# Zappa-Szép products of semigroups and their $C^*$-algebras

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(Joint with Jacqui Ramagge, Dave Robertson, Mike Whittaker)

# The internal Zappa-Szép product of semigroups

These definitions can be found in the work of Brin [1]:

Suppose $P$ is a semigroup with identity. Suppose $U, A \subseteq P$ are subsemigroups with

1. $U \cap A = \{e\}$, and
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The semigroup $P$ is an (internal) Zappa-Szép product $U \bowtie A = \{(u, a) : u \in U, a \in A\}$ with

$$(u, a)(v, b) = (u(a \cdot v), a|_v b).$$

# The external Zappa-Szép product of semigroups

Suppose $U$ and $A$ are semigroups with identity, and there are maps

$$\cdot : A \times U \to U \quad \text{and} \quad | : A \times U \to A$$

satisfying

(B1) $e_A \cdot u = u$;
(B2) $(ab) \cdot u = a \cdot (b \cdot u)$;
(B3) $a \cdot e_U = e_U$;
(B4) $a \cdot (uv) = (a \cdot u)(a|_u \cdot v)$;
(B5) $a|_{e_U} = a$;
(B6) $a|_{uv} = (a|_u)|_v$;
(B7) $e_A|_u = e_A$; and
(B8) $(ab)|_u = a|_{b \cdot u} b|_u$.

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(B8) $(ab)|_u = a|_{b \cdot u} b|_u$.

Then we can form the (External) Zappa-Szép product semigroup $U \bowtie A := \{(u, a)) : u \in U, a \in A\}$ with multiplication given by

$$(u, a)(v, b) = (u(a \cdot v), a|_v b).$$

# Hypotheses on our semigroups

We will look at the $C^*$-algebras of the Zappa-Szép product semigroups in the sense of Li [10] with the following hypotheses imposed on $U$ and $A$:

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Lemma (BRRW, Submitted 13)

*Under hypotheses (1)–(4), $U \bowtie A$ is a discrete left cancellative right LCM semigroup.*

# Example 1: Self-similar actions

Let $X$ be a finite alphabet, and $X^*$ the set of finite words. Under concatenation, $X^*$ is a semigroup with identity the empty word $\varnothing$. A faithful action of a group $G$ on $X^*$ is *self-similar* if for every $g \in G$ and $x \in X$, there exists unique $g|_x \in G$ such that

$$g \cdot (xw) = (g \cdot x)(g|_x \cdot w).$$

The pair $(G, X)$ is called a *self-similar action*. (See Nekrashevych's book [12] for more on self-similar actions.)

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## Example: Adding Machine

Consider $(\mathbb{Z} = \langle \gamma \rangle, \{0, 1, \ldots, n-1\})$ with

$$\gamma \cdot i = \begin{cases} i+1 & \text{if } i < n-1 \\ 0 & \text{if } i = n-1 \end{cases} \quad \text{and} \quad \gamma|_i = \begin{cases} e & \text{if } i < n-1 \\ \gamma & \text{if } i = n-1. \end{cases}$$

Restrictions extend to finite words, and satisfy

$$g|_{vw} = (g|_v)|_w, \quad (gh)|_v = g|_{h\cdot v} h|_v \quad \text{and} \quad (g|_v)^{-1} = g^{-1}|_{g\cdot v}.$$

The action $w \mapsto g \cdot w$ and restriction $w \mapsto g|_w$ maps satisfy (B1)–(B8), and so we can form $X^* \bowtie G$. Moreover, the semigroups $X^*$ and $G$ satisfy (1)–(4), so $X^* \bowtie G$ is a right LCM semigroup.

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## Example: Adding Machine

Note that since $\gamma|_i \in \{e, \gamma\} \subset \mathbb{N}$, we can form $X^* \bowtie \mathbb{N}$, which is a Zappa-Szép product associated to a "self-similar action of a semigroup".

# Example 2: $\mathbb{N} \rtimes \mathbb{N}^{\times}$

Consider the semidirect product $\mathbb{N} \rtimes \mathbb{N}^{\times}$, where

$$(m, a)(n, b) = (m + na, ab).$$

Laca-Raeburn [6] showed that $(\mathbb{Q} \rtimes \mathbb{Q}_{+}^{*}, \mathbb{N} \rtimes \mathbb{N}^{\times})$ is quasi-lattice ordered.

