Sharply 2-transitive linear groups.

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Outline

Basic facts about sharply 2-transitive groups

- A sharply 2-transitive group is a permutation group $\Gamma \leq \mathrm{Sym}(\Omega)$ which acts transitively and freely on pairs of distinct points.
- Sharply 2-transitive groups contain a lot of involutions (elements of order 2), and all are conjugate.
- If an involution stabilizes a point, then the conjugation action of Γ on $\mathrm{Inv}(\Gamma)$ is isomorphic to it's action on Ω.
- This gives rise to the definition of the permutation characteristic of the group, p-char(Γ):

$$p\text{-char}(\Gamma) = \begin{cases} 2 & \Gamma_x \cap \mathrm{Inv}(\Gamma) = \emptyset \\ p & \Gamma_x \cap \mathrm{Inv}(\Gamma) \neq \emptyset, \quad \mathrm{Ord}(\sigma\tau) = p \\ 0 & \Gamma_x \cap \mathrm{Inv}(\Gamma) \neq \emptyset, \quad \mathrm{Ord}(\sigma\tau) = \infty \end{cases}$$

The main example

Example

Given a field N, the affine action $x \mapsto ax + b$ is sharply 2-transitive.

An easy way to see this is using geometric interpretation (at least for $N = \mathbb{R}$). Taking (x, y) to (z, w) is equivalent to finding the unique line between (x, z) and (y, w).
Looking at this example a little closer, one can see that the same will work for a division ring or even a near-field.

Some history behind the problem

- A long standing conjecture about sharply 2-transitive groups is that every such group is the affine group of a near-field, i.e. $N^\times \ltimes N$
- In the finite case, it was completely settled by H.Zassenhaus: in his two 1936 papers he first proved this conjecture for finite groups, and later classified all finite near-fields.
- In the infinite case, much less has been done. In 1952, J.Tits proved the conjecture for locally compact connected groups. In this case all near-fields are of finite rank over $\mathbb{R}$.
- Moreover, J.Tits showed that for an infinite sharply k-transitive group, $k \leq 3$.

Statement of the main theorem

Theorem

Let F be a field and let $\Gamma \leq \mathrm{GL}_n(F)$ be a sharply 2-transitive group. Assume that $\mathrm{char}(F) \neq 2$ and that $p\text{-char}(\Gamma) \neq 2$. Then $\Gamma \cong N^{\times} \ltimes N$, where N is a near-field.

Proof strategy

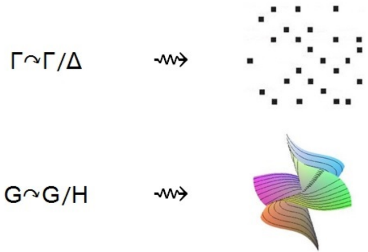
Theorem (Dixon and Mortimer, Permutation Groups, Theorem 7.6C)

Let $|\Omega| \geq 2$ *and let* $\Gamma \leq \mathrm{Sym}(\Omega)$ *be a sharply* 2*-transitive group which possesses a fixed-point free normal abelian subgroup* K*. Then there exists a near-field* N *such that* Γ *is permutation isomorphic to* $N^{\times} \ltimes N$*.*

- Using this theorem, it suffices to prove the existence of a fixed-point free normal abelian subgroup.

Passing to algebraic groups

- Let G, H be the Zariski closures of $\Gamma, \Delta = \Gamma_\omega$ respectively, in $\mathrm{GL}_n(k)$ where $k = \overline{F}$.
- We know that $\Gamma \curvearrowright \Gamma/\Delta$ sharply 2-transitively. What can we say about $G \curvearrowright G/H$? for that, we need to introduce the algebraic analogue of transitivity.



Passing to algebraic groups

Definition (Generic transitivity)

Let $\rho : G \curvearrowright X$ be an algebraic group acting algebraically on an algebraic variety X. ρ is called generically n-transitive if the action ρ^n of G on X^n admits an open dense orbit.

Idea: First, show that $G \curvearrowright G/H$ is generically 2-transitive. If under our assumptions, there is no normal abelian subgroup then $G \curvearrowright G/H$ can not be generically 2-transitive.

Passing to algebraic groups

Theorem (Jonathan Brundan)

Let G be a connected reductive group and $H < G$ a proper reductive subgroup. Then, there is no dense (H, H)-double coset in G.

Theorem (Domingo Luna)

Let $H < G \leq \mathrm{GL}_n(F)$, with $\mathrm{char}(F) = 0$, be connected reductive groups acting on an algebraic variety X. Then the generic H orbit is closed.

