_{L(\underline{G})} \circ \rho_0)$.

Then

- ρ_0 *is nonsingular in* $\mathrm{Hom}(T, \underline{G})$*.*
- $\exists$ *nonsingular Zariski dense* $\rho : T \to \underline{G}$ *in the same component of* $\mathrm{Hom}(T, \underline{G})$ *containing* ρ_0*.*
- $\dim H^1(T, \mathrm{Ad}\,|_{L(\underline{G})} \circ \rho) = \dim H^1(T, \mathrm{Ad}\,|_{L(\underline{G})} \circ \rho_0)$.

The principal homomorphism

- $\underline{G} = X(\mathbb{C})$ simple of adjoint type.
- $\exists!\ p_0^{\underline{G}} : \mathrm{SL}_2(\mathbb{C}) \to \underline{G}$ sending every nontrivial unipotent to a regular unipotent, called the **principal homomorphism.**
- It factors through $\mathrm{PSL}_2(\mathbb{C})$.
- Let $\sigma_0^{\underline{G}}$: $T \hookrightarrow \mathrm{PSL}_2(\mathbb{C}) \to \underline{G}$ induced from $p_0^{\underline{G}}$.
- $\mathrm{Ad} \circ \sigma_0^{\underline{G}}$ has no invariants on $L(\underline{G})$.
- $\dim H^1(T, \mathrm{Ad} \circ \sigma_0^{\underline{G}})$ is easy to calculate and depends only on the exponents of $W(\underline{G})$ and a, b, c.
- $\dim \mathrm{Epi}(T, \underline{G}) - \dim \underline{G} \leq \dim H^1(T, \mathrm{Ad} \circ \sigma_0^{\underline{G}})$.
- If $\underline{G} = \mathrm{PSL}_2(\mathbb{C})$ then $\dim H^1(T, \mathrm{Ad} \circ \sigma_0^{\underline{G}}) = 0$.
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The principal homomorphism

- $\underline{G} = X(\mathbb{C})$ simple of adjoint type.
- $\exists!\ p_0^G : \mathrm{SL}_2(\mathbb{C}) \to \underline{G}$ sending every nontrivial unipotent to a regular unipotent, called the **principal homomorphism.**
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A one-step ladder

- Let $\underline{G}$ of type $X = A_2$, B_n $(n \geq 4)$, C_n $(n \geq 2)$, G_2, F_4, E_7, E_8.
- Dynkin: the image of $p_0^{\underline{G}} : \mathrm{PSL}_2(\mathbb{C}) \to \underline{G}$ is maximal in $\underline{G}$.
- Get Zariski dense $\rho : T \to \underline{G}$ provided $\dim H^1(T, \mathrm{Ad}\,|_{L(G)} \circ \sigma_0^{\underline{G}}) > 0$.
- Computation: T is saturated with finite quotients of type X unless $X = A_2$ and $a = 2$, $X = C_2$ and $b = 3$, $X = G_2$ and $a = 2$, $c = 5$. (i.e. unless (a, b, c) is rigid for $\underline{G}$.)

Two and three-step ladders

- For other types X need two steps, unless $X = A_6$ or D_4 in which we need three steps.
- Two steps:
  Can choose $\underline{H} = Y(\mathbb{C})$ maximal in $\underline{G} = X(\mathbb{C})$ with $p_0^H = p_0^G$ and $\underline{H}$ of type Y treated in 1-step ladder.

X	Y	X	Y	X	Y
A_r	$B_{r/2}$ (r even)	B_3	G_2	D_r	B_{r-1}
	$C_{(r+1)/2}$ (r odd)	E_6	F_4		

  So start with $\sigma_0^H : T \to \mathrm{PSL}_2(\mathbb{C}) \to \underline{H}$ and apply [LL].

  If [LL (3)] is satisfied, get Zariski dense $\rho_1 : T \to \underline{H}$.

  Continue with $\rho_1 : T \to \underline{H} \hookrightarrow \underline{G}$ and apply [LL] a second time.
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- Same philosophy for the three-step ladders:
  $G_2(\mathbb{C}) < B_3(\mathbb{C}) < A_6(\mathbb{C})$.
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A few remarks

- If [LL (3)] is not satisfied, we cannot proceed any further with the principal homomorphism.
- Sometimes, we can proceed with some other embeddings.

Main result

Theorem (Larsen - Lubotzky - M 2013)

T is saturated with finite quotients of type X, except possibly if (X, T) is in the following table.

X	(a,b,c)	r
A_n	$(2,3,7)$	$5 \leq n \leq 19$
	$(2,3,8)$	$5 \leq n \leq 13$
	$(2,3,c)$, $c \geq 9$	$5 \leq n \leq 7$
	$(2,4,5)$	$3 \leq n \leq 13$
	$(2,4,6)$	$3 \leq n \leq 9$
	$(2,4,c)$, $c \geq 7$	$3 \leq n \leq 5$
	$(2,5,5)$	$n = 6$
	$(2,b,c)$, $b \geq 5$	$n = 3$
	$(3,3,c)$, $c \geq 4$	$n \in \{3,4,6\}$
	$(2,3,c)$, $c \geq 7$	$n \in \{3,4\}$
	$(2,b,c)$, $b = 3$, $c \geq 7$; $b = 4$, $c \geq 5$; $b,c \geq 5$	$n = 2$
	any	$n = 1$
B_3	$(2,3,c)$, $c \geq 7$; $(3,3,c)$, $c \geq 4$ $(2,4,5)$, (2,5,5)	
C_2	$(2,3,c)$, $c \geq 7$; $(3,3,c)$, $c \geq 4$	
D_n	$(2,3,7)$	$n \in \{4,5,9\}$
	$(2,3,c)$, $c \geq 8$	$n \in \{4,5\}$
	$(2,4,5)$	$n = 5$
	$(3,3,4)$	$n \in \{4,5\}$
	$(3,3,c)$, $c \geq 5$	$n = 4$
G_2	$(2,4,5)$, $(2,5,5)$	
E_6	$(2,3,c)$, $c \in \{7,8\}$; $(2,4,c)$, $4 < c < 9$	

Corollary

(i) If $1/a + 1/b + 1/c \leq 1/2$ *then* T *is saturated with finite quotients of type* X *for every* $X \neq A_1$.
(ii) Every T *is saturated with finite quotients of type* X*, except possibly if*

$$X \in Y = \{A_n : 1 \leq n \leq 19\} \cup \{B_3\} \cup \{C_2\} \cup \{D_n : n = 4, 5, 9\} \cup \{G_2\} \cup \{E_6\}.$$

Corollary

If $X \notin \{A_n : 1 \leq r \leq 7\} \cup \{B_3\} \cup \{C_2\} \cup \{D_n : n = 4, 5\}$*. Then almost every* T *is saturated with finite quotients of type* X.

Corollary

$T_{2,3,7}$ *is saturated with finite quotients of type* X*,*
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Thank you.