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Consider the following subsemigroups of $\mathbb{N} \rtimes \mathbb{N}^\times$:

$$U := \{(r, x) : x \in \mathbb{N}^\times, 0 \leq r < x\} \quad \text{and} \quad A := \{(m, 1) : m \in \mathbb{N}\}.$$

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We have $U \cap A = \{(0, 1)\}$ and each $(m, a) \in \mathbb{N} \rtimes \mathbb{N}^\times$ can be written uniquely as a product in $UA$ via

$$(m, a) = \Big(m \pmod{a}, a\Big)\Big(\frac{m - (m \pmod{a})}{a}, 1\Big).$$

So we can form $U \bowtie A$ with $\mathbb{N} \rtimes \mathbb{N}^\times \cong U \bowtie A$, and $U$ and $A$ satisfy (1)–(4).

# Example 3: $\mathbb{Z} \rtimes \mathbb{Z}^\times$

Consider the $ax + b$-semigroup over $\mathbb{Z}$, and the following subsemigroups:

$$U = \{(r, x) : x \geq 1,\, 0 \leq r < x\} \quad \text{and} \quad A = \mathbb{Z} \times \{1, -1\}.$$

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For $m, n \geq 1$ define the Baumslag-Solitar group

$$BS(m, n) = \langle a, b : ab^m = b^n a \rangle.$$

Spielberg [14] showed that $(BS(m, n), BS(m, n)^+)$ is quasi-lattice ordered.

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Spielberg [14] showed that $(BS(m,n), BS(m,n)^+)$ is quasi-lattice ordered.
Consider the subsemigroups of $BS(m,n)^+$:

$$U = \langle e, a, ba, b^2 a, \ldots, b^{n-1} a \rangle \cong \mathbb{F}_n^+ \quad \text{and} \quad A = \langle e, b \rangle \cong \mathbb{N}.$$

We have $U \cap A = \{e\}$, and we can write each element of $BS(m,n)^+$ uniquely as a product in $UA$ (the normal form of an element). So we can form $U \bowtie A$ with

$$b \cdot b^k a = b^{k+1 \pmod{n}} a \quad \text{and} \quad b|_{b^k a} = \begin{cases} e & \text{if } k < n-1 \\ b^m & \text{if } k = n-1. \end{cases}$$

Then $BS(m,n)^+ \cong U \bowtie A$, and $U$ and $A$ satisfy (1)–(4).

# Example 5: Products of self-similar actions

Let $X$ and $Y$ be finite alphabets, $G$ a group acting self-similarly on both $X$ and $Y$, and $\theta : Y \times X \to X \times Y$ a bijection. For each $(y, x) \in Y \times X$ denote by $\theta_X(y, x) \in X$ and $\theta_Y(x, y) \in Y$ the unique elements satisfying

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Every element $z \in \mathbb{F}_\theta^+$ admits a normal form $z = vw$ where $v \in X^*$ and $w \in Y^*$. We have maps $G \times \mathbb{F}_\theta^+ \to \mathbb{F}_\theta^+$ and $G \times \mathbb{F}_\theta^+ \to G$ given by

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## Proposition (BRRW,13)

*The maps given above induce a Zappa-Szép product semigroup $\mathbb{F}_\theta^+ \bowtie G$ if and only if*

$$\theta_X(y, x) = g^{-1} \cdot \theta_X(g \cdot y, g|_y \cdot x) \quad \textit{and} \quad \theta_Y(y, x) = g|^{-1}_{\theta_X(y,x)} \cdot \theta_Y(g \cdot y, g|_y \cdot x).$$

## Example: Product of Adding Machines

Fix $m, n \geq 2$. Suppose $\mathbb{Z}$ acts as the adding machine on $X := \{x_0, x_1, \cdots, x_{m-1}\}$ and $Y := \{y_0, y_1, \cdots y_{n-1}\}$. There is a bijection from $\{0, 1, \dots, n-1\} \times \{0, 1, \dots, m-1\}$ to $\{0, 1, \dots, m-1\} \times \{0, 1, \dots, n-1\}$ sending $(j, i)$ to the unique pair $(i', j')$ satisfying $j + in = i' + j'm$. This induces a bijection

$$\theta : Y \times X \to X \times Y \quad \text{given by} \quad \theta(y_j, x_i) = (x_{i'}, y_{j'}),$$

and we can form the Zappa-Szép product $\mathbb{F}_\theta^+ \bowtie \mathbb{Z}$.

