Towards an arithmetic Kac–Moody theory

Ralf Köhl*

16 & 17 April 2013

* né Gramlich

ralf.koehl@math.uni-giessen.de

What is ...
... a (split) Kac–Moody group?

- can be thought of as an infinite-dimensional Chevalley group (Kac–Peterson 1980ies)

- functor from commutative rings to groups (Tits 1987)

- group of rational points over a field of char. 0 is a group of automorphisms of a Kac–Moody algebra (Kac–Peterson 1980ies)
finite-dimensional semisimple Lie algebra $\rightsquigarrow$ Kac–Moody algebra
Cartan matrix $\rightsquigarrow$ (symmetrizable) generalized Cartan matrix
$A = DS$, S symmetric, positive definite $\rightsquigarrow$ $A = DS$, S symmetric

- in the 2-spherical case a group of rational points over a field of size at least 4 is a colimit of a graph/diagram of groups consisting of Chevalley groups of ranks 1 and 2 (Abramenko–Mühlherr 1997)

- Galois descent leads to non-split Kac–Moody groups (Rémy 2002)

Outline of mini-course

Part 1: Chevalley groups and split Lie groups as colimits

- present Chevalley groups/fields by generators and relations
- recover Lie group topology/local fields via open mapping thm

Part 2: Topological Kac–Moody groups and properties

- define Kac–Moody groups/fields by generators and relations
- define a universal Hausdorff group topology/local fields
- ... this Hausdorff topology enjoys Kazhdan's property (T)
- ... has finite-dim. torus that allows access via algebraic groups

Part 3: Rigidity of arithmetic Kac–Moody groups

- bounded generation of arithmetic groups and a fixed point thm
- strong rigidity/superrigidity of arithmetic Kac–Moody groups

Part 1:

Chevalley groups and split Lie groups as colimits of diagrams of groups

Presentations arising from group actions on simply connected simplicial complexes (cf. Wortman's talks)

Theorem 1 (Simplicial geometric group theory)

Let
Δ *simply connected finite-dim. coloured simpl. complex,*
$G \to \mathrm{Aut}(\Delta)$ *colour-preserving simplicial rigid action, transitive on maximal simplices,*
c *maximal simplex,* I *index set for vertices of* c*,*
$(G_J)_{\emptyset \neq J \subseteq I}$ *family of pointwise stabilizers of non-empty subsimplices of* c*,*
$\phi_{J,J'} : G_J \hookrightarrow G_{J'}$ *canonical embedding for* $J \supseteq J'$*.*

Then

$$G \cong \left\langle \bigcup_{\emptyset \neq J \subseteq I} G_J \;\middle|\; \begin{array}{c} \text{all relations in the } G_J \text{ plus} \\ \text{all identifications via the } \phi_{J,J'} \end{array} \right\rangle .$$

Terminology: $(G_J)_{\emptyset \neq J \subseteq I}$ together with the connecting morphisms is a **diagram of groups**. The group G is called a **colimit**.

Theorem 2 (Non-simplicial version)

Let

X *simply connected topological space,*

$G \to \text{Homeo}(X)$ *action,*

U *a connected weak fundamental domain (i.e., X = G.U),*

$\Sigma = \{g \in G \mid U \cap g.U \neq \emptyset\}$,

$R = \{xy = (xy) \mid x, y \in \Sigma, \Sigma \cap x\Sigma \cap xy\Sigma \neq \emptyset\}$.

Then $G \cong \langle \Sigma \mid R \rangle$.

Theorem 2 implies Theorem 1:

Define U as an ϵ-neighbourhood of the maximal simplex c.

Example 3

Let Sym_4 act naturally on the barycentric subdivision of a 4-simplex considered as a 2-dimensional simplicial complex.

Let c be the maximal simplex consisting of the vertex 1, the barycentre of the edge $\{1,2\}$, and the barycentre of the face $\{1,2,3\}$.

Then
$G_1 = \mathrm{Sym}\{2,3,4\}$
$G_{\{1,2\}} = \mathrm{Sym}\{1,2\} \times \mathrm{Sym}\{3,4\}$
$G_{\{1,2,3\}} = \mathrm{Sym}\{1,2,3\}$.
The other stabilizers arise as intersections.