Corollary

If G, H are both reductive then $G \curvearrowright G/H$ can not be generically 2-transitive.

Proof of the corollary

- Assume that H admits an open orbit in it's action on G/H then at least one of the H^0 orbits is open as well, since any H-orbit is a finite union of H^0-orbits. So we have $\overline{O} = H^0 gH$ open.
- $\overline{O}$ is connected and so is contained in the connected component $\overline{X} = G^0 gH$.
- The natural map $\varphi : G/H^0 \to G/H$ restricted to $X = G^0 gH^0$ is a covering map (since $[H : H^0]$ is finite), so $O = \varphi^{-1}(\overline{O}) \cap X = H^0 gH^0$ is open (and hence dense) in X. This contradicts Brundan's theorem.
- If $\mathrm{char}(F) = 0$, we can use Luna's theorem instead: the generic H^0-orbit in G^0/H^0 is closed. But we have just seen that there exists an open orbit. Hence the action $H^0 \curvearrowright G^0/H^0$ has to be transitive.

Proving splitting

Theorem

Let $\Gamma < \mathrm{GL}_n(F)$ *be a sharply 2-transitive group. Assume that* $\mathrm{char}(F) \neq 2$ *and that* $p\text{-char}(\Gamma) \neq 2$. *Then there exist a non-trivial abelian normal subgroup* $N \lhd \Gamma$.

Proposition (1)

Let Γ *be as in the assumptions of the theorem. If the conclusion of the theorem fails, then there exists an algebraically closed field* k *and a faithful representation* $\rho : \Gamma \to \mathrm{GL}_n(k)$ *such that* $\mathbb{G} = \overline{\rho(\Gamma)}^{Z}$ *is reductive.*

Proof of Proposition 1

- Take a faithful representation $\rho_0 : \Gamma \to \mathrm{GL}_n(k)$ for $k = \overline{F}$.
- Let $\mathbb{G}_0 = \overline{\rho(\Gamma)}^Z$. Let $\mathbb{G}_u$ be the unipotent radical of $\mathbb{G}_0$ and $N = \rho(\Gamma) \cap \mathbb{G}_u$.
- Since N is nilpotent, it's penultimate element of the lower central series is a characteristic subgroup of N and hence is a normal abelian subgroup of $\rho(\Gamma)$ - a contradiction.
- So we can divide by $\mathbb{G}_u$ and obtain the required representation ρ.

Generic 2-transitivity

Proposition (2)

Let $\Gamma, \Delta = \Gamma_\omega$ be as before and denote $G = \overline{\Gamma}^Z, H = \overline{\Delta}^Z$. Let $\sigma \in \Delta$ be the unique involution. Then σ is semi-simple, $H \leq C_G(\sigma)$ and G acts generically 2-transitively on G/H.

- Fix the unique involution $\sigma \in \Delta$. Since Δ centralizes σ, so does H.
- Take any $\gamma \in \Gamma$ not in Δ. Δ acts transitively on $\Gamma/\Delta \setminus \{\Delta\}$, so $\Gamma/\Delta = \Delta \sqcup \Delta\gamma\Delta$.
- the set $H \sqcup H\gamma H \subseteq G/H$ is dense, since it contains the dense set $\Gamma H = H \sqcup \Delta\gamma H$ and locally closed, hence open.
- It follows that the orbit of $(H, \gamma H)$ is open in $G/H \times G/H$:

Generic 2-transitivity

- Take the natural map $\pi : G/H \times G/H \to G/H$.
- The intersection $G(H, \gamma H) \cap \pi^{-1}(H)$ is open and dense in the fiber.
- Since the action is transitive, this is true for any fiber.
- Hence the G-orbit $G(H, \gamma H)$ is dense. It is also locally closed, as an orbit of an algebraic action - and thus open.

Existence of a normal abelian subgroup

- By the previous results we see that if there is no such subgroup, then $G, C = C_G(\sigma)$ are reductive and $G \curvearrowright G/H$ is generically 2-transitive.
- Since $H \leq C$ and $G \curvearrowright G/H$ is generically 2-transitive, then so is $G \curvearrowright G/C$.
- But G, C are both reductive, which contradicts the Corollary.

Concluding the proof

- Let $N \lhd \Gamma$ be the normal abelian subgroup we just obtained.
- For any $\omega \in \Omega$, $[N_\omega, N] = \langle e \rangle$ and hence $N_\omega = \langle e \rangle$.
- So N is a fixed-point free normal abelian subgroup.

Thank You

Thank You!