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and we can form the Zappa-Szép product $\mathbb{F}_\theta^+ \bowtie \mathbb{Z}$.

Note that $\mathbb{F}_\theta^+$ is right LCM if and only if $\gcd(m, n) = 1$.
Consider $m = pa$ and $n = pb$. Then

$$p + 0.pb = p + 0.pa \quad \text{and} \quad p + a.pb = p + b.pa,$$

So

$$y_p x_0 = x_p y_0 \quad \text{and} \quad y_p x_m = x_p y_n,$$

and $x_p$ and $y_p$ have two incomparable right common multiples.

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- $x_i \mapsto (i,m)$ and $y_j \mapsto (j,n)$ is an isomorphism of $\mathbb{F}_\theta^+$ onto the subsemigroup of $\mathbb{N} \rtimes \mathbb{N}^\times$ generated by the elements $(0,m),\ldots,(m-1,m),(0,n),\ldots,(n-1,n)$.

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- $\mathbb{F}_\theta^+$ and $\mathbb{Z}$ satisfy (1)–(4).
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Also note that since $\mathbb{N}$ is invariant under restrictions, we can form $\mathbb{F}_\theta^+ \bowtie \mathbb{N}$. If $\gcd(m, n) = 1$, then $\mathbb{F}_\theta^+ \bowtie \mathbb{N}$ is isomorphic to the subsemigroup of $\mathbb{N} \rtimes \mathbb{N}^\times$ generated by $(1, 1), (0, m), (0, n)$.

# Semigroup $C^*$-algebras

The following is due to Li [10]:

Let $P$ be a discrete left cancellative semigroup. Let $\mathcal{J}(P)$ be the smallest collection of right ideals containing $P$ and $\varnothing$, closed under left multiplication ($pX := \{px : x \in X\}$) and pre-images under left multiplication ($p^{-1}X = \{y \in P : py \in X\}$), and closed under finite intersections.

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The *full semigroup $C^*$-algebra* $C^*(P)$ is the universal $C^*$-algebra generated by isometries $\{v_p : p \in P\}$ and projections $\{e_X : X \in \mathcal{J}(P)\}$ satisfying

$$v_p v_q = v_{pq} \qquad e_P = 1 \text{ and } e_\varnothing = 0$$

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Note that when $P$ is right LCM, $\mathcal{J}(P)$ consists only of principal right ideals; that is, $\mathcal{J}(P) = \{pP : p \in P\}$. Moreover,

$$C^*(P) = \overline{\operatorname{span}}\{v_p v_q^* : p, q \in P\}.$$

# The $C^*$-algebra $C^*(U \bowtie A)$

Let $U$ and $A$ be semigroups such that $U \bowtie A$ exists, and satisfying (1)–(4).

**Theorem (BRRW,13)**

*The $C^*$-algebra $C^*(U \bowtie A)$ is the universal $C^*$-algebra generated by isometric representations $\{s_a : a \in A\}$ and $\{t_u : u \in U\}$ satisfying*

$$t_u^* t_v = t_{u'} t_{v'}^* \text{ whenever } uU \cap vU = (uu')U = (vv')U,\ uu' = vv'; \quad (1)$$

*and*

(K1) $s_a t_u = t_{a \cdot u} s_{a|_u}$; *and*

(K2) $s_a^* t_u = t_z s_{a|_z}^*$, *where $z \in U$ satisfies $a \cdot z = u$.*

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(K1) $s_a t_u = t_{a \cdot u} s_{a|_u}$; *and*

(K2) $s_a^* t_u = t_z s_{a|_z}^*$, *where $z \in U$ satisfies $a \cdot z = u$.*

Note that when $(G, P)$ is quasi-lattice ordered, we can always form $P \bowtie \{e\}$ satisfying our hypotheses. Then (1) says

$$t_u^* t_v = t_{u^{-1}(u \vee v)} t_{v^{-1}(u \vee v)} \quad \text{(if } u \vee v \text{ exists)},$$

which is Nica covariance. So $C^*(P \bowtie \{e\}) = C^*(G, P)$.