Theorem 1 states that

$$\begin{aligned} \mathrm{Sym}_4 &\cong \langle G_{\{1,2,3\}} \cup G_{\{1,2\}} \cup G_1 \mid \text{all relations in these groups} \rangle \\ &\cong \langle s_1, s_2, s_3 \mid s_i^2 = 1, (s_i s_{i+1})^3 = 1, s_1 s_3 = s_3 s_1 \rangle \end{aligned}$$

(Think $s_1 = (12)$, $s_2 = (23)$, $s_3 = (34)$.)

Note that the application of Theorem 1 can be iterated if the links of the simplicial complex are also simply connected:

Example 4

$$\begin{aligned}
\mathrm{Sym}_5 &\overset{1}{\cong} \langle G_1 \cup G_{\{1,2\}} \cup G_{\{1,2,3\}} \cup G_{\{1,2,3,4\}} \mid \text{their relations}\rangle \\
&\overset{1}{\cong} \langle G_{1,\{1,2\}} \cup G_{1,\{1,2,3\}} \cup \cdots \cup G_{\{1,2,3\},\{1,2,3,4\}} \mid \text{relations}\rangle \\
&\cong \langle \mathrm{Sym}\{3,4,5\} \cup \mathrm{Sym}\{2,3\} \times \mathrm{Sym}\{4,5\} \cup \cdots \\
&\qquad \cdots \cup \mathrm{Sym}\{1,2,3\} \mid \text{their relations}\rangle.
\end{aligned}$$

Presentations for split Lie groups

Example 5

Let
$G = \mathsf{SL}_n(\mathbb{R})$
$V = \mathbb{R}^n$
Δ simplicial complex with underlying vertex set

$$\{(U, W) \mid V = U \oplus W \text{ with } U,\ W \text{ nontrivial}\}$$

and adjacency

$$(U, W) \sim (U', W') \quad \Longleftrightarrow \quad \begin{array}{c} U \leq U', W \geq W' \text{ or} \\ U \geq U', W \leq W' \end{array}$$

Δ is a finite-dimensional coloured simplicial complex (colour=dimension of first component)

action of G is colour-preserving and transitive (G transitive on bases)

Theorem 6

Let e_1, e_2, e_3, e_4 *be a basis of* $\mathbb{R}^4$. *Then*

$$\mathsf{SL}_4(\mathbb{R}) \cong \langle \mathsf{SL}\langle e_1, e_2, e_3\rangle \cup \mathsf{SL}\langle e_1, e_2\rangle \times \mathsf{SL}\langle e_2, e_3\rangle \cup \mathsf{SL}\langle e_2, e_3, e_4\rangle \mid \textit{their relations}\rangle$$

Proof. Δ is 2-dimensional.

The three given groups are the stabilizers (up to the torus) of the three vertices corresponding to the direct decompositions $\langle e_1, e_2, e_3\rangle \oplus \langle e_4\rangle$, $\langle e_1, e_2\rangle \oplus \langle e_3, e_4\rangle$, $\langle e_1\rangle \oplus \langle e_2, e_3, e_4\rangle$.

Δ is simply connected (by direct computation or by Tits 1990).

Therefore the claim follows from Theorem 1.

(Reconstruct torus from rank 2 pieces.) □

The Curtis–Tits theorem

Theorem 7 (Curtis 1965, Tits 1962, cf. GLS 1998)
Let $\mathbb{F}$ be a field containing at least four elements and let G be a simply connected Chevalley group over $\mathbb{F}$.

Then G is generated by its fundamental rank one subgroups and defined by the relations contained in its fundamental rank two subgroups.

In other words, the group G is determined by taking

- a (spherical) Dynkin diagram Δ,
- a field $\mathbb{F}$ with $|\mathbb{F}| \geq 4$,
- a group $G_\alpha \cong \mathrm{SL}_2(\mathbb{F})$ for each node α of Δ,
- a simply connected Chevalley group $G_{\alpha,\beta}$ over $\mathbb{F}$ for each pair of nodes α, β of Δ according to their type,
- embeddings $G_\alpha \hookrightarrow G_{\alpha,\beta}$ as fundamental rank one subgroups.

Recovering the topology

Theorem 8 (Glöckner, Hartnick, K. 2010)
Let $\mathbb{F}$ be a local field and let G be a Chevalley group over $\mathbb{F}$.

Then the Lie group topology on G equals the finest group topology making the embeddings of the fundamental rank one subgroups – endowed with their Lie group topology – continuous.

