The full automorphism group of a Cayley graph

Gabriel Verret

The University of Western Australia

Banff, Canada, July 22nd, 2013

Digraphs

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An automorphism of Γ is a permutation of $\mathcal{V}$ which preserves the the relation $\mathcal{A}$.

Cayley digraphs

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Note that $\mathrm{Cay}(R, S)$ may be disconnected and may have loops.

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Babai and Godsil proved the conjecture for nilpotent groups of odd order.

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Conclusion : if A is an abelian group and $A \ncong (\mathbb{Z}_2)^n$, then no Cayley graph on A is a GRR.

Corresponding conjectures

Conjecture (Babai, Godsil, Imrich, Lóvasz, 1982)

Let R be a group of order n which is neither generalized dicyclic nor abelian. The proportion of inverse-closed subsets S of R such that $\mathrm{Cay}(R, S)$ is a GRR goes to 1 as $n \to \infty$.

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In other words, after excluding some obvious exceptional groups, almost all Cayley graphs are GRRs.

Conjecture (Babai, Godsil 1982)

Let A be an abelian group of order n. The proportion of inverse-closed subsets S of A such that $\mathrm{Aut}(\mathrm{Cay}(A, S)) = A \rtimes \langle \iota \rangle$ goes to 1 as $n \to \infty$.

A remark

If we want to count non-DRR's, we may assume that $R < G \leq \mathrm{Aut}(\mathrm{Cay}(R, S))$. Without loss of generality, R is maximal in G.

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Lemma

Let R be a group of order n. The number of subsets of R which are fixed setwise by some element of $\mathrm{Aut}(R) \setminus \{1\}$ is at most $2^{3n/4+o(n)}$.

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Lemma

Let R be a group of order n. The number of subsets of R which are fixed setwise by some element of $\mathrm{Aut}(R) \setminus \{1\}$ is at most $2^{3n/4+o(n)}$.

Using this remark, we may therefore restrict our attention to the case when R is a regular self-normalizing maximal subgroup of the permutation group G.

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Theorem (Dobson, Spiga, V., 2013)

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Let A be an abelian group of order n. The proportion of subsets S of A such that $\mathrm{Cay}(A, S)$ is a DRR goes to 1 as $n \to \infty$.

We dealt with this case by characterising permutation groups containing a maximal abelian regular self-normalizing subgroup.

The abelian graph case, I

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Theorem (Dobson, Spiga, V., 2013)

Let G be a permutation group with an abelian regular subgroup A and a proper subgroup B which is generalized dihedral on A such that $N_G(A) = B$. Then $\mathbf{Z}(G)$ is an elementary abelian 2-group contained in A and $G = U \times \mathbf{Z}(G)$ where $G_1 \leq U \cong \mathrm{PGL}(2, q)$ and $A/\mathbf{Z}(G) \cong \mathrm{C}_{q+1}$.

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Corollary (Dobson, Spiga, V., 2013)

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Future work

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In this case, G is primitive with point-stabiliser R and regular subgroup G_1.

It might be useful to obtain some results about the number of such groups. (Up to conjugacy in $\mathrm{Sym}(n)$.)

Another conjecture

There is a also the following (related) conjecture:

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Thank you!