# The boundary quotient

## Definition (BRRW,13)

We say a finite subset $F \subset \mathcal{J}(P)$ is a *foundation set* if for each $Y \in \mathcal{J}(P)$ there exists $X \in F$ with $X \cap Y \neq \varnothing$. We define $\mathcal{Q}(P)$ to be the quotient of $C^*(P)$ given by adding the relation

$$\prod_{X \in F}(1 - e_X) = 0 \quad \text{for all foundation sets } F \subseteq P.$$

# The boundary quotient

## Definition (BRRW,13)

We say a finite subset $F \subset \mathcal{J}(P)$ is a *foundation set* if for each $Y \in \mathcal{J}(P)$ there exists $X \in F$ with $X \cap Y \neq \varnothing$. We define $\mathcal{Q}(P)$ to be the quotient of $C^*(P)$ given by adding the relation

$$\prod_{X \in F} (1 - e_X) = 0 \quad \text{for all foundation sets } F \subseteq P.$$

We now give an alternative presentation for $\mathcal{Q}(U \bowtie A)$:

## Theorem (BRRW,13)

*Suppose $U$ and $A$ are semigroups such that $U \bowtie A$ exists, and satisfying (1)–(4). Then $\mathcal{Q}(U \bowtie A)$ is the universal $C^*$-algebra generated by isometric representations $\{s_a : a \in A\}$ and $\{t_u : u \in U\}$ satisfying (1), (K1), (K2) and*

(Q1) $s_a s_a^* = 1$; *and*

(Q2) $\prod_{\{u : uU \in F\}} (1 - t_u t_u^*) = 0$ *for all foundation sets $F \subseteq \mathcal{J}(U)$.*

# Example 1: $C^*$-algebras associated to self-similar actions

The $C^*$-algebra $\mathcal{O}(G, X)$ associated to a self-similar action $(G, X)$ was first considered by Nekrashevych [11].

Laca, Raeburn, Ramagge and Whittaker [7] examined the Toeplitz algebra $\mathcal{T}(G, X)$.

We can view

- $\mathcal{T}(G, X)$ as the universal $C^*$-algebra generated by a Toeplitz-Cuntz family of isometries $\{v_x : x \in X\}$ and a unitary representation $u$ of $G$ satisfying $u_g v_x = v_{g \cdot x} u_{g|_x}$; and
- $\mathcal{O}(G, X)$ as the quotient of $\mathcal{T}(G, X)$ by the ideal $I$ generated by $1 - \sum_{x \in X} v_x v_x^*$.

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### Proposition (BRRW,13)

*There are isomorphisms*

$$\mathcal{T}(G, X) \cong C^*(X^* \bowtie G) \quad \textit{and} \quad \mathcal{O}(G, X) \cong \mathcal{Q}(X^* \bowtie G).$$

# Example 2: $C^*$-algebras associated to $\mathbb{N} \rtimes \mathbb{N}^\times$

Recall from Laca and Raeburn [6] that $\mathcal{T}(\mathbb{N} \rtimes \mathbb{N}^\times)$ is the universal $C^*$-algebra generated by an isometry $s$ and isometries $v_p$ for each prime $p$ satisfying

$$v_p s = s^p v_p \qquad s^* v_p = s^{p-1} v_p s^*$$

$$v_p v_q = v_q v_p \qquad v_p^* s^k v_p = 0 \text{ for all } 1 \le k < p$$

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The boundary quotient is $\mathcal{Q}_\mathbb{N}$ from Cuntz's [2], and corresponds to adding the relations

$$ss^* = 1 \quad \text{and} \quad \sum_{k=0}^{p-1} (s^k v_p)(s^k v_p)^* = 1 \text{ for all primes } p.$$

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### Proposition (BRRW,13)

*For* $U = \{(r, x) : 0 \leq r < x\}$ *and* $A := \{(m, 1) : m \in \mathbb{N}\}$ *there are isomorphisms* $\mathcal{T}(\mathbb{N} \rtimes \mathbb{N}^\times) \cong C^*(U \bowtie A)$ *and* $\mathcal{Q}_\mathbb{N} \cong \mathcal{Q}(U \bowtie A)$*, sending* $s \mapsto s_{(1,1)}$ *and* $v_p \mapsto t_{(0,p)}$ *for all primes* $p$*.*