Proof. Via an open mapping theorem. $\square$

An open mapping theorem

Proposition 9

A surjective, continuous homomorphism $f\colon G \to H$ between Hausdorff topological groups where G is σ-compact and H is a Baire space, is open; moreover, H is locally compact.

Proof. By hypothesis, $G = \bigcup_{n\in\mathbb{N}} K_n$ for certain compact sets $K_n \subseteq G$ and thus $H = \bigcup_{n\in\mathbb{N}} f(K_n)$ with $f(K_n)$ compact.

Since H is a Baire space, $f(K_n)$ has non-empty interior for some $n \in \mathbb{N}$, and thus H is locally compact. Moreover, $f|_{K_n} : K_n \to f(K_n)$ is a quotient map.

Let $q : G \to G/\ker(f)$ be the quotient homomorphism and $\phi : G/\ker(f) \to H$ be the bijective continuous homomorphism induced by f.

Then $\phi^{-1} \circ f|_{K_n} = q|_{K_n}$ is continuous, whence $\phi^{-1}|_{f(K_n)}$ is continuous. So ϕ^{-1} is a continuous homomorphism, and ϕ is a topological isomorphism. Hence f is open. □

Topologies on colimits

Let $\delta\colon \mathbb{I} \to \mathbb{LCG}$ be a diagram of σ-compact locally compact groups $G_i := \delta(i)$ for $i \in I := \text{ob}(\mathbb{I})$ and continuous homomorphisms $\phi_\alpha := \delta(\alpha) : G_i \to G_j$ for $i, j \in I$ and $\alpha \in \text{Mor}(i, j)$, with countable I.

Furthermore, let $(G, (\lambda_i)_{i\in I})$ be a colimit of the diagram δ in the category of abstract groups, with homomorphisms $\lambda_i\colon G_i \to G$.

If there exists a locally compact Hausdorff group topology $\mathcal{O}$ on G making $\lambda_i\colon G_i \to (G, \mathcal{O})$ continuous for each $i \in I$, then

$$((G, \mathcal{O}), (\lambda_i)_{i\in I})$$

is a colimit of δ in the category of topological groups, in the category of Hausdorff groups, and in the category of locally compact groups, by Proposition 9.

If each G_i is a σ-compact Lie group and $(G, \mathcal{O})$ is a Lie group, then $((G, \mathcal{O}), (\lambda_i)_{i\in I})$ also is a colimit of δ in the category $\mathbb{LIE}$ of Lie groups.

Part 2:

Topological Kac–Moody groups and their properties

Definition of 2-spherical split Kac–Moody groups

Let • Δ a 2-spherical Dynkin diagram without loops,
• $\mathbb{F}$ a field with $|\mathbb{F}| \geq 4$,
• $G_\alpha \cong \mathrm{SL}_2(\mathbb{F})$ for each node α of Δ,
• $G_{\alpha,\beta}$ a simply connected Chevalley group over $\mathbb{F}$ for each pair of nodes α, β of Δ according to their type,
• $G_\alpha \hookrightarrow G_{\alpha,\beta}$ embeddings as fundamental rank one subgroups.

Then the **Kac–Moody group** $G_\Delta(\mathbb{F})$ is defined as

$$G_\Delta(\mathbb{F}) \cong \left\langle \bigcup_{\alpha,\beta} G_{\alpha,\beta} \mid \text{their relations} \right\rangle \qquad \text{(i.e., type } F_2 \text{ if } \mathbb{F} \text{ is finite).}$$

This is well-defined, as Δ does not have loops: Using work by Goldschmidt 1980 it is possible to prove that the above system is unique up to isomorphism.

If there are loops, automorphisms of $\mathbb{F}$ may lead to ambiguity.

Work by Abramenko–Mühlherr 1997 guarantees existence.

Work by Abramenko–Mühlherr

Let
$X = (X_+, X_-, \delta_*)$ twin building of $G_\Delta(\mathbb{F})$,
$X^{\text{op}} = \{(c,d) \in X_+ \times X_- \mid c,\ d \text{ maximal simplices, } \delta_*(c,d) = 1\}$.

If $|\mathbb{F}| \geq 4$ and Δ 2-spherical, then X^{op} is simply connected.

Since $G_\Delta(\mathbb{F})$ acts transitively on X^{op} (strongly transitive action), Theorem 1 applies.