# Example 3: $C^*$-algebras associated to $\mathbb{Z} \rtimes \mathbb{Z}^\times$

Recall from [2] that $\mathcal{Q}_\mathbb{Z}$ can be viewed as the universal $C^*$-algebra generated by a unitary $s$ and isometries $\{v_a : a \in \mathbb{Z}^\times\}$ satisfying

- $v_a v_b = v_{ab}$ for all $a, b \in \mathbb{Z}^\times$;
- $v_a s = s^a v_a$ and $v_a s^* = s^{*a} v_a$ for all $a \in \mathbb{Z}^\times$; and
- $\sum_{j=0}^{|a|-1} s^j v_a v_a^* s^{*j} = 1$ for all $a \in \mathbb{Z}^\times$.

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## Proposition (BRRW,13)

*There is an isomorphism* $\phi : \mathcal{Q}_\mathbb{Z} \to \mathcal{Q}(\mathbb{Z} \rtimes \mathbb{Z}^\times)$ *satisfying* $\phi(s) = s_{(1,1)}$ *and*

$$\phi(v_a) = s_{(0,a/|a|)} t_{(0,|a|)} \text{ for all } a \in \mathbb{Z}^\times.$$

# Example 4: $C^*$-algebras associated to $BS(m,n)^+$

Recall that $BS(m,n)^+ \cong U \bowtie A$, where

$$U = \langle e, a, ba, b^2a, \ldots, b^{n-1}a \rangle \cong \mathbb{F}_n^+ \quad \text{and} \quad A = \langle e, b \rangle \cong \mathbb{N},$$

and

$$b \cdot b^k a = b^{k+1 (\text{mod } n)} a \quad \text{and} \quad b|_{b^k a} = \begin{cases} e & \text{if } k < n-1 \\ b^m & \text{if } k = n-1. \end{cases}$$

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## Proposition (BRRW,13)

*The boundary quotient $\mathcal{Q}(BS(m,n)^+$ is the universal $C^*$-algebra generated by a unitary $s$ and isometries $t_1, \ldots, t_n$ satisfyng*

- $st_i = t_{i+1}$ *for* $1 \leq i < n$*;*
- $st_n = t_1 s^m$*; and*
- $\sum_{i=1}^n t_i t_i^* = 1.$

*Moreover, $\mathcal{Q}(BS(m,n)^+)$ is isomorphic to the Spielberg's category of paths algebra from [14], and the topological graph algebra $\mathcal{O}(E_{n,m})$ from Katsura's [5].*

# Example 5: $C^*$-algebras associated to $\{0,1\}^* \bowtie \mathbb{N}$.

Let $X = \{0,1\}$ and $\mathbb{N} = \langle\gamma\rangle$. Consider $X^* \bowtie \mathbb{N}$, where

$$\begin{aligned} \gamma \cdot 0 &= 1 & \gamma|_0 &= e \\ \gamma \cdot 1 &= 0 & \gamma|_1 &= \gamma. \end{aligned}$$

# Example 5: $C^*$-algebras associated to $\{0,1\}^* \bowtie \mathbb{N}$.

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- $C^*(X^* \bowtie \mathbb{N}) \cong C^*(BS(1,2), BS(1,2)^+)$

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Then

- $X^* \bowtie \mathbb{N} \cong \mathrm{BS}(1,2)^+$
- $C^*(X^* \bowtie \mathbb{N}) \cong C^*(BS(1,2), BS(1,2)^+)$
- Recall from Larsen-Li [8] that $\mathcal{Q}_2$ is the universal $C^*$-algebra generated by a unitary $u$ and an isometry $s_2$ satisfying

$$s_2 u = u^2 s_2 \quad \text{and} \quad s_2 s_2^* + u s_2 s_2^* u^* = 1.$$

  There is an isomorphism $\phi : \mathcal{Q}_2 \to \mathcal{Q}(X^* \bowtie \mathbb{N})$ such that $\phi(u) = s_\gamma$ and $\phi(s_2) = t_0$.

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Thanks!