Example (cf. Example 5)
Frames of a vector space, i.e., opposite flags
$\langle e_1 \rangle$, $\langle e_1, e_2 \rangle$, ..., $\langle e_1, ..., e_{n-1} \rangle$ and $\langle e_n \rangle$, $\langle e_n, e_{n-1} \rangle$, ..., $\langle e_n, ..., e_2 \rangle$

Pairs of opposite minimal parabolics of an isotropic alg. group.

So:
Abramenko–Mühlherr 1997 provide a proof that generalizes and contains the classical Curtis–Tits theorem as a special case, provided $|\mathbb{F}| \geq 4$.

Topologies on 2-spherical split Kac–Moody groups

Let

- Δ a 2-spherical Dynkin diagram without loops,
- $\mathbb{F}$ a local field,
- $G_\alpha \cong \mathrm{SL}_2(\mathbb{F})$ for each node α of Δ, endowed with Lie group topology,
- $G_{\alpha,\beta}$ a simply connected split algebraic Lie group over $\mathbb{F}$ for each pair of nodes α, β of Δ according to their type,
- $G_\alpha \hookrightarrow G_{\alpha,\beta}$ embeddings as fundamental rank one subgroups.

Then the **Kac–Peterson topology** τ_{KP} on the Kac–Moody group $G_\Delta(\mathbb{F})$ is defined as the finest group topology that makes the embeddings $G_\alpha \hookrightarrow G_\Delta(\mathbb{F})$ continuous.

Theorem 10 (Hartnick, K., Mars)
The Kac–Peterson topology τ_{KP} *is Hausdorff and* k_ω.

If Δ *is non-spherical, then* τ_{KP} *is not locally compact, not metrizable, in particular not Polish.*

Question

Does there exist a non-zero σ-finite left-invariant Borel measure on $(G_\Delta(\mathbb{F}), \tau_{\mathsf{KP}})$?

It is known that a *Polish* topological group admits such a measure if and only if it is locally compact. Our question is beyond that setting.

$(G_\Delta(\mathbb{F}), \tau_{\mathsf{KP}})$ is Kazhdan

A topological group is called **Kazhdan**, if there exist a compact subset Q and $\epsilon > 0$ such that, whenever a unitary representation π of that group has a (Q, ϵ)-invariant vector, then it has a non-trivial fixed vector.

A vector ξ of the Hilbert space is (Q, ϵ)-invariant if

$$\sup_{x \in Q} \|\pi(x)\xi - \xi\| < \epsilon \|\xi\|.$$

Theorem 11 (Harnick, K., preprint)
Let $\mathbb{F}$ be a local field and let G be an irreducible 2-spherical split Kac–Moody group. Then $(G(\mathbb{F}), \tau_{\mathsf{KP}})$ is Kazhdan.

Irreducible means that the defining diagram Δ is connected.

Question

Does this also hold for the discrete fin. gen. subgroup $G(\mathbb{Z})$?

Proof by induction

Let Δ be a 2-spherical diagram and let $G(\mathbb{F}) = G_{\Delta}(\mathbb{F})$. Let $n := |\Delta|$.

For $n = 2$ the group G is a Chevalley group and by Theorem 8 the topology τ_{KP} equals the Lie group topology. Hence the claim follows from the fact that Lie groups of rank 2 are Kazhdan.

Let now $n \geq 3$ and assume that the claim has been proved for each irreducible 2-spherical Kac–Moody group of lower rank.

Let α_1, ..., α_n be the simple root of G and let H be the fundamental subgroup of rank $n-1$ corresponding to the set α_2, ..., α_n of simple roots, i.e., $H := \langle G_{\alpha_2}, ..., G_{\alpha_n} \rangle$.

Up to a change of enumeration of the simple roots of G we can assume that

- H is irreducible,
- G_{α_1,α_2} is irreducible.

The restriction of τ_{KP} yields the Kac–Peterson topology on H and the Lie group topology on G_{α_1,α_2}.

Let ρ be a unitary representation of G on some Hilbert space W almost having invariant vectors.

Since H is Kazhdan by induction hypothesis, there exists a non-zero $\rho(H)$-invariant vector $w \in W$.

Hence any element x of the Torus T_{α_2} of the fundamental rank one subgroup $G_{\alpha_2} \cong \mathsf{SL}_2(\mathbb{F})$ satisfies $\rho(x)w = w$.

There exist $x \in T_{\alpha_2}$ such that, moreover, for each $y \in U_{\alpha_1}$ one has

$$\lim_{n\to\infty} x^n y x^{-n} = 1$$

and for each $y \in U_{-\alpha_1}$ one has

$$\lim_{n\to\infty} x^{-n} y x^n = 1.$$

Example: $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda^n & 0 \\ 0 & 0 & \lambda^{-n} \end{pmatrix} \begin{pmatrix} 1 & y & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda^{-n} & 0 \\ 0 & 0 & \lambda^n \end{pmatrix} = \begin{pmatrix} 1 & \lambda^{-n}y & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Mautner's Lemma implies $\rho(U_{\alpha_1})w = w = \rho(U_{-\alpha_1})w$.

As $G = \langle G_{\alpha_1}, H \rangle = \langle U_{\alpha_1}, U_{-\alpha_1}, H \rangle$ this implies $\rho(G).w = w$.

Tori and isogenies

There exist several isogeneous versions of a Kac–Moody group, the extremal ones being the simply connected verion G^{sc} and the adjoint version G^{ad}.

The group $G_{\Delta}(\mathbb{F})$ is by construction a group of rational points of a simply connected Kac–Moody group G^{sc}.

Indeed, the presentation/definition does not contain any relations between torus elements that are not visible locally. This can be remedied by also prescribing the structure of the torus in the construction of $G_{\Delta}(\mathbb{F})$ (cf. proof of Theorem 6).

A Kac–Moody group, considered as a functor, comes along with a torus functor that distinguishes between the different isogeneous versions of a Kac–Moody group.

Details can be found in Rémy 2002 (Chapters 7, 8, 9).

This torus is an algebraic torus.

Indeed:
- group of characters Λ of Kac–Moody group is free abelian of finite rank (Rémy 2002, 7.1.1)
- torus functor T defined as the group functor $\mathrm{Hom}_{\mathbb{Z}\text{-alg}}(\mathbb{Z}[\Lambda], -)$ (Rémy 2002, 8.2.1)

($\mathbb{Z}[\Lambda]$: group ring of Λ, i.e., polyn. ring in free gen.s and inverses)
$\rightsquigarrow$ diagonalizable (cf. Waterhouse 1979, 2.2)
$\rightsquigarrow$ algebraic torus (actually split)

Consider the adjoint map

$$\mathsf{Ad} : G^{\mathsf{sc}} \to G^{\mathsf{ad}}$$

defined via a suitable $\mathbb{Z}$-form of the universal enveloping algebra of the complex Kac–Moody algebra (cf. Rémy 2002, 9.5)

This induces a surjective morphism of tori

$$T^{\mathsf{sc}} \to T^{\mathsf{ad}}$$

with finite kernel (i.e., an isogeny)

Indeed:
- exists a natural embedding $\Lambda^{\mathsf{ad}} \to \Lambda^{\mathsf{sc}}$
- yields an injective $\mathbb{Z}$-algebra homomorphism $\mathbb{Z}[\Lambda^{\mathsf{ad}}] \to \mathbb{Z}[\Lambda^{\mathsf{sc}}]$
- yields a surjective morphism $\mathsf{Hom}_{\mathbb{Z}\text{-alg}}(\mathbb{Z}[\Lambda^{\mathsf{sc}}], -) \to \mathsf{Hom}_{\mathbb{Z}\text{-alg}}(\mathbb{Z}[\Lambda^{\mathsf{ad}}], -)$
- as $\dim(T^{\mathsf{sc}}) = \dim(T^{\mathsf{ad}})$ the kernel has dim. 0, i.e., is finite

Index of $\mathrm{Ad}\, G^{\mathrm{sc}}(\mathbb{F})$ in $G^{\mathrm{ad}}(\mathbb{F})$

$$[G^{\mathrm{ad}}(\mathbb{F}) : \mathrm{Ad}\, G^{\mathrm{sc}}(\mathbb{F})] = [T^{\mathrm{ad}}(\mathbb{F}) : \mathrm{Ad}\, T^{\mathrm{sc}}(\mathbb{F})],$$

because kerAd lies in the centre of $G^{\mathrm{sc}}(\mathbb{F})$ (Rémy 2002, 9.6.2).

The exact sequence

$$1 \to F \to T^{\mathrm{sc}} \overset{\mathrm{Ad}}{\to} T^{\mathrm{ad}} \to 1$$

yields an exact sequence

$$1 \to F(\mathbb{F}) \to T^{\mathrm{sc}} \overset{\mathrm{Ad}}{\to} T^{\mathrm{ad}}(\mathbb{F}) \to H^1(\mathbb{F}, F) \to H^1(\mathbb{F}, T^{\mathrm{sc}}).$$

Since $H^1(\mathbb{F}, T^{\mathrm{sc}}) = 1$ (cf. Platonov, Rapinchuk 1994, 2.4), the index is therefore given by $|H^1(\mathbb{F}, F)|$.

Therefore, over local fields, the index is finite (cf. Milne 1986).

Part 3:

Rigidity of arithmetic Kac–Moody groups

A fixed-point theorem

Theorem 12 (Caprace, Monod 2009)

Let L be an irreducible Chevalley group of rank at least 2, let X be a complete CAT(0) polyhedral complex with finitely many isometry types of polyhedra, and let $\phi : L(\mathbb{Z}) \to \mathrm{Isom}(X)$ be an action via cellular and rigid isometries.

Then $\phi(L(\mathbb{Z}))$ has a fixed point.

Corollary 13

Let L be an irreducible Chevalley group of rank at least 2, let $G(\mathbb{R})$ be a Kac–Moody group, and let $\phi : L(\mathbb{Z}) \to G(\mathbb{R})$ be a group homomorphism.

Then $\phi(L(\mathbb{Z}))$ is a bounded subgroup, i.e., lies in the intersection of two spherical parabolic subgroups of G of opposite sign, i.e., lies in an algebraic subgroup of G.

Proof. Apply Theorem 12 to the Davis realization of the twin building of $G(\mathbb{R})$ and use Rémy 2002, 10.3. $\square$

Towards a proof of the fixed-point theorem

Lemma 14 (Lubotzky, Mozes, Raghunathan 2000)
Let Σ be the (finite) set of unit root group elements of $L(\mathbb{Z})$, let δ_Σ be the word metric of $L(\mathbb{Z})$, and let $\gamma \in \Sigma$. Then

$$l_\Sigma(\gamma^n) := \delta_\Sigma(1, \gamma^n) = O(\log(n)).$$

Lemma 15
Let $x \in X$ and $\gamma \in \Sigma$. Then for all $n \in \mathbb{N}$

$$d_X(x, \gamma^n.x) \leq l_\Sigma(\gamma^n) \cdot \max\{d_X(x, s.x) \mid s \in \Sigma\}.$$

Proof. Use triangle inequality and action by isometries. □

Lemma 16 (cf. Bridson, Haefliger 1999)
Let $\gamma \in \Sigma$ and define $|\gamma| := \inf\{d_X(x, \gamma.x) \mid x \in X\}$ (translation length). Then

$$|\gamma| = \lim_{n\to\infty} \frac{d_X(x, \gamma^n.x)}{n}.$$

Lemma 17
Let $\gamma \in \Sigma$. Then $\phi(\gamma)$ fixes a point of X.

Here, as above, $\phi : L(\mathbb{Z}) \to \mathrm{Isom}(X)$ is an action via cellular and rigid isometries.

Proof. We compute

$$\begin{aligned} |\gamma| &= \inf\{d_X(x, \gamma.x) \mid x \in X\} \\ &\overset{16}{=} \lim_{n\to\infty} \frac{d_X(x, \gamma^n.x)}{n} \\ &\overset{15}{\leq} \lim_{n\to\infty} \frac{l_\Sigma(\gamma^n) \cdot \max\{d_X(x, s.x) \mid s \in \Sigma\}}{n} \\ &\overset{14}{=} \max\{d_X(x, s.x) \mid s \in \Sigma\} \lim_{n\to\infty} \frac{O(\log(n))}{n} \\ &= 0. \end{aligned}$$

Since $\phi(\gamma)$ cannot be parabolic, it is elliptic, i.e., $\phi(\gamma)$ fixes a point of X. □

Proof of the fixed-point theorem 12

Proof. By Tavgen 1991 the group $L(\mathbb{Z})$ is boundedly generated by root group elements. The group $L(\mathbb{Z})$ is therefore boundedly generated by the (finite) family of (cyclic) root subgroups.

Lemma 17 implies that each generator of a root subgroup and, hence, each root subgroup has a fixed point.

Any group boundedly generated by a finite family of groups with fixed point has itself a fixed point. $\square$

Application of Corollary 13:
Strong and superrigidity of arithmetic Kac–Moody groups

Theorem 18 (Farahmand, K.)

Mostow–Margulis strong rigidity: *Let* $G(\mathbb{R})$ *and* $G'(\mathbb{R})$ *be irreducible* 2*-spherical Kac–Moody groups and let* $\phi : G(\mathbb{Z}) \to G'(\mathbb{Z})$ *be an isomorphism.*

Then ϕ *uniquely extends to a topological isomorphism*

$$G(\mathbb{R}) \to G'(\mathbb{R}).$$

Theorem 19 (Farahmand, Horn, K.)

Margulis superrigidity: *Let* $G(\mathbb{R})$ *and* $G'(\mathbb{R})$ *be irreducible* 2*-spherical Kac–Moody groups and let* $\phi : G(\mathbb{Z}) \to G'(\mathbb{R})$ *be a homomorphism of groups.*

Then ϕ *uniquely extends to a continuous homomorphism*

$$G(\mathbb{R}) \to G'(\mathbb{R}).$$

Second main ingredient of proof

Proposition 20 (Local superrigidity)
Let H be an irreducible Chevalley group of rank at least 2 and let G be a Kac–Moody group (functor).

Then any group homomorphism $\phi : \Gamma := H(\mathbb{Z}) \to G(\mathbb{R})$ uniquely extends to a continuous group homomorphism

$$\psi : (H(\mathbb{R}), \tau_{\text{Lie}}) \to (G(\mathbb{R}), \tau_{\text{KP}}).$$

Proof:

- Corollary 13: $\phi(\Gamma)$ lies in an algebraic subgroup A of G.
- Margulis 1991, IX.6.15:
Zariski closure $H' := \overline{\phi(\Gamma)}$ in A is semisimple
- Margulis 1991, VII.5.9 + "semisimple = almost direct product of simple factors": exists rational (hence continuous) homomorphism of algebraic groups $\psi : H(\mathbb{R}) \to H'(\mathbb{R}) \hookrightarrow G(\mathbb{R})$
- Γ Zariski dense in $H(\mathbb{R})$: ψ uniquely determined by ϕ.

Margulis superrigidity of arithmetic Kac–Moody groups

Theorem 19 (Farahmand, Horn, K.)
Let $G(\mathbb{R})$ and $G'(\mathbb{R})$ be irreducible 2-spherical Kac–Moody groups and let $\phi : G(\mathbb{Z}) \to G'(\mathbb{R})$ be a homomorphism of groups.

Then ϕ uniquely extends to a continuous homomorphism

$$G(\mathbb{R}) \to G'(\mathbb{R}).$$

Proof of Theorem 19 (superrigidity)

- For distinct simple roots α_i, α_j consider the fundamental subgroups G_{α_i,α_j} of rank 2.
- Define the restriction $\phi_{i,j} : G_{\alpha_i,\alpha_j}(\mathbb{Z}) \to G'(\mathbb{R}) : g \mapsto \phi(g)$.
- By Proposition 20 there exist unique continuous extensions

$$\psi_{i,j} : G_{\alpha_i,\alpha_j}(\mathbb{R}) \to G'(\mathbb{R})$$

for adjacent α_i, α_j.

- Zariski density implies that $\psi_{i,j}$ and $\psi_{j,k}$ coincide on G_{α_j}.
- Universality of $G(\mathbb{R}) \cong \left\langle \bigcup_{\alpha_i,\alpha_j} G_{\alpha_i,\alpha_j} \mid \text{their relations} \right\rangle$ provides a (unique) continuous extension

$$\psi : G(\mathbb{R}) \to G'(\mathbb{R}).$$

Mostow–Margulis strong rigidity of arithmetic Kac–Moody groups

Theorem 18 (Farahmand, K.)
Let $G(\mathbb{R})$ and $G'(\mathbb{R})$ be irreducible 2-spherical Kac–Moody groups and let $\phi : G(\mathbb{Z}) \to G'(\mathbb{Z})$ be an isomorphism.

Then ϕ uniquely extends to a topological isomorphism

$$G(\mathbb{R}) \to G'(\mathbb{R}).$$

Corollary 21 (Solution of the isomorphism problem)
Let G and G' be irreducible 2-spherical Kac–Moody group functors and let $G(\mathbb{Z}) \to G'(\mathbb{Z})$ be a group isomorphism.

Then $G = G'$.

Proof:

- Theorem 18: exists an isomorphism $G(\mathbb{R}) \cong G'(\mathbb{R})$.
- Caprace 2009: $G = G'$.

Proof of Theorem 18 (strong rigidity)

- Consider the compositions

$$G(\mathbb{Z}) \stackrel{\phi}{\to} G'(\mathbb{Z}) \hookrightarrow G'(\mathbb{R}), \qquad G'(\mathbb{Z}) \stackrel{\phi^{-1}}{\to} G(\mathbb{Z}) \hookrightarrow G(\mathbb{R}).$$

- Theorem 19 provides unique continuous extensions

$$\psi : G(\mathbb{R}) \to G'(\mathbb{R}), \qquad \psi' : G'(\mathbb{R}) \to G(\mathbb{R}).$$

- Both $\mathrm{id}_{G(\mathbb{R})}$ and $\psi' \circ \psi$ extend the embedding $G(\mathbb{Z}) \hookrightarrow G(\mathbb{R})$, so by uniqueness $\mathrm{id}_{G(\mathbb{R})} = \psi' \circ \psi$.

- By symmetry ψ is a topological isomorphism.

Further research

- Does rigidity hold over arbitrary rings of S-integers? Probably yes. (PhD project Farahmand)

- Let G be an irreducible 2-spherical Kac–Moody functor. Is $G(\mathbb{Z})$ finitely presented?

$\mathrm{SL}_2(\mathbb{Z})$ acts transitively on rational points of $P_1(\mathbb{Q})$:
$\begin{pmatrix} a & p \\ b & q \end{pmatrix}$ maps 0 to $\frac{a \cdot 0 + p}{b \cdot 0 + q} = \frac{p}{q}$
Euclidean algorithm gives $a, b \in \mathbb{Z}$ with $ap - bq = \mathrm{gcd}(p, q)$.

Therefore $G(\mathbb{Z})$ acts transitively on each of the buildings X_+, X_- of $G(\mathbb{Q})$, by a combinatorial local-to-global argument.

Hence Theorem 1 provides a presentation of $G(\mathbb{Z})$.

However, the stabilizers have huge unipotent parts in the Kac–Moody group that I currently do not understand.

(This is why over fields one takes X^{op} instead of X_+.)

A combinatorial local-to-global argument

Proposition 22

Let X be a set endowed with a family of equivalence relations

$$(\sim_i)_{i \in I} \subseteq X \times X$$

such that

$$\text{transitive hull}\left(\bigcup_{i \in I} \sim_i\right) = X \times X.$$

Let G be a group permuting X preserving each $\sim_i$.

If there exists $p \in X$ such that for each $i \in I$ the stabilizer $G_{[p]_i}$ is transitive on $[p]_i$, then G is transitive on X.

Apply to building with the common-face relations as $(\sim_i)_{i \in I}$.

Some references

Abramenko–Mühlherr 1997: Présentations de certaines BN-paires jumelées comme sommes amalgamées, C.R. Acad. Sci. Paris Sér. I Math. 325

Glöckner–G.–Hartnick 2010: Final group topologies, Kac-Moody groups and Pontryagin duality, Israel J. Math. 177

Gorenstein–Lyons–Solomon 1998:
The classification of the finite simple groups. Number 3. Part I. Chapter A. Almost simple K-groups, AMS Math. Surveys Monogr. 40.3

Hartnick–K.: Two-spherical topological Kac–Moody groups are Kazhdan, `https://www.mittag-leffler.se/preprints/files/IML-1112s-27.pdf`

Lubotzky–Mozes–Raghunathan 2000: The word and Riemannian metrics on lattices of semisimple groups, Inst. Hautes Études Sci. Publ. Math. 91

Margulis 1991: Discrete subgroups of semisimple Lie groups

Rémy 2002:
Groupes de Kac-Moody déployés et presque déployés, Astérisque 277

Tavgen 1991: Bounded generation of Chevalley groups over rings of algebraic S-integers, Math. USSR-Izv. 36

Tits 1987:
Uniqueness and presentation of Kac-Moody groups over fields. J. Alg. 105